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Z2-EQUIVARIANT SYSTEMS

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BI-CENTER PROBLEM FOR SOME CLASSES OF Z_2-EQUIVARIANT SYSTEMS

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Abstract. We investigate the simultaneous existence of two centers (bi-center) for two families of planar Z_2-equivariant differential systems. First we present necessary and sufficient conditions for the existence of an isochronous bi-center for a planar Z_2-equivariant cubic system having two centers at the points $(-1, 0)$ and $(1, 0)$, completing the study done by Liu and Li (2011). Next, we give conditions for the existence of a bi-center and study its isochronicity for a planar Z_2-equivariant quintic system having two weak foci or centers at the points $(-1, 0)$ and $(1, 0)$. We also give an example of a cubic system with three isochronous centers.

1. Introduction

In a real planar analytic differential system a singular point with pure imaginary eigenvalues of the matrix of the linear approximation can be either a focus or a center. The problem to distinguish between a center or a focus is called the *center-focus problem*. If the singular point is a center the next arising problem is to determine whether the center is isochronous.

Although the center-focus problem have been studied during more than hundred years by many authors it is unresolved even for planar systems with cubic nonlinearities. For polynomial quadratic systems it is solved in [11, 17, 18, 36], were the authors presented necessary and sufficient conditions for the existence of a center. For polynomial cubic systems only particular families were investigated, see e.g. [1, 4, 21, 22, 33, 34, 35, 37] and references therein. There are also some works on the center problem for families of polynomial systems of higher degrees, see e.g. [12, 13, 14] and references given there.

The studies of isochronicity of polynomial systems goes back to Loud [27], who found the necessary and sufficient conditions for isochronicity of the quadratic system. Latter on, the isochronicity problem was solved for the linear center perturbed by homogeneous polynomials of degree three [29] and degree five [31]. There are also many works devoted to the investigation of particular families of polynomial systems, see e.g. [2, 3, 6, 28, 33] and references therein.

The existence of two simultaneous centers in planar differential systems was investigated only for very few particular families of systems. Kirnitskaya and Sibirskii in [19] and Li in [20] studied the bi-center problem for quadratic systems. They presented necessary and sufficient conditions for a planar quadratic differential system to have two centers simultaneously. Conti [7] investigated a particular family of cubic systems obtaining the first cubic system possessing a bi-center. Chen, Lu and Wang [5] studied other particular family of cubic systems called Kukles system, characterizing when such systems have a bi-center.

Recently Liu and Li [24] studied the Z_2-equivariant cubic system of the form

$$\dot{x} = X_1(x, y) + X_3(x, y), \quad \dot{y} = Y_1(x, y) + Y_3(x, y), \tag{1.1}$$

where $X_i(x,y), Y_i(x,y)$, $(i=1,3)$, are homogeneous polynomials of degree i, in the variables x and y and having two weak foci or centers at the points $(-1,0)$ and $(1,0)$. They presented necessary and sufficient conditions for system (1.1) to have a bi-center at these points. Du [10] also studied a system having Z_2-equivariant symmetry. He obtained necessary and sufficient conditions for the existence of a bi-center and an isochronous bi-center for a particular family of Z_2-equivariant system of degree seven.

The main purpose of this paper is to find conditions for the existence of bi-centers and isochronous bi-centers for two families of Z_2-equivariant systems. This paper has three main results: Theorem 3.1 which gives the necessary and sufficient conditions for the existence of an isochronous bi-center in system (1.1) at the points $(-1,0)$ and $(1,0)$, Theorem 4.2 and Theorem 4.4, which characterize the existence of a bi–center and an isochronous bi–center, respectively, in the Z_2-equivariant quintic system of the form

$$\dot{x} = X_1(x,y) + X_5(x,y) = X(x,y), \quad \dot{y} = Y_1(x,y) + Y_5(x,y) = Y(x,y), \tag{1.2}$$

where $X_i(x,y), Y_i(x,y)$, $(i=1,5)$ are homogeneous polynomials of degree i in the variables x and y and (1.2) has two weak foci or centers at the points $(-1,0)$ and $(1,0)$. The reason for choosing homogeneous polynimials of degree five in (1.2) (rather than polynomials of degree four) is to assure the existence of Z_2-equivariant symmetry, which can appear if the polynomials defining the system have only odd degree monomials. So, if we replace in (1.2) $X_5(x,y)$ and $Y_5(x,y)$ with homogeneous polynomials of degree four, then the system can not have a bi-center at $(-1,0)$ and $(1,0)$.

2. Preliminaries

In this section we introduce some definitions and basic results which are used in this paper.

2.1. **The center and integrability problems.** Consider a real polynomial differential system on $\mathbb{R}^2$ having a weak focus or an elementary center at the point (x_0,y_0). Moving the point (x_0,y_0) to the origin and applying a linear change of coordinates and a time rescaling we can write the system in the form

$$\dot{x} = -y + \sum_{p+q\geq 2}^{n} a_{p,q}x^p y^q = P(x,y), \quad \dot{y} = x + \sum_{p+q\geq 2}^{n} b_{p,q}x^p y^q = Q(x,y), \tag{2.1}$$

where $a_{p,q}, b_{p,q} \in \mathbb{R}$. We will denote by $\mathfrak{X}$ its corresponding vector field

$$\mathfrak{X} = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}.$$

A local first integral of system (2.1) is a nonconstant differentiable function Φ from a neighbourhood of the origin in $\mathbb{R}^2$ into $\mathbb{R}$ which is constant on trajectories of (2.1), equivalently,

$$\mathfrak{X}\Phi = P\Phi_x + Q\Phi_y \equiv 0.$$

A formal first integral of system (2.1) is a formal power series Φ in the variables x and y satisfying $\mathfrak{X}\Phi \equiv 0$.

The next theorem due to Poincaré and Lyapunov ([23, 30]) characterizes when a system of the form (2.1) posses a center at the origin.

Theorem 2.1. *System* (2.1) *has a center at the origin if and only if it admits a local analytic first integral of the form*

$$\Phi(x,y) = x^2 + y^2 + \cdots . \tag{2.2}$$

Moreover, the existence of a formal first integral Φ *of the form* (2.2) *implies the existence of a local analytic first integral of the same form.*

For system (2.1) it is always possible to find a formal power series of the form $\Psi = x^2 + y^2 + \sum_{k+j\geq 3} u_{k,j}x^k y^j$, such that

$$\mathfrak{X}\Psi = v_1(xy)^2 + v_2(x^2y)^2 + v_3(x^3y)^2 + \cdots + v_k(x^ky)^2 + \cdots,$$

where v_k are polynomials in the parameters $a_{p,q}$ and $b_{p,q}$ of system (2.1) called *focus quantities.* The polynomials v_k represent obstacles for the existence of the first integral of the form (2.2), that is, system (2.1) admits a first integral of the form (2.2) if and only if $v_k = 0$, for all $k \geq 1$. Thus, the simultaneous vanishing of all focus quantities provide conditions which characterize when a system of the form (2.1) has a center at the origin. The ideal defined by the focus quantities, $\mathcal{B} = \langle v_1, v_2, \dots \rangle \subset \mathbb{C}[a,b]$, where a and b represents all the parameters $a_{p,q}$ and $b_{p,q}$, respectively, of system (2.1) is called *Bautin ideal.* The affine variety $V_{\mathcal{C}} = \mathbf{V}(\mathcal{B})$, is called the *center variety* of system (2.1).

By the Hilbert Basis Theorem there exist a positive integer k such that $\mathcal{B} = \mathcal{B}_k = \langle v_1, ..., v_k\rangle$. Note that the inclusion $V_{\mathcal{C}} = \mathbf{V}(\mathcal{B}) \subset \mathbf{V}(\mathcal{B}_k)$ holds for any $k \geq 1$. The opposite inclusion is verified finding the irreducible decomposition of $\mathbf{V}(\mathcal{B}_k)$ and then checking that any point of each component of the decomposition corresponds to a system having a center at the origin. To find the irreducible decomposition of $\mathbf{V}(\mathcal{B}_k)$ we performed computations with the routine `minAssGTZ` [9] (which is based on the algorithm of [16]) of the computer algebra system SINGULAR [8].

We now recall few results from the Darboux theory of integrability that allow to find first integrals of polynomial systems (see e.g. [25, 26, 33] for more details).

Consider the general polynomial differential system

$$\dot{x} = U(x,y), \quad \dot{y} = V(x,y), \tag{2.3}$$

where $x, y \in \mathbb{R}$, U and V are polynomials in the variables x, y without constant terms that have no common factor, and $m = max(deg(U), deg(V))$.

A *Darboux factor* of system (2.3) is a polynomial $f(x,y)$ satisfying

$$\frac{\partial f}{\partial x}U + \frac{\partial f}{\partial y}V = Kf,$$

where $K(x,y)$ is a polynomial of degree at most $m-1$, called the *cofactor of* f.

A Darboux first integral of system (2.3) is a first integral of system (2.3) of the form $H = f_1^{\alpha_1}\cdots f_k^{\alpha_k}$, where $f_1, f_2, ..., f_k$ are Darboux factors of system (2.3) and $\alpha_i \in \mathbb{R}$, $1 \leq i \leq k$. If a first integral of system (2.3) cannot be found, then we turn our attention to possible existence of an integrating factor. A Darboux integrating factor of system (2.3) is an integrating factor of the form $\mu = f_1^{\beta_1}\cdots f_k^{\beta_k}$, where $f_1, f_2, ..., f_k$ are Darboux factors of system (2.3) and $\beta_i \in \mathbb{R}$, $1 \leq i \leq k$.

Suppose that $f_1, f_2, ..., f_k$ are Darboux factors of system (2.3) with respective cofactors $K_1, K_2, ..., K_k$. Then it is easy to see that

1. If there exist constants $\alpha_i \in \mathbb{R}$, $1 \le i \le k$, satisfying

$$\sum_{i=1}^{k} \alpha_i K_i = 0, \tag{2.4}$$

then $H = f_1^{\alpha_1} \cdots f_k^{\alpha_k}$ is a Darboux first integral of (2.3).

2. If there exist constants $\beta_i \in \mathbb{R}$, $1 \le i \le k$, satisfying

$$\sum_{i=1}^{k} \beta_i K_i + \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0, \tag{2.5}$$

then $\mu = f_1^{\beta_1} \cdots f_k^{\beta_k}$ is a Darboux integrating factor of system (2.3).

Finally, we recall that a differential system possess a *time-reversible symmetry* with respect to a line L if its phase portrait is invariant after reflecting with respect to the line and reversing the sense of all orbits (reversing of time). If we know that a singular point on the line L is a center or a focus, the presence of time-reversible symmetry with respect to L prevents this singularity to be a focus, consequently forcing it to be a center. When the line L is the x-axis, system (2.3) posses time–reversible symmetry if and only if obviously,

$$U(x, -y) = -U(x, y), \quad V(x, -y) = V(x, y). \tag{2.6}$$

2.2. **Isochronicity and linearizability problems.** If the singular point at the origin of (2.1) is known to be a center we say that this center is *isochronous* if all periodic solutions of (2.1) in a neighbourhood of the origin have the same period. System (2.1) is said to be *linearizable* if there is an analytic change of coordinates

$$x_1 = x + \sum_{m+n\ge 2} c_{m,n} x^m y^n, \qquad y_1 = y + \sum_{m+n\ge 2} d_{m,n} x^m y^n, \tag{2.7}$$

that reduces (2.1) to the canonical linear center,

$$\dot{x}_1 = -y_1, \qquad \dot{y}_1 = x_1. \tag{2.8}$$

The following theorem which goes back to Poincaré and Lyapunov shows that there is an intimate relation between linearizability and isochronicity. A proof can be found e.g. in [33].

Theorem 2.2. *The origin of system* (2.1) *is an isochronous center if and only if the system is linearizable.*

It follows from Theorem 2.2 that solving the isochronicity problem is equivalent to solving the linearizability problem, but the investigation of the latter problem is computationally much simpler.

Differentiating with respect to t both sides of each equation of (2.7) we obtain

$$\begin{aligned} \dot{x}_1 &= \dot{x} + \left(\sum_{m+n\ge 2} m c_{m,n} x^{m-1} y^n\right) \dot{x} + \left(\sum_{m+n\ge 2} n c_{m,n} x^m y^{n-1}\right) \dot{y}, \\ \dot{y}_1 &= \dot{y} + \left(\sum_{m+n\ge 2} m d_{m,n} x^{m-1} y^n\right) \dot{x} + \left(\sum_{m+n\ge 2} n d_{m,n} x^m y^{n-1}\right) \dot{y}. \end{aligned} \tag{2.9}$$

Thus, the substitution (2.7) linearizes system (2.1) if it holds that
(2.10)

$$\begin{aligned}&\sum_{m+n\geq 2} d_{m,n}x^m y^n + \sum_{p+q\geq 2}^{n} a_{p,q}x^p y^q + \left(\sum_{m+n\geq 2} mc_{m,n}x^{m-1}y^n\right)\left(-y+\sum_{p+q\geq 2}^{n} a_{p,q}x^p y^q\right)\\&+\left(\sum_{m+n\geq 2} nc_{m,n}x^m y^{n-1}\right)\left(x+\sum_{p+q\geq 2}^{n} b_{p,q}x^p y^q\right)\equiv 0,\\&-\sum_{m+n\geq 2} c_{m,n}x^m y^n + \sum_{p+q\geq 2}^{n} b_{p,q}x^p y^q + \left(\sum_{m+n\geq 2} md_{m,n}x^{m-1}y^n\right)\left(-y+\sum_{p+q\geq 2}^{n} a_{p,q}x^p y^q\right)\\&+\left(\sum_{m+n\geq 2} nd_{m,n}x^m y^{n-1}\right)\left(x+\sum_{p+q\geq 2}^{n} b_{p,q}x^p y^q\right)\equiv 0.\end{aligned}$$

The left hand sides of two equations in (2.10) can be written in the form $\sum_{k,l\geq 2} h_1^{(k,l)}x^k y^l$, and $\sum_{k,l\geq 2} h_2^{(k,l)}x^k y^l$, respectively, where $h_1^{(k,l)}$ and $h_2^{(k,l)}$ are polynomials in the parameters $a_{p,q}, b_{p,q}$, $p+q\geq 2$, of system (2.1) and $c_{m,n}$, $d_{m,n}$ $m+n\geq 2$, of (2.7). It is clear that both equations in (2.10) are satisfied if and only if $h_i^{(k,l)}=0$, $i=1,2$, for all $k,l\in\mathbb{N}_0$, $k+l\geq 2$.

Using the computer algebra system Mathematica we can construct a linearizing transformation (2.7). The process starts from solving the polynomial system $h_i^{(k,l)}=0$, $i=1,2$ such that $k+l=2$. At this step we are able to determine the coefficients $c_{m,n}$, $d_{m,n}$, $m+n=2$ of (2.7) in terms of the parameters $a_{p,q}$'s and $b_{p,q}$'s of system (2.1). The next step is to solve the polynomial system $h_i^{(k,l)}=0$, $i=1,2$ such that $k+l=3$, determining the coefficients $c_{m,n}$, $d_{m,n}$, $m+n=3$, of (2.7) in terms of the parameters $a_{p,q}$'s and $b_{p,q}$'s of system (2.1). In general case the polynomial system $h_i^{(k,l)}=0$, $i=1,2$ such that $k+l=3$, cannot be solved. However dropping two suitable equations we obtain a system that has a solution. We denote the two dropped polynomials on the left hand sides of these two equations by i_1 and j_1.

Proceeding step-by-step we obtain that the polynomial system $h_i^{(k,l)}=0$, $i=1,2$, $k+l=r$, can not be solved when $r=k+l$ is odd number. Dropping on each step two suitable equations (and denoting by $i_{(r-1)/2}$ and $j_{(r-1)/2}$ the corresponding polynomials), we obtain a system that has a solution.

The polynomials i_k and j_k obtained at each odd step of the process are polynomials in the parameters $a_{p,q}$ and $b_{p,q}$ of system (2.1) called the *linearizability quantities*. They represent obstacles for the existence of a linearizing change of coordinates (2.7), that is, system (2.1) admits a linearizing change of coordinates (2.7) if and only if $i_k=j_k=0$, for all $k>1$. Thus, the simultaneous vanishing of all linearizability quantities provide conditions which characterize when a system of the form (2.1) is linearizable (equivalently it has an isochronous center at the origin). The ideal defined by the linearizability quantities, $\mathcal{L}=\langle i_1,j_1,i_2,j_2,...\rangle\subset\mathbb{C}[a,b]$, is called *linearizability ideal* and its affine variety, $V_{\mathcal{L}}=\mathbf{V}(\mathcal{L})$ is called the *linearizability variety.*

By the Hilbert Basis Theorem there exists a positive integer k such that $\mathcal{L}=\mathcal{L}_k=\langle i_1,j_1,...,i_k,j_k\rangle$. Computing the irreducible decomposition of the variety $\mathbf{V}(\mathcal{L}_k)$ (using the routine minAssGTZ of the computer algebra system Singular) we find necessary conditions for the existence of the linearizable change of coordinates of the form (2.7). To prove the sufficiency of the obtained conditions we check that any point of each component of $\mathbf{V}(\mathcal{L}_k)$ corresponds to a linearizable system.

To find a linearizing change of coordinates we use the method of Darboux linearization. To construct Darboux linearization it is convenient to complexify the real system (2.1). To this end, we introduce the complex variable $z = x + iy$, obtaining from (2.1) the system of the form

$$\dot{z} = i(z - X(z, \bar{z})), \tag{2.11}$$

where $i = \sqrt{-1}$, $X = (P + iQ)/i$ and P and Q are evaluated at $((z + \bar{z})/2, (z - \bar{z})/2i)$. Adjoining to equation (2.11) its complex conjugate, $\dot{\bar{z}} = -i(z - \overline{X(z, \bar{z})})$, and replacing $\bar{z}$ by w, we obtain the pair of equations

$$\dot{z} = i(z - X(z, w)), \qquad \dot{w} = -i(w - Y(z, w)), \tag{2.12}$$

where $Y(z, \bar{z}) = \overline{X(z, \bar{z})}$. System (2.12) is the complexification of the real system (2.1).

The linearizability problem for the complex system (2.12) is to decide whether it can be transformed to the linear system $\dot{z_1} = iz_1$, $\dot{w_1} = -iw_1$ by an analytic change of coordinates of the form

$$z_1 = z + Z_1(z, w), \qquad w_1 = w + W_1(z, w). \tag{2.13}$$

Remark 2.3. If the system

$$\dot{z} = z - X(z, w), \qquad \dot{w} = -w + Y(z, w), \tag{2.14}$$

is transformed into the linear system $\dot{z_1} = z_1$, $\dot{w_1} = -w_1$ by transformation (2.13), then (2.12) is reduced to $\dot{z_1} = iz_1$, $\dot{w_1} = -iw_1$ by the same transformation. Conversely, if (2.13) linearizes (2.12), then it also linearizes (2.14). Therefore systems (2.12) and (2.14) are equivalent with regard to the problem of linearizability.

A *Darboux linearization* of system (2.14) is an analytic change of coordinates, $z_1 = Z_1(z, w)$, $w_1 = W_1(z, w)$, such that

$$Z_1(z, w) = \prod_{j=0}^{m} f_j^{\alpha_j}(z, w) = z + Z_1'(z, w),$$
$$W_1(z, w) = \prod_{j=0}^{n} g_j^{\beta_j}(z, w) = w + W_1'(z, w),$$

where $f_j, g_j \in \mathbb{C}[z, w]$, $\alpha_j, \beta_j \in \mathbb{C}$, and Z_1' and W_1' have neither constant nor linear terms.

A system is said to be Darboux linearizable if it admits a Darboux linearization. The next theorem provides a way to construct a Darboux linearization using Darboux factors (see e.g. [33] for a proof).

Theorem 2.4. *System* (2.14) *is Darboux linearizable if and only if there exits $s + 1 \geq 1$ Darboux factors $f_0, ..., f_s$ with corresponding cofactors $K_0, ..., K_s$ and $t + 1 \geq 1$ Darboux factors $g_0, ..., g_t$ with corresponding cofactors $L_0, ..., L_t$ with the following properties:*

a. $f_0(z, w) = z + \cdots$ *but* $f_j(0, 0) = 1$ *for* $j \geq 1$*;*
b. $g_0(z, w) = w + \cdots$ *but* $g_j(0, 0) = 1$ *for* $j \geq 1$*; and*

c. *there are* $s+t$ *constants* $\alpha_1,...,\alpha_s,\beta_1,...,\beta_t\in\mathbb{C}$ *such that*

$$K_0+\alpha_1K_1+\cdots+\alpha_sK_s=1 \quad \text{and} \quad L_0+\beta_1L_1+\cdots+\beta_tL_t=-1. \tag{2.15}$$

The Darboux linearization is then given by

$$z_1=H_1(z,w)=f_0f_1^{\alpha_1}\cdots f_s^{\alpha_s}, \qquad w_1=H_2(z,w)=g_0g_1^{\beta_1}\cdots g_t^{\beta_t}.$$

3. Isochronicity and linearizability problems for system (1.1)

In [24] Liu and Li studied the bi-center problem for a Z_2-equivariant cubic system of the form (1.1) and found the necessary and sufficient conditions for existence of a bi-center at the points $(-1,0)$ and $(1,0)$. After a change of coordinates, they obtained from (1.1) the following standard form of the system (see Theorem 7 of [24]):

$$\begin{aligned}\dot{x}&=-(a_1+1)y+a_1x^2y+a_2xy^2+a_3y^3,\\ \dot{y}&=-\frac{1}{2}x-a_4y+\frac{1}{2}x^3+a_4x^2y+a_5xy^2+a_6y^3.\end{aligned}\tag{3.1}$$

The following eleven necessary and sufficient conditions for the existence of a bi-center at the points $(-1,0)$ and $(1,0)$ for system (3.1) are given in Theorem 11 of [24]:

(1) $a_4=0$, $a_1=-a_5$, $a_6=-a_2/3$;
(2) $a_4=0$, $a_1+a_5\neq 0$, $a_2=a_6=0$;
(3) $a_1+a_5\neq 0$, $a_6=-(a_2+2a_1a_2-2a_4+2a_2a_5-2a_4a_5)/3$, $2(1+a_1)(a_1+a_5)^2-a_4^2(1+2a_1+2a_5)=0$, $3(a_1+a_5)(-a_3+2(1+a_1)(1+a_5))-2a_4(2a_4(1+a_5)+a_2(2+a_1+a_5))=0$;
(4) $a_3=2(1+a_1)(1+a_5)$, $a_6=-(a_2+2a_1a_2-2a_4+2a_2a_5-2a_4a_5)$, $2(1+a_5)a_4+(2+a_1+a_5)a_2=0$;
(5) $a_4\neq 0$, $a_1=-(2-3a_4^2)/3$, $a_2=a_4$, $a_3=a_24(1-a_4^2+a_5)$, $a_6=a_4(1-a_4^2)$;
(6) $a_4\neq 0$, $a_1=-(8-5a_4^2)/8$, $a_2=a_4/2$, $a_5=-(8+a_4^2)/8$, $a_3=-5a_4^4/32$, $a_6=a_4(2-a_4^2)/4$;
(7) $a_4\neq 0$, $a_1=-(32+15a_4^2)/32$, $a_2=a_4/4$, $a_3=a_4^2(64+15a_4^2)/512$, $a_5=-(96+17a_4^2)/32$, $a_6=-3a_4(4+a_4^2)/16$;
(8) $a_4\neq 0$, $a_1=-(50+21a_4^2)/50$, $a_2=a_4/5$, $a_3=a_4^2/1250(250+63a_4^2)$, $a_5=-(200+39a_4^2)/50$, $a_6=-a_4/25(35+9a_4^2)$;
(9) $a_4\neq 0$, $a_1=-(9+4a_4^2)/9$, $a_2=0$, $a_3=0$, $a_6=2a_4/3(1+a_5)$;
(10) $a_4\neq 0$, $a_1=-(8+3a_4^2)/8$, $a_2=-a_4/2$, $a_3=3a_4^2(4+a_4^2+4a_5)/16$, $a_6=a_4(4-a_4^2+8a_5)/8$;
(11) $a_4\neq 0$, $a_1=-(32+15a_4^2)/32$, $a_2=-a_4/4$, $a_3=a_4^2/512(832+495a_4^2)$, $a_5=(160+111a_4^2)/32$, $a_6=a_4(76+45a_4^2)/16$.

To complete the study of bi-centers started by Liu and Li in [24] we investigate the existence of isochronous bi–centers for system (3.1) and obtain the following result.

Theorem 3.1. *System (3.1) has an isochronous bi-center at the points* $(1,0)$ *and* $(-1,0)$ *if and only if one of the following conditions is satisfied:*

(i) $a_1=-3/2$, $a_2=0$, $a_3=1/2$, $a_4=0$, $a_5=-3/2$, $a_6=0$;
(ii) $a_1=-3$, $a_2=0$, $a_3=0$, $a_4=0$, $a_5=-9$, $a_6=0$.

Proof. To compute the linearizability quantities we move the singular point (1,0) to the origin. Applying the translation $u = x - 1$, $v = y$, we obtain from (3.1) the system

$$\begin{aligned} \dot{u} &= -v + a_1(2+u)uv + a_2(1+u)v^2 + a_3v^3, \\ \dot{v} &= u + \frac{3}{2}u^2 + \frac{1}{2}u^3 + a_4(2+u)uv + a_5(1+u)v^2 + a_6v^3. \end{aligned} \tag{3.2}$$

Using the computer algebra system MATHEMATICA we computed the first seven pairs of the linearizability quantities for system (3.2) using the procedure described in Subsection 2.2. Their expressions are very large, so we present here only the first pair:

$$\begin{aligned} i_1 &= 18 + 4a_1^2 + 10a_2^2 + 9a_3 - 2a_2a_4 + 4a_4^2 + 12a_5 - 10a_1a_5 + 4a_5^2, \\ j_1 &= a_2 + 2a_1a_2 - 2a_4 + 2a_2a_5 - 2a_4a_5 + 3a_6. \end{aligned}$$

The reader can easily compute the others quantities using any available computer algebra system.

The next computational step is to compute the irreducible decomposition of the variety $\mathbf{V}(\mathcal{L}_7) = \mathbf{V}(\langle i_1, j_1, ..., i_7, j_7\rangle)$. For this purpose we use the routine `minAssGTZ` of the computer algebra system SINGULAR. Since an isochronous center must be a center, to make the computations easier we investigated the existence of an isochronous center using the eleven cases found by Liu and Li in [24] and given above. Thus, our proof is split in eleven cases corresponding to these eleven conditions.

Case (1): Computation with `minAssGTZ` of SINGULAR shows that

$$I = \langle \mathcal{L}_7, a_4, a_1 + a_5, a_6 + a_2/3\rangle,$$

is a primary ideal whose minimal associate prime is

$$I_1 = \langle a_4,\, a_6^2 + 1,\, 2a_5 - 3,\, a_1 + a_5,\, -2a_4a_5 + a_2 - 2a_4 + 3a_6, f\rangle,$$

where $f = \frac{40}{9}a_4^2a_5^2 + \frac{76}{9}a_4^2a_5 - \frac{40}{3}a_4a_5a_6 + \frac{40}{9}a_4^2 + 2a_5^2 - \frac{38}{3}a_4a_6 + 10a_6^2 + a_3 + \frac{4}{3}a_5 + 2.$

Analysing this variety we see that the corresponding polynomial system has only complex solutions, so $\mathbf{V}(I)$ is the empty set in $\mathbb{R}^6$. Thus system (3.1) can not have an isochronous center if condition (1) of Theorem 11 of [24] holds.

Case (2): The irreducible decomposition of the variety of the ideal

$$\langle \mathcal{L}_7, a_2, a_4, a_6\rangle$$

computed using the routine `minAssGTZ` over the field of rational numbers consists of the varieties of the following two ideals

$$\begin{aligned} I_1 &= \langle a_6,\, 2a_5 + 3,\, a_4,\, \frac{4}{9}a_1^2 + \frac{4}{9}a_4^2 - \frac{10}{9}a_1a_5 + \frac{4}{9}a_5^2 + a_3 + \frac{4}{3}a_5 + 2, a_2,\, 2a_1 + 3\rangle, \\ I_2 &= \langle a_6,\, a_5 + 9,\, a_4,\, \frac{4}{9}a_1^2 + \frac{4}{9}a_4^2 - \frac{10}{9}a_1a_5 + \frac{4}{9}a_5^2 + a_3 + \frac{4}{3}a_5 + 2, a_2,\, a_1 + 3\rangle. \end{aligned}$$

The varieties of the ideals I_1 and I_2 give conditions (i) and (ii) of this theorem. Thus, (i) and (ii) are the necessary conditions for existence of an isochronous bi-center at the points $(-1, 0)$ and $(1, 0)$ for system (3.1). Now we need to show that these two conditions are also sufficient for existence of an isochronous bi-center. To do so, we look for the Darboux linearizations of the corresponding systems.

Condition (i): In this case system (3.1) becomes

$$\dot{x} = y(\frac{1}{2} - \frac{3}{2}x^2 + \frac{1}{2}y^2), \quad \dot{y} = x(-\frac{1}{2} + \frac{1}{2}x^2 - \frac{3}{2}y^2). \tag{3.3}$$

Translating the point (1,0) to the origin, using the substitution $u = x - 1$, $v = y$, and then the complexification $z = u + iv$, $w = u - iv$, we obtain the system

$$\dot{z} = z(1 + \frac{3}{2}z + \frac{1}{2}z^2), \quad \dot{w} = w(-1 - \frac{3}{2}w - \frac{1}{2}w^2). \tag{3.4}$$

This system is a particular case of the system studied in [15] (namely, it satisfies condition (1) of Theorem 4 of [15]) where it was shown that this system is linearizable. Thus, system (3.4) is linearizable at the origin. Consequently, system (3.3) has an isochronous bi–center at the points $(1, 0)$ and $(-1, 0)$.

Condition (ii): The conditions of this case yield the system

$$\dot{x} = y(2 - 3x^2), \quad \dot{y} = x(-\frac{1}{2} + \frac{1}{2}x^2 - 9y^2). \tag{3.5}$$

As in the previous case, after apply the translation, $u = x-1$, $v = y$, and the complexification, $z = u + iv$, $w = u - iv$, to system (3.5) we obtain the system

$$\begin{aligned} \dot{z} &= z + \frac{33z^2}{8} - \frac{15zw}{4} + \frac{9w^2}{8} + \frac{25z^3}{16} - \frac{9z^2w}{16} - \frac{21zw^2}{16} + \frac{13w^3}{16}, \\ \dot{w} &= -w - \frac{9z^2}{8} + \frac{15zw}{4} - \frac{33w^2}{8} - \frac{13z^3}{16} + \frac{21z^2w}{16} + \frac{9zw^2}{16} - \frac{25w^3}{16}. \end{aligned} \tag{3.6}$$

This system has the following Darboux factors

$$\begin{aligned} l_1 &= z + \frac{3z^2}{8} - \frac{3zw}{4} + \frac{3w^2}{8} + \frac{z^3}{16} + \frac{3z^2w}{16} + \frac{3zw^2}{16} + \frac{w^3}{16}, \\ l_2 &= w + \frac{3z^2}{8} + \frac{3zw}{4} + \frac{3w^2}{8} + \frac{z^3}{16} + \frac{3z^2w}{16} + \frac{3zw^2}{16} + \frac{w^3}{16}, \\ l_3 &= 1 + 3z + 3w + \frac{3z^2}{4} + \frac{3zw}{2} + \frac{3w^2}{4}, \end{aligned}$$

with respective cofactors

$$\begin{aligned} k_1 &= \frac{1}{4}(4 + 18z - 18w + 9z^2 - 9w^2), \\ k_2 &= \frac{1}{4}(-4 + 18z - 18w + 9z^2 - 9w^2), \\ k_3 &= \frac{3}{2}(2z - 2w + z^2 - w^2). \end{aligned}$$

It is easy to verify that conditions (2.15) are satisfied with $f_0 = l_1$, $f_1 = l_3$, $g_0 = l_2$, $g_1 = l_3$ and constants $\alpha_1 = \beta_1 = -\frac{3}{2}$. Thus, the Darboux linearization for system (3.6) is given by the following analytic change of coordinates

$$\begin{aligned} z_1 &= \frac{16z + 6z^2 + z^3 + 12zw + 3z^2w + 6w^2 + 3zw^2 + w^3}{2(4 + 12z + 3z^2 + 12w + 6zw + 3w^2)^{3/2}}, \\ w_1 &= \frac{6z^2 + z^3 + 16w + 12zw + 3z^2w + 6w^2 + 3zw^2 + w^3}{2(4 + 12z + 3z^2 + 12w + 6zw + 3w^2)^{3/2}}. \end{aligned}$$

Thus system (3.6) is linearizable and therefore system (3.5) has an isochronous bi–center at the points $(1, 0)$ and $(-1, 0)$.

Case (3): The irreducible decomposition of the variety of the ideal

$$\langle \mathcal{L}_7, f_1, f_2, f_3 \rangle, \tag{3.7}$$

where

$$\begin{aligned} f_1 =& 3a_6 + (a_2 + 2a_1a_2 - 2a_4 + 2a_2a_5 - 2a_4a_5),\\ f_2 =& 2(1 + a_1)(a_1 + a_5)^2 - a_4^2(1 + 2a_1 + 2a_5),\\ f_3 =& 3(a_1 + a_5)(-a_3 + 2(1 + a_1)(1 + a_5)) - 2a_4(2a_4(1 + a_5) + a_2(2 + a_1 + a_5)), \end{aligned}$$

computed using the routine `minAssGTZ` over the field of rational numbers consists of the seven components presented in Appendix I. Analysing the components we see that the corresponding polynomial systems have only complex solutions, so the varieties are empty sets in $\mathbb{R}^6$. Thus, system (3.1) cannot have an isochronous center if condition (3) of Theorem 11 of [24] holds.

The remaining eight cases (Cases (4),(5),...,(11)) are analogous to Case (3). All the varieties are empty sets in $\mathbb{R}^6$. Therefore system (3.1) can not have an isochronous center if conditions (4), (5),..., (11) of Theorem 11 of [24] holds.

□

We note that for both conditions (i) and (ii) we obtain concrete systems (without parameters). Figures 1 and 2 present the behaviour of the vector field in a neighbourhood of the origin of such systems. From the pictures we can suppose that the singular points at the origin also are centers. We prove the existence of such centers at the origin and that these centers are also isochronous.

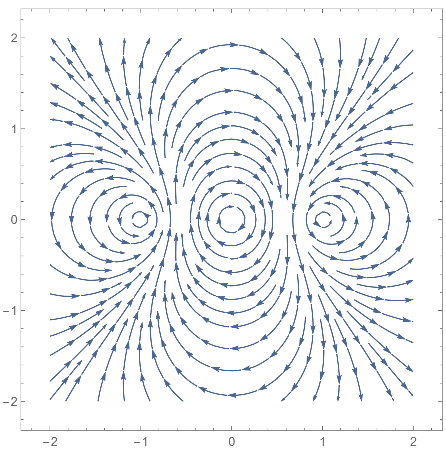

Figure 1. Condition (i).

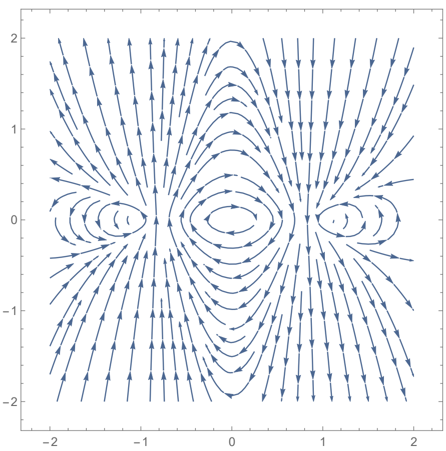

Figure 2. Condition (ii).

Proposition 3.2. *If conditions (i) or (ii) of Theorem 3.1 holds system* (3.1) *has a center at the origin.*

Proof. Under condition (i) of Theorem 3.1 system (3.1) becomes system (3.3), namely

$$\dot{x} = y(\frac{1}{2} - \frac{3}{2}x^2 + \frac{1}{2}y^2), \quad \dot{y} = x(-\frac{1}{2} + \frac{1}{2}x^2 - \frac{3}{2}y^2).$$

By a time rescaling and a linear change of coordinates system (3.3) becomes

$$\dot{x_1} = -y_1 - 3x_1^2y_1 + y_1^3, \quad \dot{y_1} = x_1 + x_1^3 - 3x_1y_1^2. \tag{3.8}$$

System (3.8) has Darboux factors $f_1 = x_1^2 + y_1^2$ and $f_2 = 1 + 2x_1^2 - 2y_1^2$, with respective cofactors $k_1 = -4x_1y_1$ and $k_2 = -8x_1y_1$. It is easy to verify that $\alpha_1 = 1$ and $\alpha_2 = 1/2$ is a solution of equation (2.4) with k_1, k_2 given above. So we obtain that

$$H = \frac{x^2 + y^2}{\sqrt{1 + 2x_1^2 - 2y_1^2}},$$

is an analytic Darboux first integral of system (3.8). Thus, the origin is a center for (3.8), and, hence, the origin of (3.3) is a center as well.

Under condition (ii) of Theorem 3.1 system (3.1) becomes system (3.5), namely

$$\dot{x} = y(2 - 3x^2), \quad \dot{y} = x(-\frac{1}{2} + \frac{1}{2}x^2 - 9y^2).$$

By a time rescaling and a linear change of coordinates system (3.5) becomes

$$\dot{x_1} = -y_1 + 3x_1^2y_1, \quad \dot{y_1} = x_1 - 2x_1^3 + 9x_1y_1^2. \tag{3.9}$$

System (3.9) has Darboux factors $f_1 = 9x_1^2 - 27y_1^2 - 4$, $f_2 = x_1 - 1/\sqrt{3}$ and $f_3 = x_1 + 1/\sqrt{3}$, which allow to construct the first integral

$$H = \frac{9x_1^2 - 27y_1^2 - 4}{(-1 + 3x_1^2)^3}.$$

Thus, the origin is a center of (3.9). Therefore the origin is a center of (3.5). □

Remark 3.3. In the proof of Proposition 3.2 it is possible to obtain first integrals using the results of [24], but we have presented a direct proof. In [24] the authors gave examples of Hamiltonian systems (1.1) having more than two centers. Here we have presented the examples of non-Hamiltonian systems with 3 centers.

It turns out all three centers are isochronous.

Proposition 3.4. *If condition (i) or (ii) of Theorem 3.1 holds, then system* (3.1) *has an isochronous center at the origin.*

Proof. As showed in the proof of Proposition 3.2, by a linear change of coordinates system (3.3) is transformed to system (3.8) and system (3.5) is transformed to system (3.9).

System (3.8) corresponds to system (S_1^*) in Table II of [28], and it was shown in [28] that it possess an isochronous center at the origin. Thus, system (3.3) also has an isochronous center at origin. System (3.9) corresponds to system (S_3^*) in Table II of [28] so it has an isochronous center at the origin. Hence, system (3.5) has an isochronous center at origin as well. □

Therefore under the conditions of Theorem 3.1 system (3.1) has three isochronous centers at points $(-1, 0)$, $(0, 0)$ and $(1, 0)$. We are unaware about other examples of cubic systems with 3 isochronous centers.

4. The center and isochronocity problems for system (1.2)

In this section we investigate the existence of a bi-center and an isochronous bi-center for another class of Z_2-invariant system, namely, for the quintic system (1.2).

4.1. **Canonical form of bi-centers for system** (1.2).

To apply the methods described in the preliminaries to study system (1.2) we look for a canonical form of the system for which the computations are simpler.

Suppose that system (1.2) satisfies the following conditions

$$X(1,0) = Y(1,0) = 0, \quad \frac{\partial X(1,0)}{\partial x} = \frac{\partial Y(1,0)}{\partial y} = 0, \quad \frac{\partial X(1,0)}{\partial y} = -1, \quad \frac{\partial Y(1,0)}{\partial x} = 1, \tag{4.1}$$

which mean that the point $(1,0)$ is a singular point of (1.2), and its linearization at the point $(1,0)$ is

$$\frac{dx}{dt} = -y, \quad \frac{dy}{dt} = x - 1. \tag{4.2}$$

According to Lemma 6 in [24], if system (1.2) has two weak foci or centers at $(-1,0)$ and $(1,0)$ then the transformation

$$x_1 = x + \left(\frac{\partial Y(1,0)}{\partial y} \Big/ \frac{\partial Y(1,0)}{\partial x}\right) y, \quad y_1 = \left(r \Big/ \frac{\partial Y(1,0)}{\partial x}\right) y, \quad t_1 = rt,$$

where

$$r = \sqrt{\frac{\partial X(1,0)}{\partial x}\frac{\partial Y(1,0)}{\partial y} - \frac{\partial X(1,0)}{\partial y}\frac{\partial Y(1,0)}{\partial x}},$$

carries system (1.2) into a system whose linear part at the point $(1,0)$ is given by (4.2) and the conditions (4.1) holds. Moreover, this linear transformation preserves the distance between the origin and the two singular points. Consequently we have the following result.

Proposition 4.1. *Suppose that system* (1.2) *has two weak foci or centers that could be arranged at the points* $(-1,0)$ *and* $(1,0)$*. Then there is a change of coordinates that maps the orbits of system* (1.2) *to the orbits of the following system*

$$\begin{aligned} \dot{x} &= -(a_1+1)y + a_1x^4y + a_2x^3y^2 + a_3x^2y^3 + a_4xy^4 + a_5y^5, \\ \dot{y} &= -\frac{1}{4}x - a_6y + \frac{1}{4}x^5 + a_6x^4y + a_7x^3y^2 + a_8x^2y^3 + a_9xy^4 + a_{10}y^5, \end{aligned} \tag{4.3}$$

where $a_i \in \mathbb{R}$, $i = 1, ..., 10$.

System (4.3) is a Z_2-equivariant quintic system, so the existence of a bi-center or an isochronous bi-center at the points $(1,0)$ and $(-1,0)$ for such system follows from the existence of a center or an isochronous center at the point $(1,0)$.

To compute the focus and linearizability quantities we have to move the singular point (1,0) to the origin. Applying the transformation $u = x-1$, $v = y$, we obtain from (4.3) the system

$$
\begin{aligned}
\dot{u} =& -v + 4a_1uv + a_2v^2 + 6a_1u^2v + 3a_2uv^2 + a_3v^3 + 4a_1u^3v + 3a_2u^2v^2 + 2a_3uv^3 + \\
&+ a_4v^4 + a_1u^4v + a_2u^3v^2 + a_3u^2v^3 + a_4uv^4 + a_5v^5, \\
\dot{v} =& u + \frac{5u^2}{2} + 4a_6uv + a_7v^2 + \frac{5u^3}{2} + 6a_6u^2v + 3a_7uv^2 + a_8v^3 + \frac{5u^4}{4} + 4a_6u^3v + \\
&+ 3a_7u^2v^2 + 2a_8uv^3 + a_9v^4 + \frac{u^5}{4} + a_6u^4v + a_7u^3v^2 + a_8u^2v^3 + a_9uv^4 + a_{10}v^5.
\end{aligned} \tag{4.4}
$$

Our goal is to find systems with bi-centers and isochronous bi-centers within the family (4.3). Unfortunately, because this system has ten parameters the computations described in Section 2 become infeasible for the whole family (4.3). So we restrict our study to a subcase. From now on we assume that system (4.3) posses the y-axis as an invariant curve, i.e, we fixed two parameters of the original system (4.3), $a_1 = -1$ and $a_5 = 0$. Thus, we look for necessary and sufficient conditions for system

$$
\begin{aligned}
\dot{x} &= -x^4y + a_2x^3y^2 + a_3x^2y^3 + a_4xy^4, \\
\dot{y} &= -\frac{1}{4}x - a_6y + \frac{1}{4}x^5 + a_6x^4y + a_7x^3y^2 + a_8x^2y^3 + a_9xy^4 + a_{10}y^5,
\end{aligned} \tag{4.5}
$$

to have a bi-center (respectively, an isochronous bi-center) at the points $(-1, 0)$ and $(1, 0)$, or, equivalently, for the system

$$
\begin{aligned}
\dot{u} =& -v - 4uv + a_2v^2 - 6u^2v + 3a_2uv^2 + a_3v^3 - 4u^3v + 3a_2u^2v^2 + 2a_3uv^3 + a_4v^4 + \\
&- u^4v + a_2u^3v^2 + a_3u^2v^3 + a_4uv^4, \\
\dot{v} =& u + \frac{5u^2}{2} + 4a_6uv + a_7v^2 + \frac{5u^3}{2} + 6a_6u^2v + 3a_7uv^2 + a_8v^3 + \frac{5u^4}{4} + 4a_6u^3v + \\
&+ 3a_7u^2v^2 + 2a_8uv^3 + a_9v^4 + \frac{u^5}{4} + a_6u^4v + a_7u^3v^2 + a_8u^2v^3 + a_9uv^4 + a_{10}v^5,
\end{aligned} \tag{4.6}
$$

to have a center (respectively, an isochronous center) at the origin.

4.2. **Bi-center conditions for the Z_2-equivariant system** (4.5)**.** In this subsection we find conditions for existence of a bi-center for the Z_2-equivariant system (4.5).

Theorem 4.2. *System (4.5) has a bi-center at the points $(-1, 0)$ and $(1, 0)$ if one of the following conditions holds:*

(1) $a_6 = 0$, $a_8 = \frac{1}{3}(a_2 - 2a_2a_7)$, $a_9 = \frac{1}{2}(a_3 - a_3a_7)$, $a_{10} = \frac{1}{5}(3a_4 - 2a_4a_7)$;
(2) $a_2 = -4a_6$, $a_4 = 4a_3a_6$, $a_8 = 4a_6a_7$, $a_{10} = 4a_6a_9$;
(3) $a_4 = 4(a_3a_6 - 4a_2a_6^2 - 16a_6^3)$, $a_8 = \frac{1}{3}(a_2 + 4a_6 - 2a_2a_7 + 4a_6a_7)$,
$a_9 = \frac{1}{6}(3a_3 - 4a_2a_6 - 16a_6^2 - 3a_3a_7 - 4a_2a_6a_7 - 16a_6^2a_7)$,
$a_{10} = 2a_6(-a_3 + 4a_2a_6 + 16a_6^2)(-1 + a_7)$;
(4) $a_7 = -1$, $a_8 = a_2$, $a_9 = a_3$, $a_{10} = a_4$.

Proof. Using the computer algebra system MATHEMATICA we computed the first nine nonzero focus quantities for system (4.6) using the procedure described in Subsection 2.1. Their expressions are very large, so we present here only the first two:

$$
\begin{aligned}
v_1 =& -a_2 - 4a_6 + 2a_2a_7 - 4a_6a_7 + 3a_8,\\
v_2 =& 60a_{10} + 75a_2 - 40a_2a_3 - 36a_4 + 300a_6 + 80a_2^2a_6 - 16a_3a_6 + 256a_2a_6^2 - 256a_6^3\\
& - 154a_2a_7 + 40a_2a_3a_7 + 24a_4a_7 + 284a_6a_7 + 64a_3a_6a_7 + 320a_2a_6^2a_7 - 256a_6^3a_7\\
& + 148a_2a_7^2 + 544a_6a_7^{-}280a_2a_7^3 + 560a_6a_7^3 - 225a_8 - 80a_2a_6a_8 + 64a_6^2a_8 + 12a_7a_8\\
& - 420a_7^2a_8 + 80a_2a_9 + 80a_6a_9.
\end{aligned}
$$

The reader can easily compute the others quantities using any available computer algebra system.

The next computational step is to compute the irreducible decomposition of the variety $\mathbf{V}(\mathcal{B}_9) = \mathbf{V}(\langle v_1, ..., v_9\rangle)$. For this purpose we use the routine `minAssGTZ` of the computer algebra system Singular. Due to the complexity of the focus quantities the computations become infeasible over the field of rational numbers. To be able to complete our computations we computed in the field of finite characteristic 32003 and then lifted the resulting ideals to the ring of polynomials with rational coefficients using the rational reconstruction algorithm of [38] and the computational procedure of [32]. The irreducible decomposition of the variety of the ideal $\mathcal{B}_9$ computed using the routine `minAssGTZ` of the computer algebra system Singular over a field of characteristic 32003 consists of 4 irreducible components. After the rational reconstruction we obtain the 4 conditions in the statement of this theorem, which are necessary conditions for existence of a bi-center at the points $(-1,0)$ and $(1,0)$ for system (4.5).

Now we show that they are also sufficient.

Case (1): Under conditions (1) of Theorem 4.2 system (4.5) becomes

$$
\begin{aligned}
\dot{x} =& -xy(x^3 - a_2x^2y - a_3xy^2 - a_4y^3),\\
\dot{y} =& -\frac{x}{4} + \frac{x^5}{4} + a_7x^3y^2 + \left(\frac{a_2 - 2a_2a_7}{3}\right)x^2y^3\\
& + \left(\frac{a_3 - a_3a_7}{2}\right)xy^4 + \left(\frac{3a_4 - 2a_4a_7}{5}\right)y^5.
\end{aligned}
\tag{4.7}
$$

System (4.7) has the Darboux factor $l_1 = x$ with the cofactor $k_1 = -y(x^3 - a_2x^2y - a_3xy^2 - a_4y^3)$. It is easy to verify that $\beta_1 = 2a_7 - 4$ is a solution of equation (2.5). Thus, $\mu = x^{2a_7-4}$ is an integrating factor of system (4.7) and computing we obtain the first integral

$$
\begin{aligned}
H(x,y) =& x^{2a_7-3}(-(15a_7+15)x + (15a_7-15)x^5 + (60a_7^2 - 60)x^3y^2\\
& + (40a_2 - 40a_2a_7^2)x^2y^3 + (30a_3 - 30a_3a_7^2)xy^4 + (24a_4 - 24a_4a_7^2)y^5),
\end{aligned}
$$

which is analytic in a neighbourhood of the point (1,0). Thus, the points $(-1,0)$ and $(1,0)$ are centers of (4.7).

Case (2): Under conditions (2) of Theorem 4.2 system (4.5) becomes

$$
\begin{aligned}
\dot{x} =& -xy(x + 4a_6y)(x^2 - a_3y^2),\\
\dot{y} =& \frac{1}{4}(x + 4a_6y)(-1 + x^4 + 4a_7x^2y^2 + 4a_9y^4).
\end{aligned}
\tag{4.8}
$$

Note that polynomials on the right hand sides of equation (4.8) have the common factor $x + 4a_6y$, so after the reparametrization of time we obtain the system

$$\begin{aligned}\dot{x} =& - xy(x^2 - a_3y^2) = P(x, y),\\ \dot{y} =& \frac{1}{4}(-1 + x^4 + 4a_7x^2y^2 + 4a_9y^4) = Q(x, y).\end{aligned} \tag{4.9}$$

It is easy to verify that the polynomials $P(x, y)$ and $Q(x.y)$ satisfy equation (2.6). So, system (4.9) posses time-reversible symmetry with respect to the x-axis. Thus, the singular points $(-1, 0)$ and $(1, 0)$ are centers for (4.8).

Case (3): In this case the corresponding system (4.5) is written as

(4.10)

$$\begin{aligned}\dot{x} =& - xy(x + 4a_6y)(x^2 - a_2xy - 4a_6xy - a_3y^2 + 4a_2a_6y^2 + 16a_6^2y^2),\\ \dot{y} =& \frac{1}{12}(x + 4a_6y)(-3 + 3x^4 + 12a_7x^2y^2 + 4a_2xy^3 + 16a_6xy^3 - 8a_2a_7xy^3 - 32a_6a_7xy^3+\\ &+ 6a_3y^4 - 24a_2a_6y^4 - 96a_6^2y^4 - 6a_3a_7y^4 + 24a_2a_6a_7y^4 + 96a_6^2a_7y^4).\end{aligned}$$

Applying the reparametrization of time we obtain the system

$$\begin{aligned}\dot{x} =& - xy(x^2 - (a_2 + 4a_6)xy + (-a_3 + 4a_2a_6 + 16a_6^2)y^2),\\ \dot{y} =& \frac{1}{12}(-3 + 3x^4 + 12a_7x^2y^2 + (4a_2 + 16a_6 - 8a_2a_7 - 32a_6a_7)xy^3\\ &+ (6a_3 - 24a_2a_6 - 96a_6^2 - 6a_3a_7 + 24a_2a_6a_7 + 96a_6^2a_7)y^4).\end{aligned} \tag{4.11}$$

System (4.11) has the Darboux factor $l_1 = x$ with cofactor $k_1 = -y(x^2 - a_2xy - 4a_6xy - a_3y^2 + 4a_2a_6y^2 + 16a_6^2y^2)$. It is easy to verify that $\beta_1 = 2a_7 - 3$ is a solution of equation (2.5). Thus, $\mu = x^{2a_7-3}$ is an integrating factor of system (4.11) and computing we obtain the first integral

$$\begin{aligned}H(x, y) =& x^{2a_7-2}(-3 - 3a7 + (3a_7 - 3)x^4 + (12a_7^2 - 12)x^2y^2\\ &+ (8a_2 + 32a_6 - 8a_2a_7^2 - 32a_6a_7^2)xy^3\\ &+ (6a_3 - 24a_2a_6 - 96a_6^2 - 6a_3a_7^2 + 24a_2a_6a_7^2 + 96a_6^2a_7^2)y^4),\end{aligned}$$

which is analytic in a neighbourhood of the point (1,0). Thus the points $(-1, 0)$ and $(1, 0)$ are centers of (4.10).

Case (4): For this case, system (4.6) becomes

$$\begin{aligned}\dot{u} =& - v - 4uv + a_2v^2 - 6u^2v + 3a_2uv^2 + a_3v^3 - 4u^3v + 3a_2u^2v^2 + 2a_3uv^3 + a_4v^4+\\ &- u^4v + a_2u^3v^2 + a_3u^2v^3 + a_4uv^4,\\ \dot{v} =& u + \frac{5u^2}{2} + 4a_6uv - v^2 + \frac{5u^3}{2} + 6a_6u^2v - 3uv^2 + a_2v^3 + \frac{5u^4}{4} + 4a_6u^3v\\ &- 3u^2v^2 + 2a_2uv^3 + a_3v^4 + \frac{u^5}{4} + a_6u^4v - u^3v^2 + a_2u^2v^3 + a_3uv^4 + a_4v^5.\end{aligned} \tag{4.12}$$

System (4.12) has the Darboux factors $l_1 = 1 + u$ and $l_2 = 1 + u + a_6 v$, with respective cofactors

$$\begin{aligned} k_1 =& v(-1 - 3u - 3u^2 - u^3 + a_2 v + 2a_2 uv + a_2 u^2 v + a_3 v^2 + a_3 uv^2 + a_4 v^3),\\ k_2 =& 4a_6 u + 6a_6 u^2 + 4a_6 u^3 + a_6 u^4 - v - 3uv - 3u^2 v - u^3 v + a_2 v^2 +\\ & 2a_2 uv^2 + a_2 u^2 v^2 + a_3 v^3 + a_3 uv^3 + a_4 v^4. \end{aligned}$$

It is easy to verify that $\beta_1 = -5$ and $\beta_2 = -1$ is a solution of equation (2.5). Thus, $\mu = (1+u)^{-5}(1+u+a_6 v)^{-1}$ is an integrating factor of system (4.12) of the form $\mu = 1 + \cdots$. It implies the existence of an analytic first integral for (4.12). Hence the origin of (4.12) is a center and, therefore, system (4.5) has a bi-enter at the points $(-1, 0)$ and $(1, 0)$.

□

As it is mentioned in the proof of Theorem 4.2, to be able to compute the irreducible decomposition of the variety $\mathbf{V}(\mathcal{B}_9)$ it was necessary to use modular arithmetic. Since we were not able to complete the last step of the computational procedure of [32], we do not know if a component of the variety $\mathbf{V}(\mathcal{B}_9)$ was lost in the computations, that is, if conditions in Theorem 4.2 are all necessary conditions. So, the following conjecture is still open.

Conjecture 4.3. *The conditions in the statement of Theorem 4.2 are necessary and sufficient for the existence of a bi-center for system* (4.5)*.*

4.3. **Isochronicity of system** (4.5)**.** In this subsection we study the isochronicity problem for system (4.5) and obtain the following result.

Theorem 4.4. *System (4.5) does not have isochronous centers at the points* $(-1, 0)$ *and* $(1, 0)$.

Proof. Using MATHEMATICA we computed the first nine pairs of the linearizability quantities for system (4.6) using the procedure described in Subsection 2.2. Their expressions are very large, so we present here only the first pair:

$$\begin{aligned} i_1 &= \frac{1}{9}(48 + 10a_2^2 + 9a_3 - 4a_2 a_6 + 16a_6^2 + 36a_7 + 4a_7^2),\\ j_1 &= \frac{1}{3}(-a_2 - 4a_6 + 2a_2 a_7 - 4a_6 a_7 + 3a_8). \end{aligned}$$

The next computational step is to compute the irreducible decomposition of the variety $\mathbf{V}(\mathcal{L}_9) = \mathbf{V}(\langle i_1, j_1, ..., i_9, j_9\rangle)$. For this purpose we use the routine `minAssGTZ` of the computer algebra system SINGULAR. Since an isochronous center must be a center, to make the computations easier we investigate the existence of an isochronous center using the four conditions described in Theorem 4.2. Thus, our proof is split in four cases corresponding to these four conditions.

Case (1): Computing the reduced Gröbner basis we see that

$$\langle \mathcal{L}_9, a_6, 3a_8 - a_2 + 2a_2 a_7, 2a_9 - a_3 + a_3 a_7, 5a_{10} - 3a_4 + 2a_4 a_7\rangle = \langle 1\rangle.$$

Thus, system (4.5) does not have isochronous centers if condition (1) of Theorem 4.2 holds.

Case (2): The irreducible decomposition of the variety of the ideal

$$\langle \mathcal{L}_9, a_2 + 4a_6, a_4 - 4a_3 a_6, a_8 - 4a_6 a_7, a_{10} - 4a_6 a_9\rangle \tag{4.13}$$

computed using the routine `minAssGTZ` over the field of rational numbers consists of the six components in Appendix II. Analysing the components we see that the corresponding polynomial systems have only complex solutions, so the varieties are empty sets in $\mathbb{R}^8$. Thus system (4.5) cannot have an isochronous center if condition (2) of Theorem 4.2 holds.

The remaining two cases (Cases (3) and (4)) are analogous to Case (2). All the varieties are empty sets in $\mathbb{R}^8$. Therefore system (4.5) cannot have an isochronous center if conditions (3) and (4) of Theorem 4.2 holds.

□

To conclude, in the subfamily of system (1.2) we did not find any isochronous bi-centers, whereas we found two isochronous bi-centers in family (1.1). However, in both cases of isochronous centers of system (1.1) the system had also the third isochronous center at the origin. Since the subfamily of (1.2) which we have studied has y-axis as an invariant curve a center at the origin is not possible in the family. The possibility to have a center at the origin arises if we omit one of conditions $a_1 = -1$ or $a_5 = 0$, but then the computations become too laborious and we were unable to complete them at our computational facilities.

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Appendix I

Here we present the seven prime ideals, $I_1, I_2, ..., I_7$, defining the irreducible components of the the variety of the ideal (3.7), which give conditions for the existence of an isochronous bi-center for system (3.1) under condition (3) of Theorem 11 in [24].

$I_1 = \langle f_1^{(1)}, f_2^{(1)}, f_3^{(1)}, f_4^{(1)}, f_5^{(1)}, f_6^{(1)}, f_7^{(1)} \rangle$, where

$$
\begin{aligned}
f_1^{(1)} =& 16a_4^2 + 4a_5^2 - 12a_5 + 9, \quad f_2^{(1)} = 2a_2a_5 - 4a_4a_5 - 3a_2 - 6a_4, \quad f_3^{(1)} = 8a_2a_4 + 4a_5^2 - 9,\\
f_4^{(1)} =& 4a_2^2 + 4a_5^2 + 12a_5 + 9, \quad f_5^{(1)} = 2/3a_1a_2 + 2/3a_2a_5 - 2/3a_4a_5 + 1/3a_2 - 2/3a_4 + a_6,\\
f_6^{(1)} =& 4/9a_1^2 + 10/9a_2^2 - 2/9a_2a_4 + 4/9a_4^2 - 10/9a_1a_5 + 4/9a_5^2 + a_3 + 4/3a_5 + 2,\\
f_7^{(1)} =& 17671134844578061918193949 0816a_4^2a_5^{13} + 4417783711144515479548487 2704a_5^{15}\\
&+3802937789975548287443966230528a_4^2a_5^{12} + 8182009361595516074745369395 20a_5^{14}\\
&+35343630268187272246740460929024a_4^2a_5^{11} + 6083104358065908444391981522944a_5^{13}\\
&+186124816855970198178551925194752a_4^2a_5^{10} + 22162634019713341271269866606592a_5^{12}\\
&+611958345752720144505463533920256a_4^2a_5^{9} + 33276765822057728131243448856576a_5^{11}\\
&+1302493291690448907138352781529088a_4^2a_5^{8} - 28650226910444645119073997135872a_5^{10}\\
&+1791802131580824336245097947664384a_4^2a_5^{7} - 184692866386725515008166861400576a_5^{9}\\
&+1518210963847726688743552391800832a_4^2a_5^{6} - 231646381147809069732611923187968a_5^{8}\\
&-478160056922050033090560000000a_4^4a_5^3 + 655722361345603412492915306966016a_4^2a_5^5\\
&+33164054965175288516138944452096a_5^7 - 373501753169380009574400000000a_4^6\\
&-180584207014133053587456000000a_4^4a_5^2 - 297076086251379994801693216952 32a_4^2a_5^4\\
&+354767765913173682461862769739648a_5^6 + 420286738397107036815360000000a_4^4a_5\\
&-193222358437787547499641121768320a_4^2a_5^3 + 342794196369769992826332834997728a_5^5\\
&-269670211109923457433600000000a_4^4 - 100437459727707082584722497451200a_4^2a_5^2\\
&+103206647024633679074924509509872a_5^4 - 231586597160868022310056460520 00a_4^2a_5\\
&-392925669595533046003940854192 80a_5^3 - 201899742305835195027698607 0000a_4^2\\
&-395607894345777799626529047948 00a_5^2 + 199200935023669338439680000000a_1\\
&-114223589995927548222485523517 50a_5 - 984400343940413185636244664375.
\end{aligned}
$$

$I_2 = \langle f_1^{(2)}, f_2^{(2)}, f_3^{(2)}, f_4^{(2)}, f_5^{(2)}, f_6^{(2)} \rangle$, where

$$
\begin{aligned}
f_1^{(2)} =& 4a_5 + 3, \quad f_2^{(2)} = 16a_4^2 + 9,\\
f_3^{(2)} =& 2/3a_1a_2 + 2/3a_2a_5 - 2/3a_4a_5 + 1/3a_2 - 2/3a_4 + a_6,\\
f_4^{(2)} =& 4/9a_1^2 + 10/9a_2^2 - 2/9a_2a_4 + 4/9a_4^2 - 10/9a_1a_5 + 4/9a_5^2 + a_3 + 4/3a_5 + 2,\\
f_5^{(2)} =& 25152a_4a_5^3 + 312712a_4a_5^2 + 1074974a_4a_5 + 195000a_2 + 835941a_4,\\
f_6^{(2)} =& 960a_5^3 + 9592a_5^2 + 3900a_1 + 27398a_5 + 18483.
\end{aligned}
$$

$I_3 = \langle f_1^{(3)}, f_2^{(3)}, f_3^{(3)}, f_4^{(3)}, f_5^{(3)}, f_6^{(3)}\rangle$, where

$$f_1^{(3)} = 2a_5 + 9, \quad f_2^{(3)} = 9a_4^2 + 25,$$
$$f_3^{(3)} = 2/3a_1a_2 + 2/3a_2a_5 - 2/3a_4a_5 + 1/3a_2 - 2/3a_4 + a_6,$$
$$f_4^{(3)} = 4/9a_1^2 + 10/9a_2^2 - 2/9a_2a_4 + 4/9a_4^2 - 10/9a_1a_5 + 4/9a_5^2 + a_3 + 4/3a_5 + 2,$$
$$f_5^{(3)} = 25152a_4a_5^3 + 312712a_4a_5^2 + 1074974a_4a_5 + 195000a_2 + 835941a_4,$$
$$f_6^{(3)} = 960a_5^3 + 9592a_5^2 + 3900a_1 + 27398a_5 + 18483.$$

$I_4 = \langle f_1^{(4)}, f_2^{(4)}, f_3^{(4)}, f_4^{(4)}, f_5^{(4)}, f_6^{(4)}\rangle$, where

$$f_1^{(4)} = 6a_5 + 7, \quad f_2^{(4)} = a_4^2 + 4,$$
$$f_3^{(4)} = 2/3a_1a_2 + 2/3a_2a_5 - 2/3a_4a_5 + 1/3a_2 - 2/3a_4 + a_6,$$
$$f_4^{(4)} = 4/9a_1^2 + 10/9a_2^2 - 2/9a_2a_4 + 4/9a_4^2 - 10/9a_1a_5 + 4/9a_5^2 + a_3 + 4/3a_5 + 2,$$
$$f_5^{(4)} = 25152a_4a_5^3 + 312712a_4a_5^2 + 1074974a_4a_5 + 195000a_2 + 835941a_4,$$
$$f_6^{(4)} = 960a_5^3 + 9592a_5^2 + 3900a_1 + 27398a_5 + 18483.$$

$I_5 = \langle f_1^{(5)}, f_2^{(5)}, f_3^{(5)}, f_4^{(5)}, f_5^{(5)}, f_6^{(5)}\rangle$, where

$$f_1^{(5)} = 8a_5 + 31, \quad f_2^{(5)} = a_4^2 + 4,$$
$$f_3^{(5)} = 2/3a_1a_2 + 2/3a_2a_5 - 2/3a_4a_5 + 1/3a_2 - 2/3a_4 + a_6,$$
$$f_4^{(5)} = 4/9a_1^2 + 10/9a_2^2 - 2/9a_2a_4 + 4/9a_4^2 - 10/9a_1a_5 + 4/9a_5^2 + a_3 + 4/3a_5 + 2,$$
$$f_5^{(5)} = 25152a_4a_5^3 + 312712a_4a_5^2 + 1074974a_4a_5 + 195000a_2 + 835941a_4,$$
$$f_6^{(5)} = 960a_5^3 + 9592a_5^2 + 3900a_1 + 27398a_5 + 18483.$$

$I_6 = \langle f_1^{(6)}, f_2^{(6)}, f_3^{(6)}, f_4^{(6)}, f_5^{(6)}, f_6^{(6)}\rangle$, where

$$f_1^{(6)} = 2a_5 - 1, \quad f_2^{(6)} = a_4^2 + 1,$$
$$f_3^{(6)} = 2/3a_1a_2 + 2/3a_2a_5 - 2/3a_4a_5 + 1/3a_2 - 2/3a_4 + a_6,$$
$$f_4^{(6)} = 4/9a_1^2 + 10/9a_2^2 - 2/9a_2a_4 + 4/9a_4^2 - 10/9a_1a_5 + 4/9a_5^2 + a_3 + 4/3a_5 + 2,$$
$$f_5^{(6)} = 2a_4a_5 + 3a_2 + 2a_4, \quad f_6^{(6)} = 4a_1 + 2a_5 + 3.$$

$I_7 = \langle f_1^{(7)}, f_2^{(7)}, f_3^{(7)}, f_4^{(7)}, f_5^{(7)}, f_6^{(7)}\rangle$, where

$$f_1^{(7)} = 2a_5 - 7, \quad f_2^{(7)} = a_4^2 + 1,$$
$$f_3^{(7)} = 2/3a_1a_2 + 2/3a_2a_5 - 2/3a_4a_5 + 1/3a_2 - 2/3a_4 + a_6,$$
$$f_4^{(7)} = 4/9a_1^2 + 10/9a_2^2 - 2/9a_2a_4 + 4/9a_4^2 - 10/9a_1a_5 + 4/9a_5^2 + a_3 + 4/3a_5 + 2,$$
$$f_5^{(7)} = 2a_4a_5 + 3a_2 + 2a_4, \quad f_6^{(7)} = 4a_1 + 2a_5 + 3.$$

Appendix II

Here we present the six prime ideals, $I_1, I_2, ..., I_6$, defining the irreducible components of the the variety of the ideal (4.13), which give conditions for the existence of an isochronous bi-center for system (4.5) under condition (2) of Theorem 4.2.

$I_1 = \langle f_1^{(1)}, f_2^{(1)}, f_3^{(1)}, f_4^{(1)}, f_5^{(1)}, f_6^{(1)}, f_7^{(1)}, f_8^{(1)} \rangle$, where

$$f_1^{(1)} = a_9, \quad f_2^{(1)} = a_7, \quad f_3^{(1)} = 4a_6^2 + 1, \quad f_4^{(1)} = -4a_6a_9 + a_{10}, \quad f_5^{(1)} = -4a_6a_7 + a_8,$$
$$f_6^{(1)} = 256/3a_6^3 + 16/9a_6a_7^2 + 16a_6a_7 + a_4 + 64/3a_6,$$
$$f_7^{(1)} = 64/3a_6^2 + 4/9a_7^2 + a_3 + 4a_7 + 16/3, \quad f_8^{(1)} = a_2 + 4a_6.$$

$I_2 = \langle f_1^{(2)}, f_2^{(2)}, f_3^{(2)}, f_4^{(2)}, f_5^{(2)}, f_6^{(2)}, f_7^{(2)}, f_8^{(2)} \rangle$, where

$$f_1^{(2)} = 81a_9 + 16, \quad f_2^{(2)} = a_7 + 1, \quad f_3^{(2)} = 9a_6^2 + 1, \quad f_4^{(2)} = -4a_6a_9 + a_{10},$$
$$f_5^{(2)} = -4a_6a_7 + a_8, \quad f_6^{(2)} = 256/3a_6^3 + 16/9a_6a_7^2 + 16a_6a_7 + a_4 + 64/3a_6,$$
$$f_7^{(2)} = 64/3a_6^2 + 4/9a_7^2 + a_3 + 4a_7 + 16/3, \quad f_8^{(2)} = a_2 + 4a_6.$$

$I_3 = \langle f_1^{(3)}, f_2^{(3)}, f_3^{(3)}, f_4^{(3)}, f_5^{(3)}, f_6^{(3)}, f_7^{(3)}, f_8^{(3)} \rangle$, where

$$f_1^{(3)} = a_9 + 4, \quad f_2^{(3)} = a_7 - 3, \quad f_3^{(3)} = 4a_6^2 + 1, \quad f_4^{(3)} = -4a_6a_9 + a_{10},$$
$$f_5^{(3)} = -4a_6a_7 + a_8, \quad f_6^{(3)} = 256/3a_6^3 + 16/9a_6a_7^2 + 16a_6a_7 + a_4 + 64/3a_6,$$
$$f_7^{(3)} = 64/3a_6^2 + 4/9a_7^2 + a_3 + 4a_7 + 16/3, \quad f_8^{(3)} = a_2 + 4a_6.$$

$I_4 = \langle f_1^{(4)}, f_2^{(4)}, f_3^{(4)}, f_4^{(4)}, f_5^{(4)}, f_6^{(4)}, f_7^{(4)}, f_8^{(4)} \rangle$, where

$$f_1^{(4)} = a_9 - 48, \quad f_2^{(4)} = a_7 - 15, \quad f_3^{(4)} = a_6^2 + 1, \quad f_4^{(4)} = -4a_6a_9 + a_{10},$$
$$f_5^{(4)} = -4a_6a_7 + a_8, \quad f_6^{(4)} = 256/3a_6^3 + 16/9a_6a_7^2 + 16a_6a_7 + a_4 + 64/3a_6,$$
$$f_7^{(4)} = 64/3a_6^2 + 4/9a_7^2 + a_3 + 4a_7 + 16/3, \quad f_8^{(4)} = a_2 + 4a_6.$$

$I_5 = \langle f_1^{(5)}, f_2^{(5)}, f_3^{(5)}, f_4^{(5)}, f_5^{(5)}, f_6^{(5)}, f_7^{(5)}, f_8^{(5)} \rangle$, where

$$f_1^{(5)} = 4a_9 - 1, \quad f_2^{(5)} = 2a_7 + 3, \quad f_3^{(5)} = 16a_6^2 + 1, \quad f_4^{(5)} = -4a_6a_9 + a_{10},$$
$$f_5^{(5)} = -4a_6a_7 + a_8, \quad f_6^{(5)} = 256/3a_6^3 + 16/9a_6a_7^2 + 16a_6a_7 + a_4 + 64/3a_6,$$
$$f_7^{(5)} = 64/3a_6^2 + 4/9a_7^2 + a_3 + 4a_7 + 16/3, \quad f_8^{(5)} = a_2 + 4a_6.$$

$I_6 = \langle f_1^{(6)}, f_2^{(6)}, f_3^{(6)}, f_4^{(6)}, f_5^{(6)}, f_6^{(6)}, f_7^{(6)}, f_8^{(6)} \rangle$, where

$$f_1^{(6)} = a_9, \quad f_2^{(6)} = a_7 - 3, \quad f_3^{(6)} = a_6^2 + 1, \quad f_4^{(6)} = -4a_6a_9 + a_{10}, \quad f_5^{(6)} = -4a_6a_7 + a_8,$$
$$f_6^{(6)} = 256/3a_6^3 + 16/9a_6a_7^2 + 16a_6a_7 + a_4 + 64/3a_6,$$
$$f_7^{(6)} = 64/3a_6^2 + 4/9a_7^2 + a_3 + 4a_7 + 16/3, \quad f_8^{(6)} = a_2 + 4a_6.$$

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