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**Indices of Newton Non-Degenerate Vector Fields and a
Conjecture of Loewner for Surfaces in \mathbb{R}^4**

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INDICES OF NEWTON NON-DEGENERATE VECTOR FIELDS AND A CONJECTURE OF LOEWNER FOR SURFACES IN \mathbb{R}^4 .

CARLOS GUTIERREZ AND MARIA APARECIDA SOARES RUAS

ABSTRACT. We study the index of a vector field in \mathbb{R}^2 , with isolated singularity, in terms of conditions on the Newton polyhedra associated to its coordinates. When the vector field is Newton non-degenerate, we show that its index is determined by the principal part of the Newton polyhedra. As a consequence we can prove that, under very mild conditions, the index of an isolated inflection point of a locally convex surface generically embedded in \mathbb{R}^4 is the same as the index of an umbilic point of a surface immersed in \mathbb{R}^3 .

RESUMO. O objetivo do trabalho é estudar o índice de uma singularidade isolada de um campo vetorial em \mathbb{R}^2 , em termos de condições sobre poliedros de Newton associados às funções coordenadas do campo. Quando o campo vetorial é Newton não degenerado, mostramos que seu índice é determinado pela parte principal do poliedro de Newton. Como consequência, com pequenas hipóteses de genericidade, é possível mostrar que o índice de um ponto de inflexão isolado de uma superfície localmente convexa genericamente mergulhada em \mathbb{R}^4 é igual ao índice de um ponto umbílico de uma superfície imersa em \mathbb{R}^3 .

1. INTRODUCTION

This article has been motivated by the classical (local) Loewner's Conjecture which states that every umbilic of a smooth surface immersed in \mathbb{R}^3 must have index less than or equal to one. It was shown in [6] that the index of an isolated inflection point (in the sense of Little [12]) of a locally convex surface generically embedded in \mathbb{R}^4 is $\pm 1/2$.

Our first result, Theorem 1, was inspired in the result of M. Brunella and M. Miari [3], in which they proved that a Newton non-degenerate vector field in the plane possessing characteristic orbit, is topologically equivalent to its principal part. Here we deal with the index of a vector field at an isolated singularity. While our conditions are also given in terms of Newton Polyhedra, they are different from those of Brunella and Miari and, we believe, better suited for the case of indices of vector fields. Also, as it should be, proofs are much shorter.

In our second result we show that, under very mild conditions given in terms of Newton Polyhedra, the index of an isolated inflection point is the same as the index of an umbilic point of a surface immersed in \mathbb{R}^3 . It may be possible that

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the Loewner Conjecture can be extended to the case of isolated inflection points of locally convex surfaces in \mathbb{R}^4 .

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2. INDEX OF A VECTOR FIELD AT A SINGULARITY

Let \mathcal{E}_n be the set of smooth germs $g : (\mathbb{R}^n, 0) \rightarrow \mathbb{R}$. To $g \in \mathcal{E}_n$ associate its formal Taylor series expansion at 0: $\hat{g}(x) = \sum a_k x^k$. Define $\text{supp } g = \{k \in \mathbb{Z}^n : a_k \neq 0\}$. Given $\Delta \subset \mathbb{Z}^n$, we define $g|_\Delta = \sum_{k \in \Delta} a_k x^k$.

The Newton Polyhedron of $g \in \mathcal{E}_n$, denoted by $\Gamma_+(g)$, is the convex hull in \mathbb{R}_+^n of the set $\cup\{k + v : k \in \text{supp } g, v \in \mathbb{R}_+^n\}$. The Newton Diagram of g will be the union $\Gamma(g)$ of all compact faces of $\Gamma_+(g)$.

Let $X = (f, g) : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ be the germ of a vector field in the plane. We call the pair $(\Gamma(f), \Gamma(g)) = \Gamma(X)$ the Newton pair of X .

We denote by $X_\Gamma = (f|_{\Gamma(f)}, g|_{\Gamma(g)})$ the principal part of the vector field X . The Newton pair $\Gamma(X) = (\Gamma(f), \Gamma(g))$ is Newton non degenerate if for all pair of parallell faces $\Delta_1 \in \Gamma(f)$ and $\Delta_2 \in \Gamma(g)$, the following holds: Δ_1 and Δ_2 are compact and the equations $f|_{\Delta_1} = 0$ and $g|_{\Delta_2} = 0$ have no common solutions in $(\mathbb{R} \setminus 0)^2$.

Theorem 2.1. *Let $X = (f, g) : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ be the germ of a vector field in the plane. If $\Gamma(X)$ is Newton non-degenerate, then, there exists a neighborhood U of $(0, 0)$ such that, $\forall (x, y) \in U \setminus \{(0, 0)\}$ and $\forall s \in [0, 1]$,*

$$\begin{aligned} 0 &< |sX(x, y) + (1 - s)X_\Gamma(x, y)| \\ &:= |s f(x, y) + (1 - s) f|_{\Gamma(f)}(x, y)| + |s g(x, y) + (1 - s) g|_{\Gamma(g)}(x, y)| \end{aligned}$$

In particular $(0, 0)$ is an isolated singularity of both X and X_Γ and

$$\text{Index}(X, 0) = \text{Index}(X_\Gamma, 0)$$

Proof. Let $S_+^1 = \{(a, b) \in \mathbb{R}^2 : a^2 + b^2 = 1, a > 0, b > 0\}$. Given $(a, b) \in S_+^1$ and $h \in \{f, f|_{\Gamma(f)}, g, g|_{\Gamma(g)}\}$, denote by $L_h(a, b)$ the straight line orthogonal to (a, b) that meets $\Gamma(h)$ and such that one of the closed half-planes bounded by $L_h(a, b)$ contains $\Gamma_+(h)$. As $\Gamma_+(h)$ is convex, we obtain:

- (1a) $L_h(a, b) \cap \Gamma_+(h) \subset \Gamma(h)$ is either a point or a segment. In the last case it is a compact face of $\Gamma(h)$;
- (1b) if $(m_0, n_0), (m_1, n_1) \in L_h(a, b) \cap \Gamma(h)$, then $(m_0, n_0) \cdot (a, b) = (m_1, n_1) \cdot (a, b)$, where “ \cdot ” denotes the usual inner product of \mathbb{R}^2 .
- (1c) if $(m_0, n_0) \in L_h(a, b) \cap \Gamma(h)$ and $\Gamma(h) \setminus L_h(a, b)$ is not empty, then, there exists $\epsilon(a, b, h) > 0$ such that

$$(m_0, n_0) \cdot (a, b) + \epsilon(a, b, h) = \inf\{(m, n) \cdot (a, b) / (m, n) \in \Gamma_+(h) \setminus L_h(a, b)\}.$$

This proof is organized as follows: We shall find, for each $(a, b) \in S_+^1 \cup \{(0, 1), (1, 0)\}$, a small curved sector of \mathbb{R}^2 having its vertex at the origin and such that X restricted to this sector satisfies the theorem. By local compacness, we will be able to cover a neighborhood of $(0, 0) \in \mathbb{R}^2$ with finitely many of these sectors. In this way X restricted to this neighborhood of $(0, 0)$ will satisfy the theorem. We shall accomplish

this by studying, when $h \in \{f, g\}$, the following cases: (i) $L(a, b) \cap \Gamma_+(h)$ is a compact fase of $\Gamma(h)$ non-reduced to a point (and so, $(a, b) \in S_+^1$); (ii) $(a, b) \in S_+^1$ and $L(a, b) \cap \Gamma_+(h)$ is a one point set of $\Gamma(h)$; and (iii) $L_h(1, 0) \cap \Gamma_+(h)$ and $L_h(0, 1) \cap \Gamma_+(h)$ are one-point-sets.

To simplify matters, we shall continue by assuming that

$$(2) \quad \Gamma_+(h) \setminus L_h(a, b) \neq \emptyset$$

Suppose first that $L_h(a, b) \cap \Gamma(h)$ is a segment and let (M, n) and (m, N) be its endpoints. We shall assume $0 \leq m < M$ and $0 \leq n < N$. Let $\epsilon = \epsilon(u, v, h)$ be defined in $B_\delta(a, b) = \{(u, v) \in S_+^1 : |(u - a, v - b)| < \delta\}$ by

$$\begin{aligned} \epsilon(u, v, h) = & \inf\{(\mu, \nu) \cdot (u, v) : (\mu, \nu) \in \Gamma_+(h) \setminus L_h(a, b)\} \\ & - \sup\{(\mu, \nu) \cdot (u, v) : (\mu, \nu) \in L_h(a, b) \cap \Gamma(h)\}. \end{aligned}$$

If $\delta > 0$ is small enough, then $\epsilon(u, v, h)$ depends continuously on (u, v) and

$$(3)$$

$$\epsilon(u, v, h) \in \left(\frac{3}{4} \epsilon(a, b, h), \frac{5}{4} \epsilon(a, b, h)\right);$$

This implies that, for all $(u, v) \in B_\delta(a, b)$,

$$(4)$$

$$\begin{aligned} (3/4) \epsilon(a, b, h) + u \cdot M + v \cdot n & < \epsilon(u, v, h) + u \cdot M + v \cdot n \\ & \leq \epsilon(u, v, h) + \sup\{(\mu, \nu) \cdot (u, v) : (\mu, \nu) \in L_h(a, b)\} \\ & = \inf\{(\mu, \nu) \cdot (u, v) : (\mu, \nu) \in \Gamma_+(h) \setminus L_h(a, b)\} \end{aligned}$$

Therefore, if $\delta > 0$ and $\sigma > 0$ are small enough, there is a continuous function $R(t, u, v, h)$ defined for all $(t, u, v) \in [0, \sigma) \times B_\delta(a, b)$, such that:

$$(5)$$

$$\begin{aligned} h(t^u, t^v) = & a_{m, N} t^{u \cdot m + v \cdot N} + \dots + a_{M, n} t^{u \cdot M + v \cdot n} \\ & + t^{u \cdot M + v \cdot n} t^{(1/2)\epsilon(a, b, h)} R(t, u, v, h) \end{aligned}$$

Let $\rho = \rho(u)$ be defined in $\{u \leq a : (u, v) \in B_\delta(a, b)\}$ by the equation

$$(6) \quad u = a \rho(u)$$

Also, let $k = k(t, u, v)$ be defined in $[0, \sigma) \times \{(u, v) \in B_\delta(a, b) : u \leq a\}$ by the equation

$$(7)$$

$$t^v = k(t, u, v) t^{b \rho(u)}$$

As $\rho(u) \leq 1$ and $v \geq b$, we obtain that $0 < k(t, u, v) \leq 1$. Therefore, using (1b)-(1c) and (5)-(7), we obtain that, for all $(t, u, v) \in (0, \sigma) \times \{(u, v) \in B_\delta(a, b) : u \leq a\}$,

$$(8)$$

$$\begin{aligned} s h(t^u, t^v) + (1 - s) h|_{\Gamma(h)}(t^u, t^v) = & \\ = & (s h((t^\rho)^a, k(t^\rho)^b) + (1 - s) h|_{\Gamma(h)}((t^\rho)^a, k(t^\rho)^b)) \\ = & (k^n (t^\rho)^{a \cdot m + b \cdot N}) (a_{m, N} k^{N - m} + \dots + a_{M, n} + s t^{(1/2)\epsilon(a, b, h)} R(t, u, v, h) \\ & + (1 - s) t^{(1/2)\epsilon(a, b, h|_{\Gamma(h)})} R(t, u, v, h|_{\Gamma(h)})) \end{aligned}$$

This last expression shows that, under conditions above (in particular when $t > 0$ is small),

$$|s h(t^u, t^v) + (1-s) h|_{\Gamma(h)}(t^u, t^v)|$$

can only be zero nearby the real roots of the polynomial, in the variable K ,

$$a_{m,N} K^{N-m} + \dots + a_{M,n}$$

Suppose now that $L_h(a, b) \cap \Gamma(h)$ is a one-point-set. Then, by denoting $L_h(a, b) \cap \Gamma(h) = \{a_{m,N}\} = \{a_{M,n}\}$ and proceeding as above we shall obtain the following relation which corresponds to item (8) above under the same assumptions:

(8')

$$\begin{aligned} s h(t^u, t^v) + (1-s) h|_{\Gamma(h)}(t^u, t^v) &= \\ &= (s h((t^\rho)^a, k(t^\rho)^b) + (1-s) h|_{\Gamma(h)}((t^\rho)^a, k(t^\rho)^b)) \\ &= (k^n (t^\rho)^{a \cdot m + b \cdot N}) (a_{M,n} + s t^{(1/2)\epsilon(a,b,h)} R(t, u, v, h) \\ &\quad + (1-s) t^{(1/2)\epsilon(a,b,h|_{\Gamma(h)})} R(t, u, v, h|_{\Gamma(h)})) \end{aligned}$$

This expression shows us that, if $\delta > 0$ and $\sigma > 0$ are small enough, then for all $(t, u, v) \in (0, \sigma) \times \{(u, v) \in B_\delta(a, b) : u \leq a\}$,

$$|s h(t^u, t^v) + (1-s) h|_{\Gamma(h)}(t^u, t^v)| > 0$$

Therefore, using the fact that $X(\Gamma)$ is non-degenerate and by extending the arguments above to $\{(u, v) \in B_\delta(a, b) : v \leq b\}$, we obtain,

(9) If $(a, b) \in S_+^1$, then there are $\delta > 0$ and $\sigma > 0$ such that, $\forall (u, v) \in B_\delta(a, b)$ and $\forall 0 < t < \sigma$,

$$|s f(t^u, t^v) + (1-s) f|_{\Gamma(f)}(t^u, t^v)| + |s g(t^u, t^v) + (1-s) g|_{\Gamma(g)}(t^u, t^v)| > 0.$$

Now, if $L_h(1, 0) \cap \Gamma(h) = (0, N)$, then, it may be seen that there exists $\rho > 0$ such that for all $0 \leq |y| < \rho$ and $-1 \leq r \leq 1$,

$$h(r y^{N+1}, y) = a_{0,N} y^N + y^{N+1} R(y, r)$$

for some continuous real valued function $R = R(y, r)$. It follows that $|h|$ restricted to a set of the form $\{(x, y) : 0 < |y| < \rho, -\rho^{N+1} \leq x \leq \rho^{N+1}\}$ is positive.

Proceeding similarly when $L_h(0, 1) \cap \Gamma(h) = (M, 0)$ and using the fact that $X(\Gamma)$ is non-degenerate, we may conclude that there exists $\rho > 0$ such that, for all $s \in [0, 1]$, the restriction of $|s f + (1-s) g|$ to the set $\Sigma(\rho)$, which is the union of

$$\{(x, y) : 0 < |y| < \rho, -\rho^{N+1} \leq x \leq \rho^{N+1}\}$$

and

$$\{(x, y) : 0 < |x| < \rho, -\rho^{M+1} \leq y \leq \rho^{M+1}\},$$

is positive.

Using the fact that $\{(a, b) \in \mathbb{R}^2 : a^2 + b^2 = 1, a \geq 0, b \geq 0\}$ is compact and assuming that $\rho > 0$ is small enough, we may cover a neighborhood of $(0, 0)$ in $\{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$ with the union of $(0, 0) \cup \Sigma(\rho)$ with finitely many sectors of the form

$$\{(x, y) \in \mathbb{R}^2 : x = t^u, y = t^v, (u, v) \in B_\delta(a, b), 0 < t < \sigma\}$$

and for which (9) is true.

Similarly, we may cover a neighborhood of $(0, 0)$ in \mathbb{R}^2 with the union of $(0, 0) \cup \Sigma(\rho)$ with finitely many sectors of one of the forms

$$\begin{aligned} \{(x, y) \in \mathbb{R}^2 : x = t^u, y = t^v, (u, v) \in B_\delta(a, b), 0 < t < \sigma\} \\ \{(x, y) \in \mathbb{R}^2 : x = -t^u, y = t^v, (u, v) \in B_\delta(a, b), 0 < t < \sigma\} \\ \{(x, y) \in \mathbb{R}^2 : x = -t^u, y = -t^v, (u, v) \in B_\delta(a, b), 0 < t < \sigma\} \\ \{(x, y) \in \mathbb{R}^2 : x = t^u, y = -t^v, (u, v) \in B_\delta(a, b), 0 < t < \sigma\} \end{aligned}$$

and for which the statement that corresponds to (9) is true. The theorem follows from this. \square

3. LOCALLY STRICTLY CONVEX SURFACES OF \mathbb{R}^4

The asymptotic line fields associated to an embedding f of a surface M in \mathbb{R}^4 were studied in [6]. A generically embedded surface $f(M)$ always has an open region at which there are defined two line fields V_1 and V_2 of asymptotic directions. These two fields eventually collapse onto a unique one over a curve in M . Their integral lines are called asymptotic lines and their singular points are the inflection points of the embedding as defined by Little in [12]. The geometry of a generically embedded surface in the neighborhood of an inflection point was studied in [13]. The differential equation of the asymptotic lines ([6]) of M is the following binary equation:

$$(3.1) \quad \begin{vmatrix} f_{1x} & f_{2x} & f_{3x} & f_{4x} & 0 \\ f_{1y} & f_{2y} & f_{3y} & f_{4y} & 0 \\ f_{1xx} & f_{2xx} & f_{3xx} & f_{4xx} & dy^2 \\ f_{1xy} & f_{2xy} & f_{3xy} & f_{4xy} & -dxdy \\ f_{1yy} & f_{2yy} & f_{3yy} & f_{4yy} & dx^2 \end{vmatrix} = 0,$$

where (f_1, f_2, f_3, f_4) are the coordinate functions of the embedding.

In this section we study singular points of this equation when M is locally strictly convexly embedded in \mathbb{R}^4 .

Definition 3.1. A hyperplane H in \mathbb{R}^4 is a nonsingular support hyperplane to M at the point p if it is tangent to M at p , $M \cap H = \{p\}$ and p is a non-degenerate critical point of the linear projection $\pi : M \rightarrow L_H$, where L_H is the line in \mathbb{R}^4 , passing through p , orthogonal to H .

Definition 3.2. The embedding $f : M \rightarrow \mathbb{R}^4$ is locally strictly convex if every point admits a nonsingular support hyperplane.

When the embedding $f : M \rightarrow \mathbb{R}^4$ is locally strictly convex, the pair of transversal foliations on M induced by the asymptotic lines are -away from a discrete set of singularities- globally defined ([6]); their singular points are isolated and coincide with the inflection points of imaginary type.

We shall study the indices of these inflection points with respect to either of the asymptotic foliations. The index does not depend on the specific foliation \mathcal{A}_i because \mathcal{A}_1 and \mathcal{A}_2 are transversal to each other in the complement of the inflection points.

This section is motivated by a famous conjecture due to Loewner which states that -with respect to either of the foliations induced by the principal lines of curvature of an embedded surface in \mathbb{R}^3 - there are no umbilics of index bigger than one. This Loewner Conjecture has been asserted to be true for analytic surfaces by

several authors among which H. Hamburger [10], G. Bol [2], T. Klotz [11] and C. J. Titus [16]. In the following we shall show that under very mild conditions, the index of an isolated inflection point - with respect to \mathcal{A}_1 - is the same as the index of an umbilic point of a surface immersed in \mathbb{R}^3 .

The following lemma gives a useful normal form for M in a neighbourhood of an isolated inflection point of a strictly convexly embedded surface.

Lemma 3.1. *Let M be locally strictly convexly embedded in \mathbb{R}^4 and $p \in M$ be an isolated inflection point. Then, up to a rigid motion of \mathbb{R}^4 , it can be assumed that $p = (0, 0, 0, 0)$ and that M , around p , admits a parametrization of the form*

$$\alpha : (x, y) \rightarrow (x, y, \frac{1}{2}(x^2 + y^2) + F(x, y), G(x, y)),$$

where $J^2F(0, 0) = J^2G(0, 0) = 0$,

Proof. We can assume $p = (0, 0, 0, 0)$ and that the embedding is in Monge's form

$$\alpha : (x, y) \rightarrow (x, y, f_1(x, y), f_2(x, y)),$$

where $J^1f_1(0, 0) = J^1f_2(0, 0) = 0$. Since p is an inflection point of the embedding, the second fundamental form has rank 1 at p . Hence, the 2-jets of f_1 and f_2 are linearly dependent at zero. By assumption the embedding is locally strictly convex, hence by a rigid motion in \mathbb{R}^4 , we can assume that M has the desired normal form. \square

For the parametrization α of Lemma 3.1, the differential equation (3.1) takes the form

$$(3.2) \quad \begin{vmatrix} dy^2 & -dxdy & dx^2 \\ 1 + F_{xx} & F_{xy} & 1 + F_{yy} \\ G_{xx} & G_{xy} & G_{yy} \end{vmatrix} = 0,$$

This equation may be rewritten as:

$$(3.3) \quad (G_{xy} + \Delta_{12}(F, G))dx^2 + (G_{yy} - G_{xx} + \Delta_{13}(F, G)dxdy + (-G_{xy} - \Delta_{23}(F, G))dy^2 = 0,$$

where

$$\begin{aligned} \Delta_{12}(F, G) &= G_{xy}F_{xx} - G_{xx}F_{xy}, \\ \Delta_{13}(F, G) &= F_{xx}G_{yy} - G_{xx}F_{yy}, \\ \Delta_{23}(F, G) &= G_{xy}F_{yy} - G_{yy}F_{xy}. \end{aligned}$$

Theorem 3.2. *Let $F, G : \mathbb{R}^2, 0 \rightarrow (\mathbb{R}, 0)$ be germs of smooth functions such that $J^2F(0) = J^2G(0) = 0$. Let X, X_1, X_2 be germs of vector fields in $\mathbb{R}^2, 0$ defined by*

$$\begin{aligned} X(x, y) &= (G_{xx} - G_{yy}, 2G_{xy}), \\ X_1(x, y) &= (G_{xx} - G_{yy} - \Delta_{13}(F, G), 2G_{xy} + 2\Delta_{12}(F, G)), \\ X_2(x, y) &= (G_{xx} - G_{yy} - \Delta_{13}(F, G), 2G_{xy} + 2\Delta_{23}(F, G)); \end{aligned}$$

Suppose that $\Gamma(X)$ is Newton non-degenerate and that $X_\Gamma = (X_1)_\Gamma = (X_2)_\Gamma$.

Then

(i) the index at 0, of (either of the two foliations induced by) equation (3), is the same as the index at 0 of (either of the two foliations induced by) equation

$$(3.4) \quad G_{xy}dx^2 + (G_{yy} - G_{xx})dxdy - G_{xy}dy^2 = 0.$$

This common index is half of the index of X at 0.

(ii) there exists a smooth surface in \mathbb{R}^3 such that the differential equation of its principal lines of curvature -in a suitable coordinate system- is precisely equation (3.4).

Proof. By the same argument as that of the Proof of Theorem 2.1, there exists a neighborhood V of $0 \in \mathbb{R}^2$ such that, for all $s \in [0, 1]$, and -when restricted to V - the vector fields

$$(3.5) \quad \begin{aligned} X &= (G_{xx} - G_{yy}, 2G_{xy}), \\ X_1^{(s)} &= (G_{xx} - G_{yy} - \Delta_{13}(sF, G), 2G_{xy} + 2\Delta_{12}(sF, G)), \\ X_2^{(s)} &= (G_{xx} - G_{yy} - \Delta_{13}(sF, G), 2G_{xy} + 2\Delta_{23}(sF, G)), \end{aligned}$$

have 0 as the only singularity. Therefore, for all $s \in [0, 1]$ and for every point of $V \setminus \{0\}$,

$$(G_{xx} - G_{yy} + \Delta_{13}(sF, G))^2 + (2G_{xy} + \Delta_{12}(sF, G))(2G_{xy} + \Delta_{23}(sF, G)) > 0.$$

This and Homotopy Theory [8] imply that, given $s \in [0, 1]$, the index at 0 of equation

$$(3.6) \quad \begin{aligned} (G_{xy} + \Delta_{12}(sF, G))dx^2 &+ (G_{yy} - G_{xx} + \Delta_{13}(sF, G))dxdy \\ &+ (-G_{xy} - \Delta_{23}(sF, G))dy^2 = 0, \end{aligned}$$

does not depend on s . Taking $s = 0, 1$, the first statement of item (i) follows. It is not difficult to check the second statement of (i) (see, for instance, [15]). The proof of (ii) can be found in [7]. \square

In the following corollary we show that the above result always holds when the germ G is quasi-homogeneous and X is non-degenerate. We recall that the germ $G : \mathbb{R}^2, 0 \rightarrow \mathbb{R}, 0$ is quasi-homogeneous with weights $w = (w_1, w_2) \in \mathbb{N}^2$ and quasi-degree $d \in \mathbb{N}$, if $G(\lambda^{w_1}x, \lambda^{w_2}y) = \lambda^d G(x, y)$, for all $\lambda > 0$.

Corollary 3.3. *Let $G, j^2G(0, 0) = 0$, be a quasi-homogeneous polynomial defining the vector field $X = (G_{xx} - G_{yy}, 2G_{xy})$ which is Newton non degenerate. Then, the index of equation (3) does not depend on $F : \mathbb{R}^2, 0 \rightarrow \mathbb{R}, 0$, for any F such that $j^2F(0, 0) = 0$, and it is half of the index of the vector field X at 0 .*

Proof. Let us assume that G is quasi-homogeneous with weights (α, β) , $\alpha \leq \beta$, and quasi-degree d . Then, while the function $2G_{xy}$ is clearly quasi-homogeneous of type $(\alpha, \beta; d - \alpha - \beta)$, this is not always true for the function $G_{yy} - G_{xx}$. However, a direct analysis of the possibilities shows that in any case, the remainders $\Delta_{12}(F, G)$, $\Delta_{13}(F, G)$, $\Delta_{23}(F, G)$ satisfy the hypothesis of Theorem 3. \square

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