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An unknotting theorem for $S^p \times S^q$ embedded in S^{p+q+2}

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Resumo

Neste trabalho definimos mergulho não enodado de $S^p \times S^q$ em S^{p+q+2} $(p, q \geq 1)$ e caracterizamos estes mergulhos em termos de $D^{p+1} \times S^q$ (ou $S^p \times D^{q+1}$) em S^{p+q+2} cujo bordo é a imagem de um dado mergulho e em termos de um S^{p+q+1} em S^{p+q+2} que contém a imagem do mergulho.

An unknotting theorem for $S^p \times S^q$ embedded in S^{p+q+2}

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Abstract. In this note, we define unknotted embeddings of $S^p \times S^q$ into S^{p+q+2} ($p, q \geq 1$) and characterize them in terms of $D^{p+1} \times S^q$ (or $S^p \times D^{q+1}$) in S^{p+q+2} bounded by the image of a given embedding and in terms of an embedded S^{p+q+1} in S^{p+q+2} which contains the image of the embedding.

Let $f : S^p \times S^q \to S^{p+q+2}$ be a smooth embedding, where p and q are integers greater than or equal to one.

DEFINITION 1. Suppose that S^{p+q+1} is embedded in S^{p+q+2} in a standard manner. We say that f is *standard* if $f(S^p \times S^q)$ is the boundary of a tubular neighborhood of a standardly embedded p-sphere in $S^{p+q+1} (\subset S^{p+q+2})$. (Note that this is equivalent to saying that $f(S^p \times S^q)$ is the boundary of a tubular neighborhood of a standardly embedded q-sphere in S^{p+q+1}.) We say that f is

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unknotted if it is isotopic to a standard embedding in S^{p+q+2}. Note that unknotted embeddings are unique up to ambient isotopy of S^{p+q+2}. (Here we are interested in the image of the embedding $f(S^p \times S^q)$ and not the embedding map f itself.)

In [**HK**], Hosokawa and Kawauchi defined unknotted surfaces in S^4 and showed that a surface embedded in S^4 is unknotted if it is isotopic to a surface which lies on a standardly embedded S^3 in S^4. In this note, we characterize the unknotted embeddings of $S^p \times S^q$ in S^{p+q+2} in a similar manner.

For every integer $n \geq 1$, we denote by $i : SO(n) \to SO(n+1)$ the canonical inclusion. Our main result of this note is the following.

THEOREM 1. *Let $f : S^p \times S^q \to S^{p+q+2}$ be a smooth embedding. If $p, q \geq 2, q < 2p+1$ and $i_* : \pi_q(SO(p+1)) \to \pi_q(SO(p+2))$ is surjective, then the following conditions are equivalent.*

(1) *The embedding f is unknotted.*

(2) *$f(S^p \times S^q)$ bounds $D^{p+1} \times S^q$ smoothly embedded in S^{p+q+2}.*

(3) *There exists an embedding $g : S^p \times S^q \to S^{p+q+1}$ such that the image of $S^p \times S^q$ by the composite of g with the standard embedding $S^{p+q+1} \to S^{p+q+2}$ is isotopic to $f(S^p \times S^q)$ in S^{p+q+2}.*

Proof. First suppose the condition (3). Then, since $p, q \geq 2$, we see that $g(S^p \times S^q)$ bounds $D^{p+1} \times S^q$ smoothly embedded in S^{p+q+1} by [**K, W, G, LNS**]. Thus the condition (2) is satisfied.

Now suppose the condition (2). Since $D^{p+1} \times S^q$ can be embedded in S^{p+q+1}, it suffices to show that the embeddings of $D^{p+1} \times S^q$ into S^{p+q+2} are unique up to isotopy.

For the proof, we consider the following construction. Let $\varphi : S^q \to S^{p+q+2}$ be an arbitrary smooth embedding with trivial normal bundle ν_φ. We fix a trivialization $\nu_\varphi \cong S^q \times \mathbf{R}^{p+2}$ and we introduce the standard metric on $\mathbf{R}^{p+2}$. We can identify $N = \{v \in \nu_\varphi | \, ||v|| \leq 1\}$ with the tubular neighborhood of $\varphi(S^q)$ in S^{p+q+2}. For every smooth map $\xi : S^q \to SO(p+2), \xi(x) = (\xi_1(x), \cdots, \xi_{p+2}(x))$ $(x \in S^q, \xi_i(x) \in \mathbf{R}^{p+2})$, we define the embedding $\varphi_\xi : D^{p+1} \times S^q \to S^{p+q+2}$ by $\varphi_\xi((a_1, \cdots, a_{p+1}), x) = (x, a_1\xi_1(x) + \cdots + a_{p+1}\xi_{p+1}(x)) \in N \subset S^q \times \mathbf{R}^{p+2}$, where we identify D^{p+1} with the unit disk in $\mathbf{R}^{p+1}$. Note that the isotopy class of φ_ξ does not change even if we change $\xi : S^q \to SO(p+2)$ homotopically.

Now let $\zeta : D^{p+1} \times S^q \to S^{p+q+2}$ be an arbitrary smooth embedding. Then the embedding $\varphi = \zeta|_{\{0\}\times S^q}$ has a trivial normal bundle. Furthermore, it is not difficult to see that the original embedding ζ is isotopic to φ_ξ for some smooth map $\xi : S^q \to SO(p+2)$. Since $i_* : \pi_q(SO(p+1)) \to \pi_q(SO(p+2))$ is surjective by our hypothesis, we may assume that ξ is a map into $SO(p+1)$. Thus, using the trivialization $\nu_\varphi \cong S^q \times \mathbf{R}^{p+2}$, we see that the image of φ_ξ coincides with $S^q \times D \subset N$, where $D = \{(a_1, \cdots, a_{p+1}, 0) \in \mathbf{R}^{p+2} | a_1^2 + \cdots + a_{p+1}^2 \leq 1\}$. Therefore, the isotopy class of the embedding φ_ξ only depends on the embedding $\varphi : S^q \to S^{p+q+2}$. On the other hand, since $p+q+2 > 3(q+1)/2$, we see that the isotopy classes of smooth embeddings of S^q into S^{p+q+2} are unique up to isotopy by [**H**]. Thus the isotopy classes of smooth embeddings of $D^{p+1} \times S^q \to S^{p+q+2}$ are unique up to isotopy. Thus (2) implies (3). Furthermore this argument shows that the condition (2) implies the condition (1). It is clear that (1) implies (2). This completes the proof. ||

Since $SO(p+2)$ fibers over $S^{p+1} \cong SO(p+2)/SO(p+1)$ with fiber $SO(p+1)$, we have the exact sequence

$$\pi_q(SO(p+1)) \xrightarrow{i_*} \pi_q(SO(p+2)) \to \pi_q(S^{p+1}).$$

Thus, if $q < p+1$, then $i_* : \pi_q(SO(p+1)) \to \pi_q(SO(p+2))$ is surjective. Hence we have

COROLLARY 1. *Let $f : S^p \times S^q \to S^{p+q+2}$ be a smooth embedding. If $p \geq q \geq 2$, then the following conditions are equivalent.*

(1) *The embedding f is unknotted.*

(2) *$f(S^p \times S^q)$ bounds $D^{p+1} \times S^q$ smoothly embedded in S^{p+q+2}.*

(3) *There exists an embedding $g : S^p \times S^q \to S^{p+q+1}$ such that the image of $S^p \times S^q$ by the composite of g with the standard embedding $S^{p+q+1} \to S^{p+q+2}$ is isotopic to $f(S^p \times S^q)$ in S^{p+q+2}.*

REMARK 1. The above arguments show that, in general, the number of isotopy classes of embeddings of $D^{p+1} \times S^q$ into S^{p+q+2} does not exceed the number of elements of $\pi_q(SO(p+2))/i_*\pi_q(SO(p+1))$, which is smaller than or equal to the order of $\pi_q(S^{p+1})$.

REMARK 2. In Theorem 1, if $p < 2q+1$ and $i_* : \pi_p(SO(q+1)) \to \pi_p(SO(q+2))$ is surjective, then the three conditions are also equivalent to the following.

(2)' $f(S^p \times S^q)$ bounds $S^p \times D^{q+1}$ smoothly embedded in S^{p+q+2}.

For example, it is well-known that $\pi_{n+4}(S^n) = 0, n \geq 6, \pi_{n+5}(S^n) = 0, n \geq 7$ and $\pi_{n+12}(S^n) = 0, n \geq 14$. Using these facts, we see that when $(p,q) = (n+4, n-1), n \geq 6, (n+5, n-1), n \geq 7, (n+12, n-1), n \geq 14$, the three conditions in Theorem 1 are also equivalent to (2)' above.

In Theorem 1, when p or q is equal to one, we have the following.

THEOREM 2. *Let $f : S^p \times S^1 \to S^{p+3}$ be a smooth embedding. If $p = 1, 2$, then the following conditions are equivalent.*

(1) *The embedding f is unknotted.*

(2) *$f(S^p \times S^1)$ bounds $D^{p+1} \times S^1$ smoothly embedded in S^{p+3}.*

(2)' *$f(S^p \times S^1)$ bounds $S^p \times D^2$ smoothly embedded in S^{p+3}.*

(3) *There exists an embedding $g : S^p \times S^1 \to S^{p+2}$ such that the image of $S^p \times S^1$ by the composite of g with the standard embedding $S^{p+2} \to S^{p+3}$ is isotopic to $f(S^p \times S^1)$ in S^{p+3}.*

Proof. When $p = 1$, the conditions (2) and (2)' coincide and they imply (1) and (3), which we can see by using the same argument as in the proof of Theorem 1. The condition (3) implies (2) and (2)' by Alexander's torus theorem [**A**]. The condition (1) clearly implies (2) and (2)'.

Now suppose that $p = 2$. Then we see that the condition (2) implies the condition (3) by using the same argument as in the proof of Theorem 1. Suppose the condition (3). Then by [**LNS**], $f(S^p \times S^1)$ bounds a simply connected 4-manifold F homeomorphic to $S^2 \times D^2$ in S^5. (In fact, F is smoothly h-cobordant to $S^2 \times D^2$.) Now let F_0 be the standard $S^2 \times D^2$ embedded in S^5. By a finite iteration, say m, of the boundary connected sum operation of F, F_0 with the standard punctured $S^2 \times S^2$ embedded in S^5, we see that $F\natural(\natural^m S^2 \times S^2)$ is diffeomorphic to $F_0\natural(\natural^m S^2 \times S^2)$ (for example, see [**La, Q**]). Furthermore, their Seifert matrices coincide (for a definition of a Seifert matrix, see [**Le**], for example). Then by using the techniques of [**Le, S**], we see that ∂F and ∂F_0 are isotopic in S^5. Thus the condition (2)' is satisfied. Furthermore, the condition (2)' implies the condition (2), since the smooth embeddings of S^2 in S^5 are unique and $\pi_2(SO(3)) = 0$. These arguments show that the conditions (2), (2)' and (3) imply (1). It is clear that the condition (1) implies the others. This completes the proof. ||

We do not know if the conclusion of Theorem 2 holds also for $p \geq 3$. (The crucial point is that the embeddings of S^p in S^{p+3} are not necessarily unique up to isotopy if $p \geq 3$.) Note that Theorem 2 for $p = 1$ has already been obtained by Hosokawa and Kawauchi [**HK**].

REMARK 3. In Corollary 1, the two conditions (1) and (2) are not equivalent to the following condition (2)' in general.

(2)' $f(S^p \times S^q)$ bounds $S^p \times D^{q+1}$ smoothly embedded in S^{p+q+2}.

For example, consider the case where $q = p - 1$ with p odd. In this case, in the exact sequence

$$\pi_p(SO(p)) \xrightarrow{i_*} \pi_p(SO(p+1)) \xrightarrow{\pi_*} \pi_p(S^p) \longrightarrow \pi_{p-1}(SO(p)),$$

$\pi_{p-1}(SO(p))$ has finite order, since $p-1$ is even, where $\pi : SO(p+1) \longrightarrow S^p$ is the canonical projection. Thus the image of $\pi_* : \pi_p(SO(p+1)) \longrightarrow \pi_p(S^p) \cong \mathbf{Z}$ is infinite cyclic, say $k\mathbf{Z}$ ($k > 0$). For every $n \in \mathbf{Z}$, let $\xi_n : S^p \longrightarrow SO(p+1)$ be a smooth map such that $\pi \circ \xi_n$ represents the element of $\pi_p(S^p)$ corresponding to nk. Then, using the standard embedding $\varphi : S^p \longrightarrow S^{2p+1}$, construct the embedding $\zeta_n = \varphi_{\xi_n} : D^p \times S^p \longrightarrow S^{2p+1}$ as in the proof of Theorem 1. Here we take the trivialization $\nu_\varphi \cong S^p \times \mathbf{R}^{p+1}$ so that $S^p \times \{(1, 0, \cdots, 0)\} \subset N$ has linking number 0 with $S^p \times \{(0, \cdots, 0)\}$ in S^{2p+1}. Then consider the embeddings $\eta_n = \zeta_n|_{\partial(D^p \times S^p)}$ ($n \in \mathbf{Z}$). They bound $D^p \times S^p$ embedded in S^{2p+1}. We show that $\eta_n(S^{p-1} \times S^p)$ is not isotopic to $\eta_m(S^{p-1} \times S^p)$ if $|n| \neq |m|$. This is seen as follows. It is not difficult to see that the Seifert matrix of $\zeta_n(D^p \times S^p)$ is equal to $(\pm nk)$. Thus the Alexander polynomial of $\eta_n(S^{p-1} \times S^p)$ is equal to $nk(t-1)$; in other words, the p-th homology group in integral coefficients of the infinite cyclic covering space of the complement of $\eta_n(S^{p-1} \times S^p)$ in S^{2p+1} is isomorphic to Γ/I as a Γ-module, where $\Gamma = \mathbf{Z}[t, t^{-1}]$ and I is the ideal generated by $nk(t-1)$. Thus $\eta_n(S^{p-1} \times S^p)$ is isotopic to $\eta_m(S^{p-1} \times S^p)$ only if $|n| = |m|$. In particular, there exist pairwise non-isotopic infinitely many embeddings of $S^p \times S^{p-1}$ into S^{2p+1} which satisfy the condition (2)' with $q = p - 1$.

The above example also shows that the condition on $i_* : \pi_q(SO(p+1)) \to \pi_q(SO(p+2))$ is essential in Theorem 1.

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