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CLASSIFICATION OF QUADRATIC DIFFERENTIAL SYSTEMS WITH INVARIANT HYPERBOLAS ACCORDING TO THEIR CONFIGURATIONS OF INVARIANT HYPERBOLAS AND INVARIANT LINES

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Classification of quadratic differential systems with invariant hyperbolas according to their configurations of invariant hyperbolas and invariant lines

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Abstract

Let **QSH** be the whole class of non-degenerate planar quadratic differential systems possessing at least one invariant hyperbola. In this article, we study family $\mathbf{QSH}_{(\eta=0)}$ of systems in **QSH** which possess either exactly two distinct real singularities at infinity or the line at infinity filled up with singularities. We classify this family of systems, modulo the action of the group of real affine transformations and time rescaling, according to their geometric properties encoded in the configurations of invariant hyperbolas and invariant straight lines which these systems possess. The classification is given both in terms of algebraic geometric invariants and also in terms of affine invariant polynomials and it yields a total of 40 distinct such configurations. This last classification is also an algorithm which makes it possible to verify for any given real quadratic differential system if it has invariant hyperbolas or not and to specify its configuration of invariant hyperbolas and straight lines.

Key-words: quadratic differential systems, algebraic solution, configuration of algebraic solutions, affine invariant polynomials, group action

2000 Mathematics Subject Classification: 34C23, 34A34

1 Introduction and statement of the main results

We consider planar polynomial differential systems which are systems of the form

$$dx/dt = p(x,y), \quad dy/dt = q(x,y) \tag{1}$$

where $p(x,y), q(x,y)$ are polynomial in x, y with real coefficients ($p, q \in \mathbb{R}[x,y]$) and their associated vector fields

$$X = p(x,y)\frac{\partial}{\partial x} + q(x,y)\frac{\partial}{\partial y}. \tag{2}$$

We call *degree* of such a system the number $\max(\deg(p), \deg(q))$. In the case where the polynomials p and q are coprime we say that (1) is *non-degenerate*.

A real quadratic differential system is a polynomial differential system of degree 2, i.e.

$$\begin{aligned}\dot{x} &= p_0 + p_1(\tilde{a}, x, y) + p_2(\tilde{a}, x, y) \equiv p(\tilde{a}, x, y),\\ \dot{y} &= q_0 + q_1(\tilde{a}, x, y) + q_2(\tilde{a}, x, y) \equiv q(\tilde{a}, x, y)\end{aligned} \tag{3}$$

with $\max(\deg(p), \deg(q)) = 2$ and

$$\begin{aligned}&p_0 = a, \quad p_1(\tilde{a}, x, y) = cx + dy, \quad p_2(\tilde{a}, x, y) = gx^2 + 2hxy + ky^2,\\ &q_0 = b, \quad q_1(\tilde{a}, x, y) = ex + fy, \quad q_2(\tilde{a}, x, y) = lx^2 + 2mxy + ny^2.\end{aligned}$$

Here we denote by $\tilde{a} = (a, c, d, g, h, k, b, e, f, l, m, n)$ the 12-tuple of the coefficients of system (3). Thus a quadratic system can be identified with a points $\tilde{a}$ in $\mathbb{R}^{12}$.

We denote the class of all quadratic differential systems with **QS**.

Planar polynomial differential systems occur very often in various branches of applied mathematics, in modeling natural phenomena, for example, modeling the time evolution of conflicting species in biology and in chemical reactions and economics [12, 30], in astrophysics [6], in the equations of continuity describing the interactions of ions, electrons and neutral species in plasma physics [19]. Polynomial systems appear also in shock waves, in neural networks, etc. Such differential systems have also theoretical importance. Several problems on polynomial differential systems, which were stated more than one hundred years ago, are still open: the second part of Hilbert's 16th problem stated by Hilbert in 1900, the problem of algebraic integrability stated by Poincaré in 1891 [17], [18], the problem of the center stated by Poincaré in 1885 [16], and problems on integrability resulting from the work of Darboux [8] published in 1878. With the exception of the problem of the center, which was solved only for quadratic differential systems, all the other problems mentioned above, are still unsolved even in the quadratic case.

Definition 1 (Darboux)**.** *An algebraic curve $f(x,y) = 0$ where $f \in \mathbb{C}[x,y]$ is an invariant curve of the system polynomial system* (1) *if and only if there exists a polynomial $k(x,y) \in \mathbb{C}[x,y]$ such that*

$$p(x,y)\frac{\partial f}{\partial x} + q(x,y)\frac{\partial f}{\partial y} = f(x,y)k(x,y).$$

Definition 2 (Darboux)**.** *We call algebraic solution of a planar polynomial system an invariant algebraic curve over $\mathbb{C}$ which is irreducible.*

One of our motivations in this article comes from integrability problems related to the work of Darboux [8].

Theorem 1 (Darboux)**.** *Suppose that a polynomial system* (1) *has m invariant algebraic curves $f_i(x,y) = 0$, $i \le m$, with $f_i \in \mathbb{C}[x,y]$ and with $m > n(n+1)/2$ where n is the degree of the system. Then there exist complex numbers $\lambda_1, ..., \lambda_m$ such that $f_1^{\lambda_1} ... f_m^{\lambda_m}$ is a first integral of the system.*

The condition in Darboux' theorem is only sufficient for Darboux integrability (integrability in terms of invariant algebraic curves) and it is not also necessary. Thus the lower bound on the number of invariant curves sufficient for Darboux integrability stated in the theorem of Darboux is larger than necessary. Darboux' theory has been improved by including for example the multiplicity of the curves ([11]). Also, the number of invariant algebraic curves needed was reduced but by adding some conditions, in particular the condition that any two of the curves be transversal. But a deeper understanding about Darboux integrability is still lacking. Algebraic integrability, which intervenes in the open problem stated by Poincaré in 1891 ([17] and [18]), and which means the existence of a rational first integral for the system, is a particular case of Darboux integrability.

To advance knowledge on algebraic or more generally Darboux integrability it is necessary to have a large number of examples to analyze. In the literature, scattered isolated examples were analyzed but a more systematic approach was still needed. Schlomiuk and Vulpe initiated a systematic program to construct such a data base for quadratic differential systems. Since the simplest case is of systems with invariant straight lines, their first works involved only lines (see [22], [24], [25], [27], [28]). In this work we study the class **QSH** of non-degenerate, i.e. p, q are relatively prime, quadratic differential systems having an invariant hyperbola. Such systems could also have some invariant lines and in many cases the presence of these invariant curves turns them into Darboux integrable systems. We always assume here that the systems (3) are non-degenerate because otherwise doing a time rescaling, they can be reduced to linear or constant systems. Under this assumption all the systems in **QSH** have a finite number of finite singular points.

We introduced here the class **QSH** of non-degenerate quadratic systems possessing at least one invariant hyperbola. This class requires some explanation. Indeed the term hyperbola is reserved for a real irreducible affine conic which has two real points at infinity. This distinguishes it from the other two irreducible real conics: the parabola with just one real point at infinity which is double and the ellipse which has two complex points at infinity. But in the theory of Darboux the invariant algebraic curves are considered (and rightly so) over the complex field $\mathbb{C}$. We need to extend the notions of hyperbola, parabola or ellipse for conics over $\mathbb{C}$ which is easily done. We call "complex hyperbola" (respectively "complex ellipse", "complex parabola") an algebraic curve $C : f(x, y) = 0$ with $f \in \mathbb{C}[x, y]$, $\deg(f) = 2$ which is irreducible and which has two real points at infinity (respectively two complex (non-real) points at infinity, one double point at infinity (see [1])).

Let us suppose that a polynomial differential system has an algebraic solution $f(x, y) = 0$ where $f(x, y) \in \mathbb{C}[x, y]$ is of degree n, $f(x, y) = a_0 + a_{10}x + a_{01}y + ... + a_{n0}x^n + a_{n-1,1}x^{n-1}y + ... + a_{0n}y^n$ with $\hat{a} = (a_0, ..., a_{0n}) \in \mathbb{C}^N$ where $N = (n + 1)(n + 2)/2$. We note that the equation $\lambda f(x, y) = 0$ where $\lambda \in \mathbb{C}^*$ and $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ yields the same locus of complex points in the plane as the locus induced by $f(x, y) = 0$. So a curve of degree n defined by $\hat{a}$ can be identified with a point $[\hat{a}] = [a_0 : a_{10} : ... : a_{0n}]$ in $P_{N-1}(\mathbb{C})$. We say that a sequence of degree n curves $f_i(x, y) = 0$ converges to a curve $f(x, y) = 0$ if and only if the sequence of points $[a_i] = [a_{i0} : a_{i10} ... : a_{i0n}]$ converges to $[\hat{a}] = [a_0 : a_{10} : ... : a_{0n}]$ in the topology of $P_{N-1}(\mathbb{C})$.

On the class **QS** acts the group of real affine transformations and time rescaling and due to this, modulo this group action quadratic systems ultimately depend on five parameters. This group also acts on **QSH** and modulo this action the systems in this class depend on three parameters.

We observe that if we rescale the time $t' = \lambda t$ by a non-zero real constant λ the geometry of the systems (1) does not change. So for our purposes we can identify a system (1) of degree n with a point in $[a_0 : a_{10} : ... : b_{0n}]$ in $P_{N-1}(\mathbb{R})$ with $N = (n+1)(n+2)$.

Definition 3. *(i) We say that an invariant curve* $\mathcal{L} : f(x,y) = 0$, $f \in \mathbb{C}[x,y]$ *for a polynomial system* (S) *of degree* n *has* ***multiplicity*** m *if there exists a sequence of real polynomial systems* (S_k) *of degree* n *converging to* (S) *in the topology of* $P_{N-1}(\mathbb{R})$, $N = (n+1)(n+2)$, *such that each* (S_k) *has* m *distinct invariant curves* $\mathcal{L}_{1,k} : f_{1,k}(x,y) = 0, \dots, \mathcal{L}_{m,k} : f_{m,k}(x,y) = 0$ *over* $\mathbb{C}$, $deg(f) = deg(f_{i,k}) = r$, *converging to* $\mathcal{L}$ *as* $k \to \infty$, *in the topology of* $P_{R-1}(\mathbb{C})$, *with* $R = (r+1)(r+2)/2$ *and this does not occur for* $m+1$.

(ii) We say that the line at infinity $\mathcal{L}_\infty : Z = 0$ *of a polynomial system* (S) *of degree* n *has* ***multiplicity*** m *if there exists a sequence of real polynomial systems* (S_k) *of degree* n *converging to* (S) *in the topology of* $P_{N-1}(\mathbb{R})$, $N = (n+1)(n+2)$, *such that each* (S_k) *has* $m-1$ *distinct invariant lines* $\mathcal{L}_{1,k} : f_{1,k}(x,y) = 0, \dots, \mathcal{L}_{m,k} : f_{m,k}(x,y) = 0$ *over* $\mathbb{C}$, *converging to the line at infinity* $\mathcal{L}_\infty$ *as* $k \to \infty$, *in the topology of* $P_2(\mathbb{C})$ *and this does not occur for* m.

Definition 4. (a) *Suppose a planar polynomial system* (S) *has a finite number of algebraic solutions* $\mathcal{L}_i$ $i \leq k$ *with corresponding multiplicities* n_i *and the line at infinity* $\mathcal{L}_\infty$ *is not filled up with singularities and it has multiplicity* n_∞. *We call* ***total multiplicity*** *of these algebraic solutions, including the multiplicity* n_∞ *of the line at infinity* $\mathcal{L}_\infty$, *the sum* $TMC_{(S)} = n_1 + ... + n_k + n_\infty$.

(b) *Suppose the system* (S) *has a finite number of real distinct singularities* $s_1, ..., s_l$, *finite or infinite, which are located on the algebraic solutions, and having the corresponding multiplicities* $m_1, ..., m_l$. *We call* ***total multiplicity of the real singularities on the invariant curves*** *of* (S) *the sum* $TMS_{(S)} = m_1 + ... + m_l$ *and* TMS *is the function defined by this expression.*

An important ingredient in this work is the notion of *configuration of invariant curves* of a polynomial differential system. This notion appeared for the first time in [22].

Definition 5. *Consider a planar polynomial system which has a finite number of algebraic solutions and a finite number of singular points, finite or infinite. By* ***configuration of algebraic solutions*** *of this system we mean the set of algebraic solutions over* $\mathbb{C}$ *of the system, each one of these curves endowed with its own multiplicity and together with all the real singular points of this system located on these curves, each one of these singularities endowed with its own multiplicity.*

In the family **QSH** we could have systems which have an infinite number of algebraic solutions. In this particular case we show that we also have a finite number of invariant straight lines and a finite number of finite singularities and we can use this fact to define a notion of configuration including only the affine invariant lines of the system. In case such a system has a finite number of singularities at infinity (respectively an infinite number of singularities at infinity) we call *configuration of lines* of the system, the set of all invariant lines (respectively the set of invariant affine lines), each endowed with its own multiplicity together with the set of all real singularities of the systems located on these lines. We associate to each system in **QSH** its *configuration of invariant hyperbolas and/or straight lines.*

We may have two distinct systems which may even be non-equivalent modulo the action of the group but which may have "the same configuration" of invariant hyperbolas and straight lines. We need to say when two configurations are "the same" or equivalent.

Definition 6. *Suppose we have two configurations C_1, C_2 of hyperbolas and lines of systems $(S_1), (S_2)$ in* **QSH** *with a finite number of such curves and a finite number of real singular points. We say that they are equivalent if there is a one-to-one correspondence ϕ_h between the hyperbolas of C_1 and C_2 and a one to one correspondence ϕ_l between the lines of C_1 and C_2 such that:*
(i) the correspondences conserve the multiplicities of the hyperbolas and/or lines,
(ii) for each hyperbola $\mathcal{H}$ of C_1 (respectively each line $\mathcal{L}$) we have a one-to-one correspondence between the real singular points on $\mathcal{H}$ (respectively on $\mathcal{L}$) and the real singular points on $\phi_h(H)$ (respectively $\phi_l(\mathcal{L})$) conserving their multiplicities, their location on branches and their order on these branches.
In case the systems have an infinite number of hyperbolas we only need to have the one-to-one correspondences between their lines (affine lines in case (S_1) and (S_2) have the line at infinity filled up with singularities) with their associated conditions (i) and (ii) above.

In [13] the authors provide necessary and sufficient conditions for a non-degenerate quadratic differential system to have at least one invariant hyperbola and these conditions are expressed in terms of the coefficients of the systems. In [14] the family of quadratic systems in **QSH** which possess three distinct real singularities at infinity was considered. The authors classified this family of systems, modulo the action of the group of real affine transformations and time rescaling, according to their geometric properties encoded in the configurations of invariant hyperbolas and invariant straight lines which these systems possess. As a result 162 distinct such configurations where detected as well as the necessary and sufficient affine invariant conditions for the realization of each one of them where constructed.

This article is a continuation of [14]. We denote by $\mathbf{QSH}_{(\eta=0)}$ the class of non-degenerate quadratic differential systems possessing an invariant hyperbola and either exactly two distinct real singularities at infinity or the line at infinity filled up with singularities. The goal of this article is to produce a similar classification of the family $\mathbf{QSH}_{(\eta=0)}$.

As we want this classification to be intrinsic, independent of the normal form given to the systems, we use here geometric invariants and invariant polynomials for the classification. For example it is clear that the configuration of algebraic solutions of a system is an affine invariant. The classification is done according to the configurations of invariant hyperbolas and straight lines encountered in systems belonging to **QSH**. In particular the notion of multiplicity in Definition 3 is invariant under the group action, i.e. if a quadratic system S has an invariant curve $\mathcal{L} = 0$ of multiplicity m, then each system S' in the orbit of S under the group action has a corresponding invariant line $\mathcal{L}' = 0$ of the same multiplicity m. To distinguish configurations of algebraic solutions we need some geometric invariants which are introduced in Section 2. In the second part of our Main Theorem we use invariant polynomials which are also introduced in our Section 2.

Main Theorem. *Consider the class* $\mathbf{QSH}_{(\eta=0)}$ *of all non-degenerate quadratic differential systems* (3) *possessing an invariant hyperbola and either exactly two distinct real singularities at infinity or the line at infinity filled up with singularities.*

(A) This family is classified according to the configurations of invariant hyperbolas and of invariant straight lines of the systems, yielding 40 distinct such configurations. This geometric classification appears in Diagrams 1 and 2. More precisely:

(A_1) There are exactly 9 configurations with an infinity of invariant hyperbolas. These configu-

rations could have up to 3 distinct affine invariant lines which could have multiplicities up to at most 3. The configurations are split as follows:

(a) *2 of them with exactly two infinite singularities (Config.$\tilde{H}_{32}$, Config.$\tilde{H}_{33}$) distinguished by the type of the invariant lines divisor ILD (as defined in Section 2);*

(b) *7 of them with the line at infinity filled up with singularities (Config.$\tilde{H}_i$, $34 \leq i \leq 40$). The type of the ILD splits these configurations in three groups:* ***Group 1:*** *Config.$\tilde{H}.i$, $34 \leq i \leq 36$, first distinguished by the number of finite singularities (3 for Config.$\tilde{H}.36$ and 2 for Config.$\tilde{H}.i$, $i \in \{34, 35\}$). The last two configurations are distinguished by the number of finite singularities not located on the invariant hyperbolas (1 for i=34, 0 for i=35).* ***Group 2:*** *Config.$\tilde{H}.i$ with $i \in \{37, 38\}$ and* ***Group 3:*** *Config.$\tilde{H}.i$ with $i \in \{39, 40\}$ The configurations in these groups are distinguished by the type of the zero-cycle MS_{0C};*

(A_2) *The remaining 31 configurations could have up to a maximum of 2 distinct invariant hyperbolas, real or complex, and up to 3 distinct invariant straight lines, real or complex, including the line at infinity.*

-We have exactly 12 distinct configurations *of systems with exactly one hyperbola which is simple, and no invariant affine lines. These are classified by the total multiplicity of the real singularities of the systems located on the algebraic solutions (TMS) as follows:*

(a) *only one configuration (Config.$\tilde{H}_1$) with $TMS = 3$;*

(b) *5 configurations with $TMS = 5$ grouped as follows by the number of their singularities and their multiplicities:*
-one with only two singularities, both multiple and both at infinity (Config.$\tilde{H}_2$);
-two with an additional finite singularity (Config.$\tilde{H}_3$, Config.$\tilde{H}_4$) but with distinct multiplicities;
-two with two additional finite simple singularities (Config.$\tilde{H}_5$,Config.$\tilde{H}_6$) distinguished using the proximity divisor PD defined in Section 2;

(c) *4 with $TMS = 6$: one with only one finite singularity (Config.$\tilde{H}_7$); 3 with two finite singularities with the same multiplicities, distinguished by the invariant O defined in Section 2 (Config.$\tilde{H}_i$, $8 \leq i \leq 10$);*

(d) *2 with $TMS = 7$ distinguished by the multiplicities of their two finite singularities (Config.$\tilde{H}_{11}$, Config.$\tilde{H}_{12}$).*

-We have exactly 6 configurations *with a unique simple invariant hyperbola and a unique simple invariant line:*

(a) *one with no finite singularity (Config.$\tilde{H}_{13}$);*

(b) *one with only one finite singularity (Config.$\tilde{H}_{14}$);*

(c) *one with two finite singularities (Config.$\tilde{H}_{15}$);*

(d) *one with three finite singularities (Config.$\tilde{H}_{16}$);*

(e) *two with four simple finite singularities (Config.$\tilde{H}_{17}$, Config.$\tilde{H}_{18}$), configurations distinguished by the proximity divisor PD (see Section 2);*

-We have exactly 9 configurations *with a simple invariant hyperbola and invariant lines, including the line at infinity, of total multiplicity $3 \leq TML \leq 5$:*

(a) 5 configurations have exactly three distinct simple invariant lines (Config.$\tilde{H}_i$*,* $19 \le i \le 23$*) distinguished by the types of* ICD, MS_{0C} *and the proximity divisor* PD*;*

(b) 4 configurations with exactly two invariant lines, one of them being multiple (Config.$\tilde{H}_i$*,* $24 \le i \le 27$*). They are distinguished by the multiplicities of the two invariant lines.*

- We have exactly 4 configurations *with invariant hyperbolas of total multiplicity 2:*

(a) two with two distinct hyperbolas, one with real hyperbolas (Config.$\tilde{H}_{28}$*) and one with complex (non-real) hyperbolas (Config.*$\tilde{H}_{29}$*),*

(b) two of them with a double hyperbola, one with 3 finite singularities (Config.$\tilde{H}_{30}$*)and one without any finite singularity (Config.*$\tilde{H}_{31}$*);*

(B) *Diagram 3 is the bifurcation diagram in the space* $\mathbb{R}^{12}$ *of the coefficients of the system in* $\mathbf{QSH}_{(\boldsymbol{\eta}=\mathbf{0})}$ *according to their configurations of invariant hyperbolas and invariant straight lines. Moreover Diagram 3 gives an algorithm to compute the configuration of a system with an invariant hyperbola for any quadratic differential system, presented in any normal form.*

Remark 1. *In the above Theorem we indicated that the 40 configurations obtained for the family* $\mathbf{QSH}_{(\boldsymbol{\eta}=\mathbf{0})}$ *are distinct due to the types of* ICD, ILD, MS_{0C} *and* PD*. We define in Section 2 such functions on the family* $\mathbf{QSH}_{(\boldsymbol{\eta}=\mathbf{0})}$ *and prove that they define a complete set of geometric invariants for the configurations of the family* $\mathbf{QSH}_{(\boldsymbol{\eta}=\mathbf{0})}$*.*

Remark 2. *The invariant polynomials which appear in Diagram 3 are introduced in Section 2. Moreover in this diagram we denote by* $(\mathfrak{C}_1)$ *the following condition*

$$(\mathfrak{C}_1): \ (\beta_6 = 0, \beta_{11}\mathcal{R}_{11} \neq 0) \cap \big((\beta_{12} \neq 0, \gamma_{15} = 0) \cup (\beta_{12} = \gamma_{16} = 0)\big).$$

2 Basic concepts, proof of part A of the Main Theorem and auxiliary results

In this section we define all the invariants we use in the Main Theorem and we state some auxiliary results. A quadratic system possessing an invariant hyperbola could also possess invariant lines. We classified the systems possessing an invariant hyperbola in terms of their configurations of invariant hyperbolas and invariant lines. Each one of these invariant curves has a multiplicity in the sense of Definition 3 (see also in [7]). We encode this picture in the multiplicity divisor of invariant hyperbolas and lines. We first recall the algebraic-geometric definition of an r-cycle on an irreducible algebraic variety of dimension n.

Definition 7. *Let* V *be an irreducible algebraic variety of dimension* n *over a field* K*. A cycle of dimension* r *or* r*-cycle on* V *is a formal sum* $\Sigma_W n_W W$*, where* W *is a subvariety of* V *of dimension* r *which is not contained in the singular locus of* V*,* $n_W \in Z$*, and only a finite number of* n_W*'s are non-zero. We call* ***degree of an*** r***-cycle*** *the sum* Σn_W*. An* $(n-1)$*-cycle is called a* ***divisor****.*

Definition 8. *Let* V *be an irreducible algebraic variety over a field* K*. The* ***support of a cycle*** C *on* V *is the set* $Supp(C) = \{W | n_W \neq 0\}$*. We denote by* $Max(C)$ *the maximum value of the*

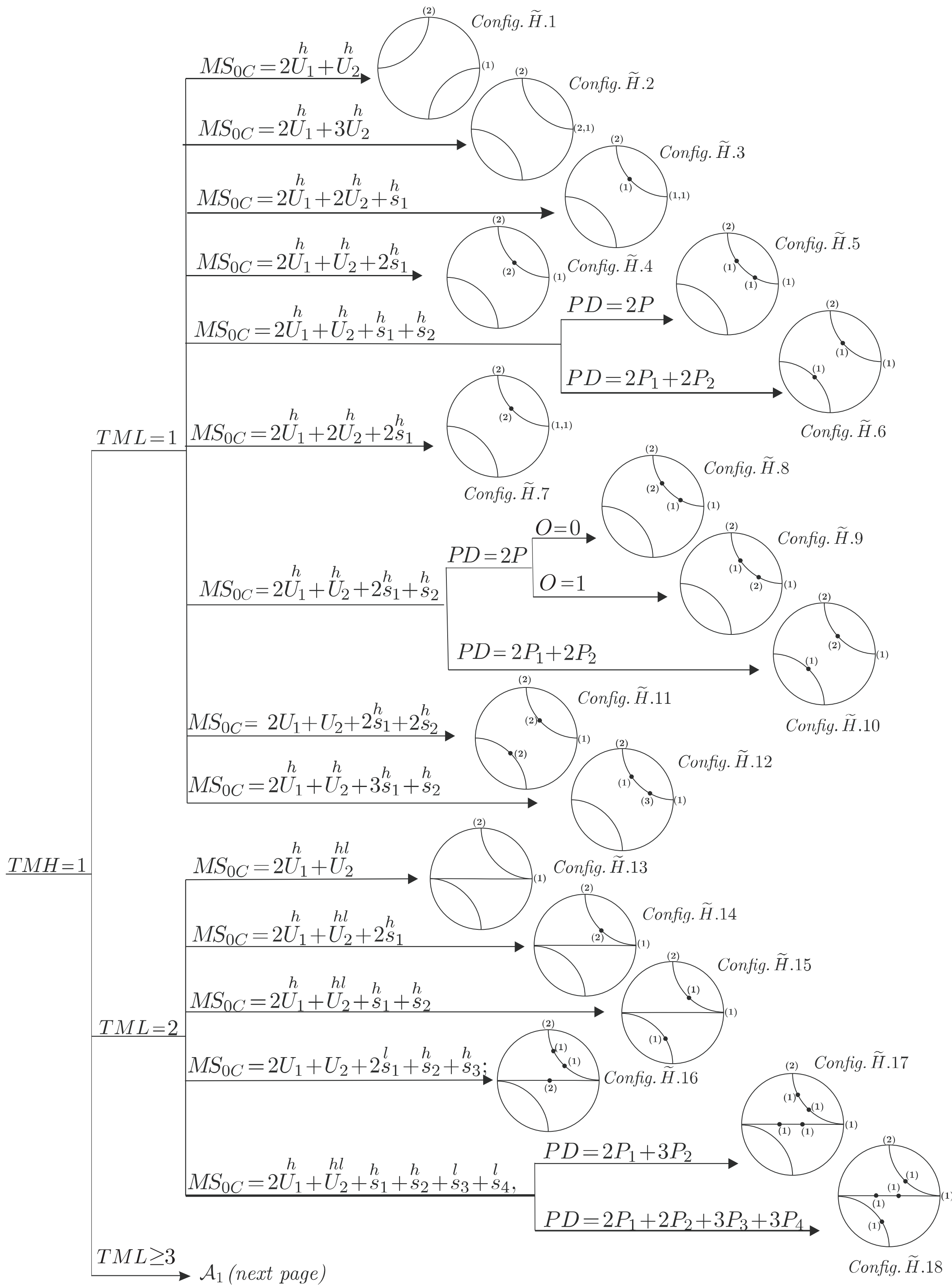

Diagram 1: **Diagram of configurations with one simple hyperbola**

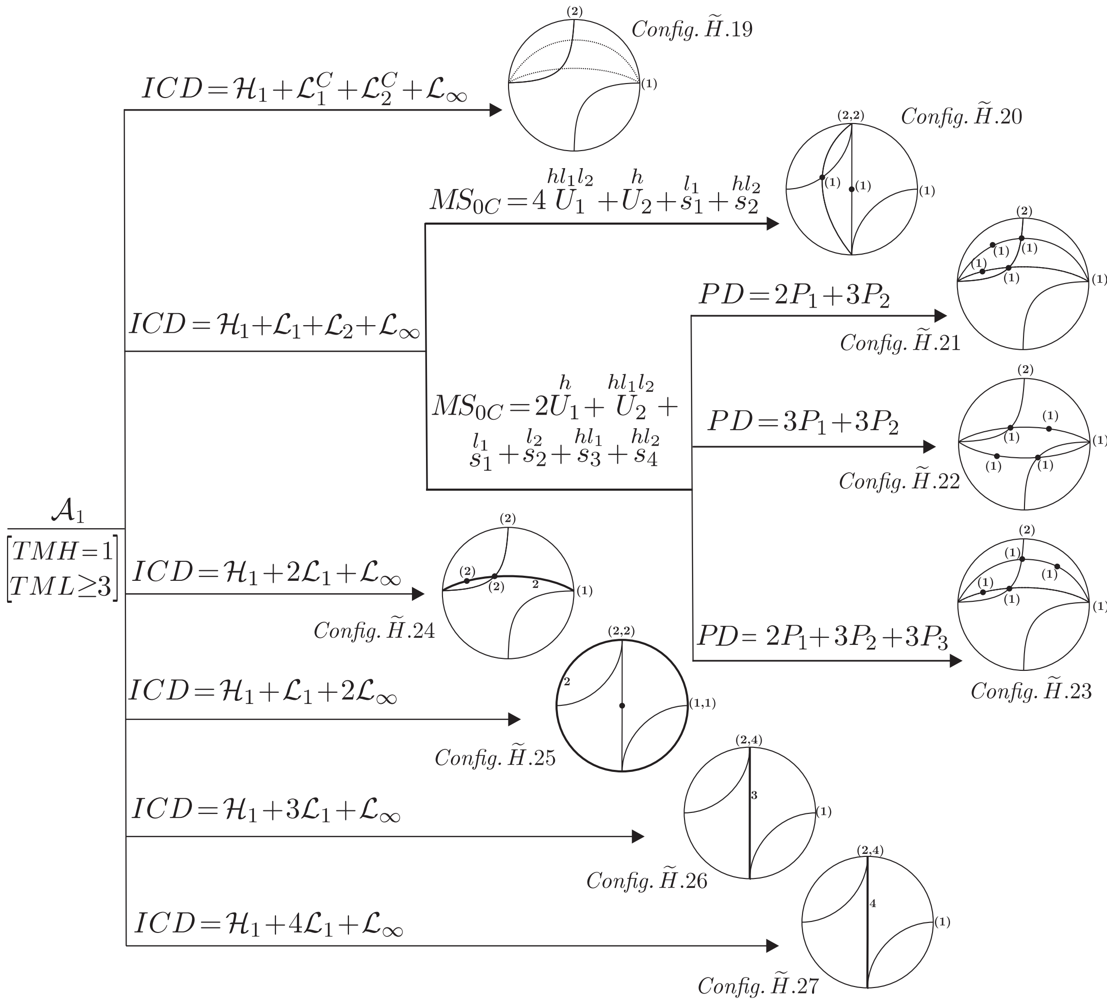

DIAGRAM 1: (*Cont.*) **Diagram of configurations with one simple hyperbola**

coefficients n_W in C. For every $m \leq Max(C)$ let $s(m)$ be the number of the coefficients n_W in C which are equal to m. We call ***type of the cycle*** *C the set of ordered couples $(s(m), m)$ where $1 \leq m \leq Max(C)$.*

For a non-degenerate polynomial differential systems (S) possessing a finite number of algebraic solutions $f_i(x, y) = 0$, each with multiplicity n_i and a finite number of singularities at infinity, we define the *algebraic solutions divisor* (or the invariant curves divisor) on the projective plane, $ICD = \Sigma_{n_i} n_i \mathcal{C}_i + n_\infty \mathcal{L}_\infty$ (the invariant curves divisor) where $\mathcal{C}_i : F_i(X, Y, Z) = 0$ are the projective completions of $f_i(x, y) = 0$, n_i is the multiplicity of the curve $\mathcal{C}_i = 0$ and n_∞ is the multiplicity of the line at infinity $\mathcal{L}_\infty : Z = 0$. It is well known (see [2]) that the maximum number of invariant straight lines for polynomial systems of degree $n \geq 2$ is 3n (including the line at infinity).

In the case we consider here, we have a particular instance of the divisor ICD because the invariant curves will be invariant hyperbolas and invariant lines of a quadratic differential system, in case these are in finite number. In case we have an infinite number of hyperbolas we use only the invariant lines to construct the divisor.

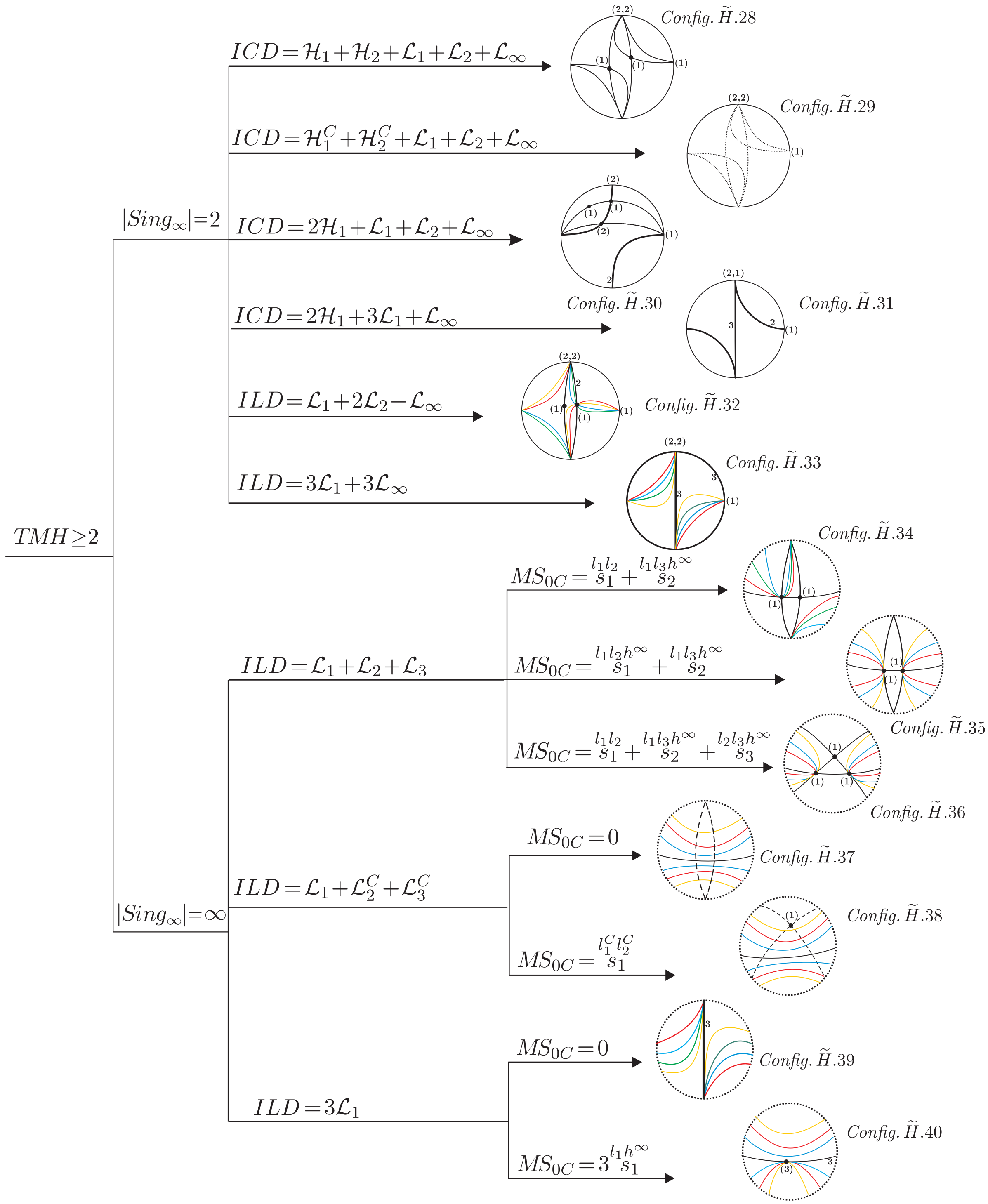

Diagram 2: **Diagram of configurations with $TMH \geq 2$**

Another ingredient of the configuration of algebraic solutions are the real singularities situated on these curves. We also need to use here the notion of multiplicity divisor of real singularities of a system located on the algebraic solutions of the system.

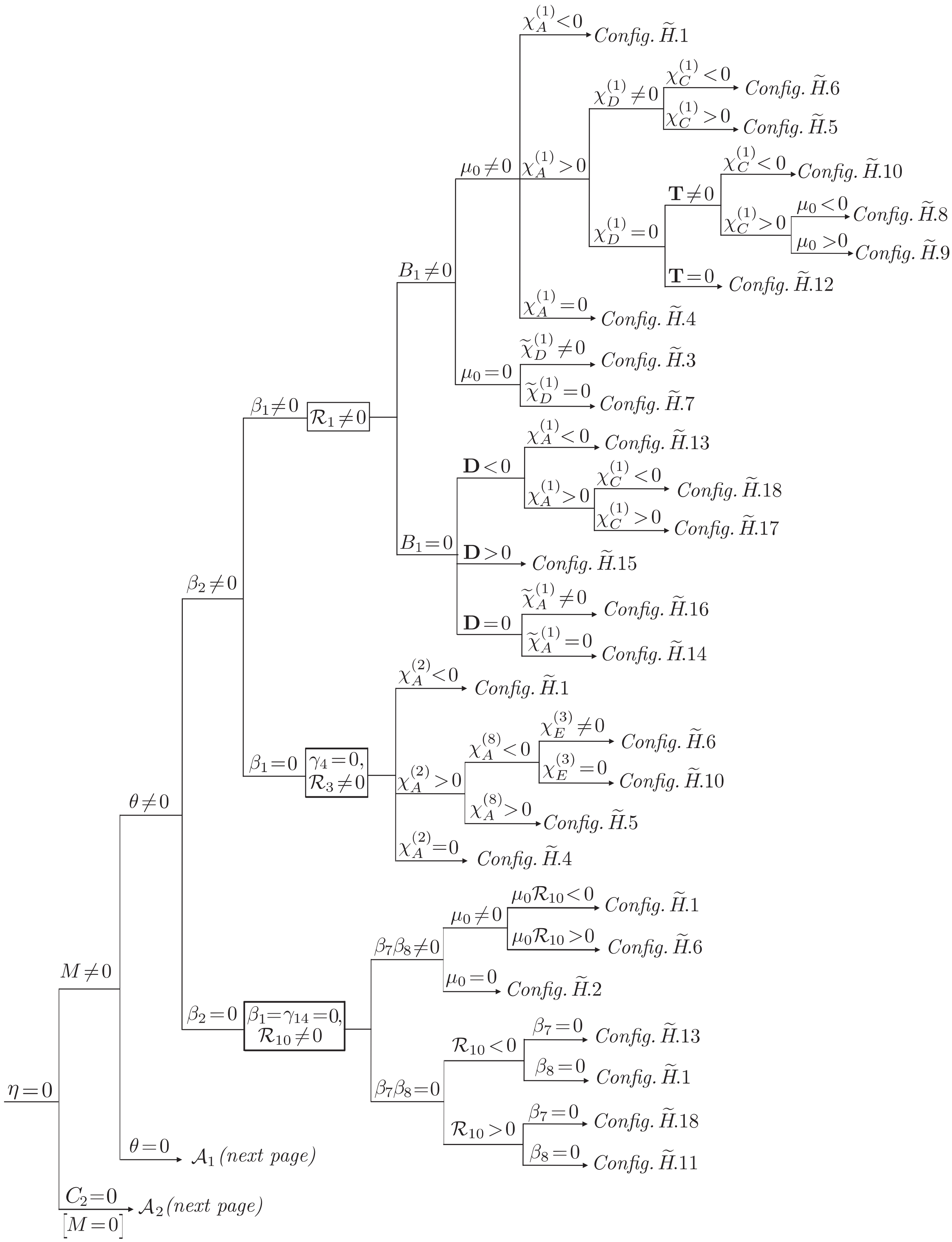

DIAGRAM 3: **Bifurcation diagram in $\mathbb{R}^{12}$ of the configurations: Case $\boldsymbol{\eta=0}$**

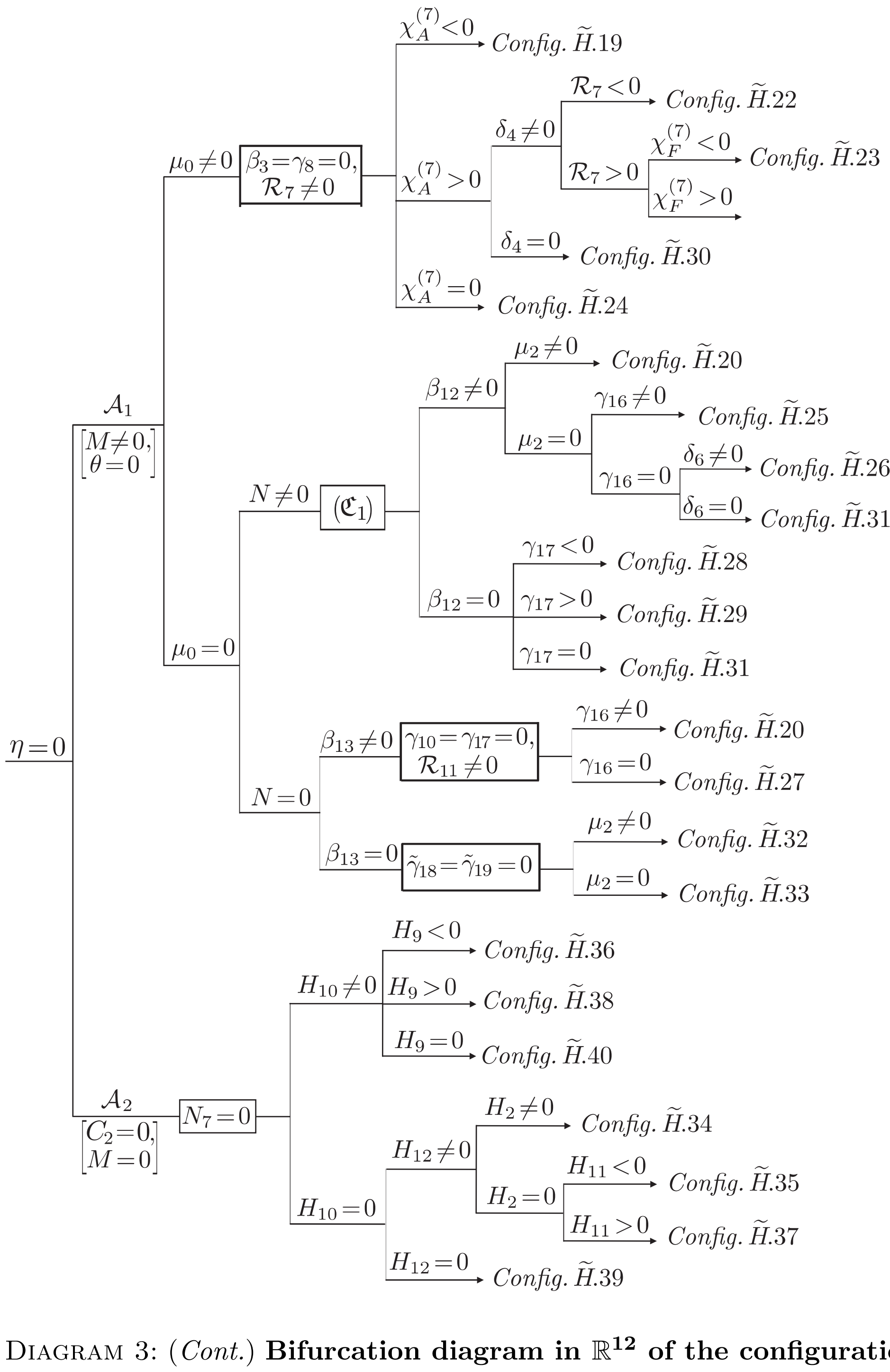

DIAGRAM 3: (*Cont.*) **Bifurcation diagram in $\mathbb{R}^{12}$ of the configurations: Case $\boldsymbol{\eta=0}$**

Definition 9. *1. Suppose a real quadratic system has a finite number of invariant hyperbolas $\mathcal{H}_i : f_i(x,y)=0$ and a finite number of affine invariant lines $\mathcal{L}_i$. We denote the line at infinity $\mathcal{L}_\infty : Z=0$. Lets assume that on the line at infinity we have a finite number of singularities. The divisor of invariant hyperbolas and invariant lines on the complex projective plane of the* *system is the following:*

$$ICD = n_1\mathcal{H}_1 + \ldots + n_k\mathcal{H}_l + m_1\mathcal{L}_1 + \ldots + m_k\mathcal{L}_k + m_\infty\mathcal{L}_\infty,$$

where n_j *(respectively* m_i*) is the multiplicity of the hyperbola* $\mathcal{H}_j$ *(respectively of the line* $\mathcal{L}_i$*), and* m_∞ *is the multiplicity of* $\mathcal{L}_\infty$*. We also mark the complex (non-real) invariant hyperbolas (respectively lines) denoting them by* $\mathcal{H}_i^C$ *(respectively* $\mathcal{L}_i^C$*). We define the total multiplicity* TMH *of invariant hyperbolas as the sum* $\sum_i n_i$ *and the total multiplicity* TML *of invariant line as the sum* $\sum_i m_i$*. We denote by* IHD *(respectively* ILD*) the invariant hyperbolas divisor (respectively the invariant lines divisor) i.e.* $IHD = n_1\mathcal{H}_1 + \ldots + n_k\mathcal{H}_l$ *(respectively* $ILD = m_\infty\mathcal{L}_\infty + m_1\mathcal{L}_1 + \ldots + m_k\mathcal{L}_k$*).*

2. *The zero-cycle on the real projective plane, of real singularities of a system* (3) *located on the configuration of invariant lines and invariant hyperbolas, is given by:*

$$MS_{0C} = l_1U_1 + \ldots + l_kU_k + m_1s_1 + \ldots + m_ns_n,$$

where U_i *(respectively* s_j*) are all the real infinite (respectively finite) such singularities of the system and* l_i *(respectively* m_j*) are their corresponding multiplicities.*

In the family $\mathbf{QSH}_{(\boldsymbol{\eta}=\mathbf{0})}$ we have configurations which have an infinite number of hyperbolas. These are of two kinds: those with a finite number of singular points at infinity, namely two, and those with the line at infinity filled up with singularities. To distinguish these two cases we define $|Sing_\infty|$ to be the cardinality of the set of singular points at infinity. In the first case we have $|Sing_\infty| = 2$ and in the second case $|Sing_\infty|$ is the continuum and we simply write $|Sing_\infty| = \infty$. Since in both cases the systems admit a finite number of affine invariant straight lines we can use them to distinguish the configurations.

Definition 10. 1. *In case we have an infinite number of hyperbolas and just two singular points at infinity but we have a finite number of invariant straight lines we define* $ILD = m_1\mathcal{L}_1 + \ldots + m_k\mathcal{L}_k + m_\infty\mathcal{L}_\infty$ *(see Definition 9);*

2. *In case we have an infinite number of hyperbolas, a finite number of affine lines and the line at infinity is filled up with singularities, we define* $ILD = m_1\mathcal{L}_1 + \ldots + m_k\mathcal{L}_k$*;*

As the Main Theorem indicates, we have nine cases with an infinite number of hyperbolas and since we have a finite number of invariant lines, the systems are classified by their configurations of invariant straight lines encoded in the invariant lines divisor.

Attached to the divisors and the zero-cycle we defined, we have their *types* which are clearly affine invariants. So although the cycles ICD and MS_{0C} are not themselves affine invariants, they are used in the classification because we can read on them several specific invariants, such as for example their types, TMS, TMC, etc.

The above defined divisor ICD and zero-cycle MS_{0C} contain several invariants such as the number of invariant lines and their total multiplicity TML, the number of invariant hyperbolas (in case these are in finite number) and their total multiplicity TMH, the number of complex invariant hyperbolas of a real system, etc.

Given a system in **QSH**, there are two compactifications which intervene in the classification of **QSH** according to the configurations of the systems: the compactification in the Poincaré disk and the compactification of its associated foliation with singularities on the real projective plane $P_2(\mathbb{R})$. We also have the compactification of its associated (complex) foliation with singularities on the complex projective plane. Each one of these compactifications plays a role in the classification. In the compactified system the line at infinity of the affine plane is an invariant line. The system may have singular points located at infinity which are not points of intersection of invariant curves, points also denoted by U_r.

The points at infinity which are intersection point of two or more invariant algebraic curves we denote by $\overset{j}{U}_r$, where $j \in \{h, l, hh, hl, ll, llh^\infty, \ldots\}$. Here h (respectively $l, hh, hl, ll, llh^\infty, \ldots$) means that the intersection of the infinite line with a hyperbola (respectively with a line, or with two hyperbolas, or with a hyperbola and a line, or with two lines, or with two line and infinity number of hyperbolas etc.).

In case we have a real finite singularity located on the invariant curves we denote it by $\overset{j}{s}_r$, where $j \in \{h, l, hh, hl, ll, llh^\infty, \ldots\}$. Here h (respectively $l, hh, hl, ll, llh^\infty, \ldots$) means that the singular point s_r is located on a hyperbola (respectively located on a line, on the intersection of two hyperbolas, on the intersection of a hyperbola and a line, on the intersection of two lines, on the intersection of two line and a infinity number of hyperbolas etc.). In other words, whenever the symbor h^∞ appears in the divisor MS_{0C} it means that the singularity lies on infinity number of hyperbolas.

Suppose the real invariant hyperbolas and lines of a system (S) are given by equations $f_i(x, y) = 0$, $i \in \{1, 2, ..., k\}$, $f_i \in \mathbb{R}[x, y]$. Let us denote by $F_i(X, Y, Z) = 0$ the projection completion of the invariant curves $f_i = 0$ in $P_2(\mathbb{R})$.

Definition 11. *We call* total invariant curve of (S) *in* $P_2(\mathbb{C})$, *the curve* $\mathcal{T}(S) : \prod F_i(X, Y, Z)Z = 0$.

We use the above notion to define the *basic curvilinear polygons determined by the total curve* $\mathcal{T}(S)$. Consider the Poincaré disk and remove from it the (real) points of the total curve $\mathcal{T}(S)$. We are left with a certain number of 2-dimensional connected components.

Definition 12. *We call* basic polygon determined by $\mathcal{T}(S)$ *the closure of anyone of these components associated to* $\mathcal{T}(S)$.

Although a basic polygon is a 2-dimensional object, we shall think of it as being just its border.

Regarding the singular points of the systems situated on $T(S)$, they are of two kinds: those which are simple (or smooth) points of $\mathcal{T}(S)$ and those which are multiple points of $\mathcal{T}(S)$.

Remark 3. *To each singular point of the system we have its associated multiplicity as a singular point of the system. In addition, we also have the multiplicity of these points as points on the total curve. Through a singular point of the systems there may pass several of the curves* $F_i = 0$ *and* $Z = 0$. *Also we may have the case when this point is a singular point of one or even of several of the curves in case we work with invariant curves with singularities. This leads to the multiplicity of the point as point of the curve* $\mathcal{T}(S)$. *The simple points are those of multiplicity one. They are also the smooth points of this curve.*

The real singular points of the system which are simple points of $\mathcal{T}(S)$ are useful for defining some geometrical invariants, helpful in the geometrical classification, besides those which can be read from the zero-cycle defined further above.

We now introduce the notion of *minimal proximity polygon* of a singular point of the total curve. This notion plays a major role in the geometrical classification of the systems.

Definition 13. *Let p be a real singular point of a system lying on $\mathcal{T}(S)$ and in the Poincaré disk. Then p may belong to several basic polygons. We call* minimal proximity polygon of *p a basic polygon on which p is located and which has the minimum number of vertices, among the basic polygons to which p belongs. In case we have more than one polygon with the minimum number of vertices, we take all such polygons as being* minimal proximity polygons of *p.*

Remark 4. *We observe that for systems in $\mathbf{QSH}_{(\eta=0)}$ we have a finite basic polygon only in one case (Config. H.36) and the polygon is a triangle. All other polygons have at least one vertex at infinity.*

For a configuration C, consider for each real singularity p of the system which is a simple point of the curve $\mathcal{T}(S)$, its minimal proximity basic polygons. We construct some formal finite sums attached to the Poincaré disk, analogs of the algebraic-geometric notion of divisor on the projective plane. For this we proceed as follows:

We first list all real singularities of the systems on the Poincaré disk which are simple points (*ss* points) of the total curve. In case we have such points U_i's located on the line at infinity, we start with those points which are at infinity. We obtain a list $U_1, \ldots, U_n, s_1, \ldots, s_k$, where s_i's are finite points. Associate to $U_1, \ldots, U_n$ their minimal proximity polygons $\mathcal{P}_1, \ldots, \mathcal{P}_m$. In case some of them coincide we only list once the polygons which are repeated. These minimal proximity polygons may contain some finite points from the list $s_1, \ldots, s_k$. We remove all such points from this list. Suppose we are left with the finite points $s'_1, \ldots, s'_r$. For these points we associate their corresponding minimal proximity polygons. We observe that for a point s'_j we may have two minimal proximity polygons in which case we consider only the minimal proximity polygon which has the maximum number of singularities s_j, simple points of the total curve. If the two polygons have the same maximum number of simple (*ss*) points then we take the two of them. We obtain a list of polygons and we retain from this list only that polygon (or those polygons) which have the maximal number of *ss* points and add these polygons to the list $\mathcal{P}_1, \ldots, \mathcal{P}_n$. We remove all the *ss* points which appear in this list of polygons from the list of points $s'_1, \ldots, s'_r$ and continue the same process until there are no points left from the sequence $s_1, \ldots, s_k$ which have not being included or eliminated. We thus end up with a list $\mathcal{P}_1, \ldots, \mathcal{P}_r$ of proximity polygons which we denote by $\mathcal{P}(C)$.

Definition 14. *We denote by PD the proximity "divisor" of the Poincaré disk*

$$PD = v_1\mathcal{P}_1 + \cdots + v_r\mathcal{P}_r,$$

over $P_2(\mathbb{R})$, associated to the list $\mathcal{P}(C)$ of the minimal proximity polygons of a configuration, where $\mathcal{P}_i$ are the minimal proximity polygons from this list and v_i are their corresponding number of vertices.

We used the word *divisor* of the Poincaré disk in analogy with divisor on the projective plane, also thinking of polygons as the borders of the 2-dimensional polygons.

Definition 15. *We define a function O (for "order"), O: $\mathbf{QSH} \to \{1, 0, -1\}$ as follows: Suppose a system (S) in $\mathbf{QSH}$ has two singular points at infinity, one simple U_1 and the other double U_2. Suppose the system has only one invariant hyperbola and only two real finite singular points s_1 and s_2 lying on a branch of an invariant hyperbola connecting U_1 with U_2 such that s_2 is double and s_1 is simple. We have only two possibilities: either the segment of hyperbola connecting the two double singularities U_2 and s_2 contains s_1 in which case we write $O(S) = 1$ or it does not contain s_1 and then we write $O(S) = 0$. In case we have a configuration where this specific situation does not occur we write $O(S) = -1$.*

Proof of part (A) of the Main Theorem.

Part (B) of the Main Theorem is proved in Section 3 and here we assume that part (B) occurs.

Summing up all the concepts introduced in order to define the invariants, we end up with the list: ICD, ILD, MS_{0C}, TMH, TML, PD, O and $|Sing_\infty|$. We note that TMH, TML, O are invariants under the group action because the multiplicities of the hyperbolas, or of the lines and of the singularities of the systems are conserved and furthermore by continuity the order of the real singular points on a branch of a hyperbola is also conserved. In general real singularities are also conserved as well as the simple singularities on an algebraic solution. As a consequence the types of the divisor $ICD, PD, ...$ on $P_2(\mathbb{C})$ and of the zero-cycle MS_{0C} on $P_2(\mathbb{R})$ are invariants under the group. The number of vertices of a basic polygon is conserved under the group action basically because the intersection points of the various invariant curves is conserved. The number of ss points on a basic polygon is also conserved. So the coefficients of PD are also conserved. The concepts involved above yield all the invariants we need. To prove that the 40 configurations obtained in Section 3 are distinct we evaluate for each configuration these divisors and zero-cycle, read on them their types and use the additional invariants O and $|Sing_\infty|$ whenever necessary.

More precisely, we start with the $TMH = 1$ and $TML = 1$ and list all the corresponding configurations for this case. We next write the values of the main divisor ICD. In many cases, just using the invariants which we can read on ICD and the zero-cycle MS_{0C} (TMH, TML and the corresponding types), suffices for distinguishing the configurations in a group of configurations. In other cases more invariants are needed and we introduce the necessary additional invariants, to distinguish the configurations of the following groups. Then we continue in a similar way with the other cases starting with $TMH = 1$ and $TML = 2$. Furthermore we consider the case $TMH \geq 2$. Here we have two possibilities: either $|\mathrm{Sing}_\infty|$ is finite or it is infinite. In the first case we list in order of increasing values of the maximum multiplicity occurring in IHD, ILD for the configurations we obtained for this case. We end up with the Diagrams 1 and 2 in which all configurations are distinguished by the system of invariants mentioned in the diagrams or those which could be read on the divisor ICD and on the zero-cycle MS_{0C} such as their types for example.

These calculations and the corresponding stratification is exhibited in the Diagram 1 and Diagram 2 which show that the 40 configurations are distinct yielding the geometric classification of the class $\mathbf{QSH}_{(\boldsymbol{\eta=0})}$ according to the configurations of invariant hyperbolas and lines of the systems. This proves statement *(A)* of the Main Theorem, using its part (B) proved in Section 3 where the configurations are obtained. ∎

A few more definitions and results which play an important role in the proof of the part (B) of the Main Theorem are needed. We do not prove these results here but we indicate where they can be found.

Consider the differential operator $\mathcal{L} = x \cdot \mathbf{L}_2 - y \cdot \mathbf{L}_1$ constructed in [4] and acting on $\mathbb{R}[\tilde{a}, x, y]$, where

$$\begin{aligned} \mathbf{L}_1 = &\ 2a_{00}\tfrac{\partial}{\partial a_{10}} + a_{10}\tfrac{\partial}{\partial a_{20}} + \tfrac{1}{2}a_{01}\tfrac{\partial}{\partial a_{11}} + 2b_{00}\tfrac{\partial}{\partial b_{10}} + b_{10}\tfrac{\partial}{\partial b_{20}} + \tfrac{1}{2}b_{01}\tfrac{\partial}{\partial b_{11}}, \\ \mathbf{L}_2 = &\ 2a_{00}\tfrac{\partial}{\partial a_{01}} + a_{01}\tfrac{\partial}{\partial a_{02}} + \tfrac{1}{2}a_{10}\tfrac{\partial}{\partial a_{11}} + 2b_{00}\tfrac{\partial}{\partial b_{01}} + b_{01}\tfrac{\partial}{\partial b_{02}} + \tfrac{1}{2}b_{10}\tfrac{\partial}{\partial b_{11}}. \end{aligned}$$

Using this operator and the affine invariant $\mu_0 = \operatorname{Res}_x\big(p_2(\tilde{a}, x, y), q_2(\tilde{a}, x, y)\big)/y^4$ we construct the following polynomials

$$\mu_i(\tilde{a}, x, y) = \frac{1}{i!}\mathcal{L}^{(i)}(\mu_0),\ i = 1, .., 4,$$

where $\mathcal{L}^{(i)}(\mu_0) = \mathcal{L}(\mathcal{L}^{(i-1)}(\mu_0))$ and $\mathcal{L}^{(0)}(\mu_0) = \mu_0$.

These polynomials are in fact comitants of systems (3) with respect to the group $GL(2, \mathbb{R})$ (see [4]). Their geometrical meaning is revealed in the next lemma.

Lemma 1. ([3],[4]) *Assume that a quadratic system (S) with coefficients $\tilde{a}$ belongs to the family* (3). *Then:*

(i) Let λ be an integer such that $\lambda \leq 4$. The total multiplicity of all finite singularities of this system equals $4 - \lambda$ if and only if for every $i \in \{0, 1, \ldots, \lambda - 1\}$ we have $\mu_i(\tilde{a}, x, y) = 0$ in the ring $\mathbb{R}[x, y]$ and $\mu_\lambda(\tilde{a}, x, y) \neq 0$. In this case, the factorization $\mu_\lambda(\tilde{a}, x, y) = \prod_{i=1}^{\lambda}(u_i x - v_i y) \neq 0$ over $\mathbb{C}$ indicates the coordinates $[v_i : u_i : 0]$ of those finite singularities of the system (S) which "have gone" to infinity. Moreover, the number of distinct factors in this factorization is less than or equal to three (the maximum number of infinite singularities of a quadratic system in the projective plane) and the multiplicity of each one of the factors $u_i x - v_i y$ gives us the number of the finite singularities of the system (S) which have coalesced with the infinite singular point $[v_i : u_i : 0]$.

(ii) The system (S) is degenerate (i.e. $\gcd(P, Q) \neq \text{const}$) if and only if $\mu_i(\tilde{a}, x, y) = 0$ in $\mathbb{R}[x, y]$ for every $i = 0, 1, 2, 3, 4$.

The following zero-cycle on the complex plane was introduced in [10] based on previous work in [20].

Definition 16. *We define $\mathcal{D}_{\mathbb{C}^2}(S) = \sum_{s \in \mathbb{C}^2} n_s s$ where n_s is the intersection multiplicity at s of the curves $p(x, y) = 0$, $q(x, y) = 0$, p, q being the polynomials defining the equations* (1) *for system (S).*

Proposition 1. ([31]) *The form of the zero-cycle $\mathcal{D}_{\mathbb{C}^2}(S)$ for non-degenerate quadratic systems* (3) *is determined by the corresponding conditions indicated in* TABLE *1, where we write $p + q + r^c + s^c$ if two of the finite points, i.e. r^c, s^c, are complex but not real, and*

$$\begin{aligned} \mathbf{D} &= \Big[3\big((\mu_3, \mu_3)^{(2)},\ \mu_2\big)^{(2)} - \big(6\mu_0\mu_4 - 3\mu_1\mu_3 + \mu_2^2,\ \mu_4\big)^{(4)}\Big]/48, \\ \mathbf{P} &= 12\mu_0\mu_4 - 3\mu_1\mu_3 + \mu_2^2, \\ \mathbf{R} &= 3\mu_1^2 - 8\mu_0\mu_2, \\ \mathbf{S} &= \mathbf{R}^2 - 16\mu_0^2\mathbf{P}, \\ \mathbf{T} &= 18\mu_0^2(3\mu_3^2 - 8\mu_2\mu_4) + 2\mu_0(2\mu_2^3 - 9\mu_1\mu_2\mu_3 + 27\mu_1^2\mu_4) - \mathbf{PR}, \\ \mathbf{U} &= \mu_3^2 - 4\mu_2\mu_4, \\ \mathbf{V} &= \mu_4. \end{aligned} \tag{4}$$

Table 1

No.	Zero–cycle $\mathcal{D}_{\mathbb{C}^2}(S)$	Invariant criteria	No.	Zero–cycle $\mathcal{D}_{\mathbb{C}^2}(S)$	Invariant criteria
1	$p+q+r+s$	$\mu_0 \neq 0, \mathbf{D} < 0, \mathbf{R} > 0, \mathbf{S} > 0$	10	$p+q+r$	$\mu_0 = 0, \mathbf{D} < 0, \mathbf{R} \neq 0$
2	$p+q+r^c+s^c$	$\mu_0 \neq 0, \mathbf{D} > 0$	11	$p+q^c+r^c$	$\mu_0 = 0, \mathbf{D} > 0, \mathbf{R} \neq 0$
3	$p^c+q^c+r^c+s^c$	$\mu_0 \neq 0, \mathbf{D} < 0, \mathbf{R} \leq 0$ $\mu_0 \neq 0, \mathbf{D} < 0, \mathbf{S} \leq 0$	12	$2p+q$	$\mu_0 = \mathbf{D} = 0, \mathbf{PR} \neq 0$
4	$2p+q+r$	$\mu_0 \neq 0, \mathbf{D} = 0, \mathbf{T} < 0$	13	$3p$	$\mu_0 = \mathbf{D} = \mathbf{P} = 0, \mathbf{R} \neq 0$
5	$2p+q^c+r^c$	$\mu_0 \neq 0, \mathbf{D} = 0, \mathbf{T} > 0$	14	$p+q$	$\mu_0 = \mathbf{R} = 0, \mathbf{P} \neq 0, \mathbf{U} > 0$
6	$2p+2q$	$\mu_0 \neq 0, \mathbf{D} = \mathbf{T} = 0, \mathbf{PR} > 0$	15	p^c+q^c	$\mu_0 = \mathbf{R} = 0, \mathbf{P} \neq 0, \mathbf{U} < 0$
7	$2p^c+2q^c$	$\mu_0 \neq 0, \mathbf{D} = \mathbf{T} = 0, \mathbf{PR} < 0$	16	$2p$	$\mu_0 = \mathbf{R} = 0, \mathbf{P} \neq 0, \mathbf{U} = 0$
8	$3p+q$	$\mu_0 \neq 0, \mathbf{D} = \mathbf{T} = 0, \mathbf{P} = 0, \mathbf{R} \neq 0$	17	p	$\mu_0 = \mathbf{R} = \mathbf{P} = 0, \mathbf{U} \neq 0$
9	$4p$	$\mu_0 \neq 0, \mathbf{D} = \mathbf{T} = 0, \mathbf{P} = \mathbf{R} = 0$	18	0	$\mu_0 = \mathbf{R} = \mathbf{P} = 0, \mathbf{U} = 0, \mathbf{V} \neq 0$

The next result is stated in [13] and it gives us the necessary and sufficient conditions for the existence of at least one invariant hyperbola for non-degenerate systems (3) and also their multiplicities. The invariant polynomials which appears in the statement of the next theorem and in the corresponding diagrams are constructed in [13] and we present them further below.

Theorem 2. ([13]) ***(A)*** *The conditions $\gamma_1 = \gamma_2 = 0$ and either $\eta \geq 0$, $M \neq 0$ or $C_2 = 0$ are necessary for a quadratic system in the class* **QS** *to possess at least one invariant hyperbola.*

(B) *Assume that for a system in the class* **QS** *the condition $\gamma_1 = \gamma_2 = 0$ is satisfied.*

- ***(B₁)*** *If $\eta > 0$ then the necessary and sufficient conditions for this system to possess at least one invariant hyperbola are given in* Diagram *4, where we can also find the number and multiplicity of such hyperbolas.*
- ***(B₂)*** *In the case $\eta = 0$ and either $M \neq 0$ or $C_2 = 0$ the corresponding necessary and sufficient conditions for this system to possess at least one invariant hyperbola are given in* Diagram *5, where we can also find the number and multiplicity of such hyperbolas.*

(C) *The* Diagrams *4 and 5 actually contain the global bifurcation diagram in the 12-dimensional space of parameters of the coefficients of the systems belonging to family* **QS***, which possess at least one invariant hyperbola. The corresponding conditions are given in terms of invariant polynomials with respect to the group of affine transformations and time rescaling.*

Remark 5. *An invariant hyperbola is denoted by $\mathcal{H}$ if it is real and by $\overset{c}{\mathcal{H}}$ if it is complex. In the case we have two such hyperbolas then it is necessary to distinguish whether they have parallel or* *non-parallel asymptotes in which case we denote them by* $\mathcal{H}^p$ ($\overset{c}{\mathcal{H}}{}^p$) *if their asymptotes are parallel and by* $\mathcal{H}$ *if there exists at least one pair of non-parallel asymptotes. We denote by* $\mathcal{H}_k$ ($k=2,3$) *a hyperbola with multiplicity* k*; by* $\mathcal{H}_2^p$ *a double hyperbola, which after perturbation splits into two* $\mathcal{H}^p$*; and by* $\mathcal{H}_3^p$ *a triple hyperbola which splits into two* $\mathcal{H}^p$ *and one* $\mathcal{H}$*.*

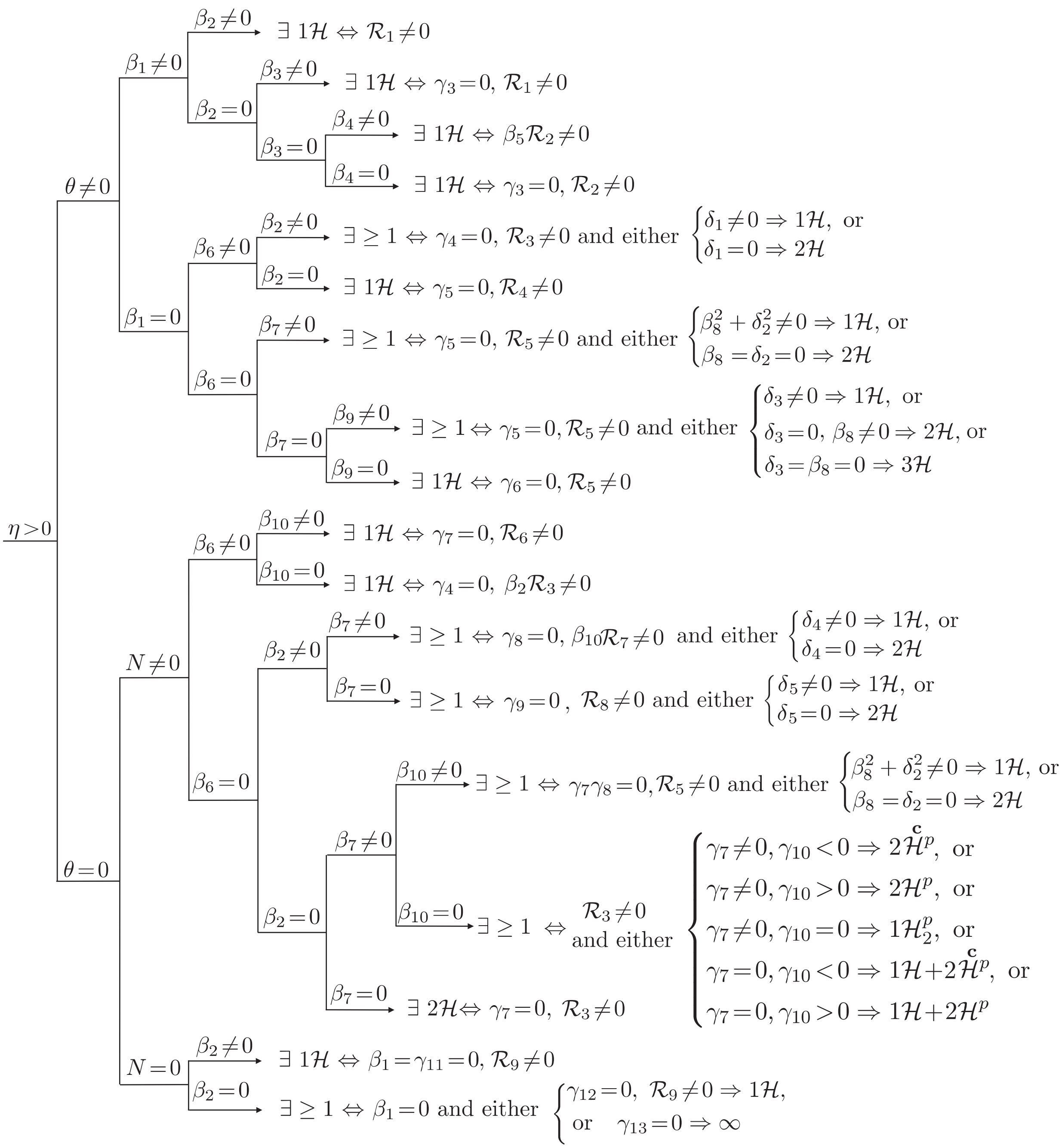

Diagram 4: **Existence of invariant hyperbolas: the case $\eta>0$**

Following [13] we present here the invariant polynomials which according to Diagrams 4 and 5 are responsible for the existence and the number of invariant hyperbolas which systems (3) could

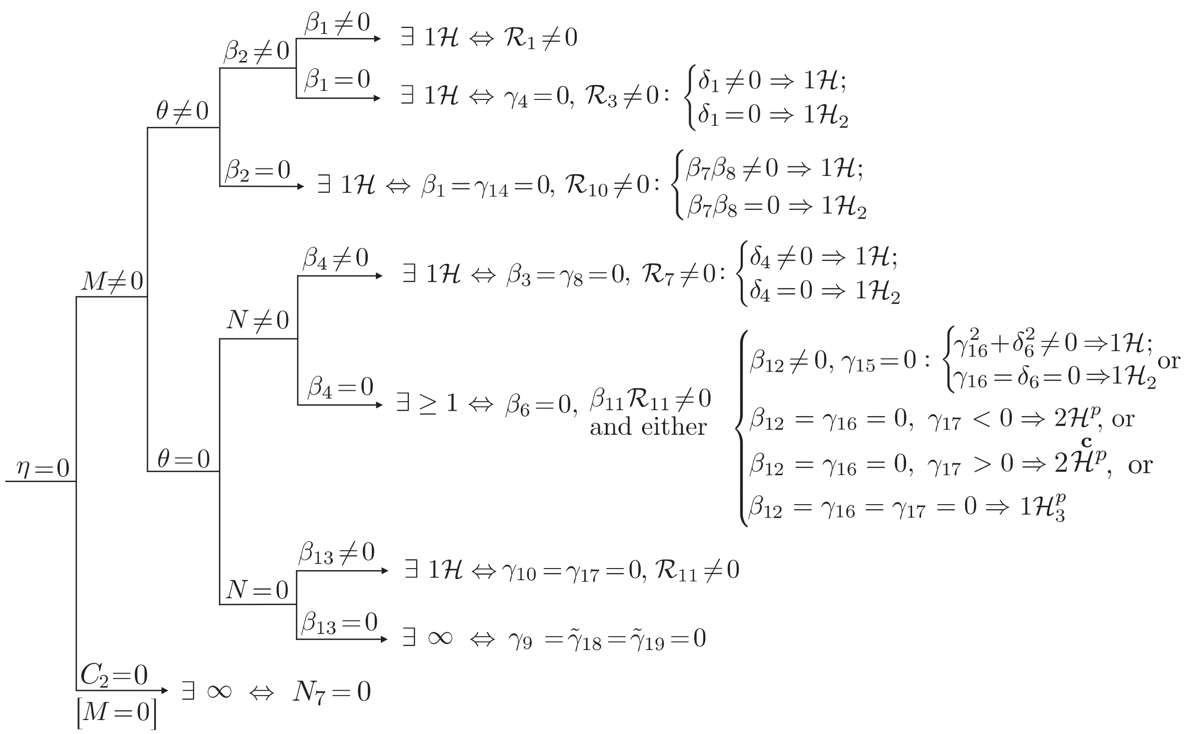

Diagram 5: **Existence of invariant hyperbolas: the case $\boldsymbol{\eta = 0}$**

possess.

First we single out the following five polynomials, basic ingredients in constructing invariant polynomials for systems (3):

$$\begin{aligned} &C_i(\tilde{a}, x, y) = yp_i(x, y) - xq_i(x, y),\ (i = 0, 1, 2)\\ &D_i(\tilde{a}, x, y) = \frac{\partial p_i}{\partial x} + \frac{\partial q_i}{\partial y},\ (i = 1, 2). \end{aligned} \tag{5}$$

As it was shown in [29] these polynomials of degree one in the coefficients of systems (3) are GL–comitants of these systems. Let $f,\ g \in \mathbb{R}[\tilde{a}, x, y]$ and

$$(f, g)^{(k)} = \sum_{h=0}^{k} (-1)^h \binom{k}{h} \frac{\partial^k f}{\partial x^{k-h} \partial y^h}\ \frac{\partial^k g}{\partial x^h \partial y^{k-h}}.$$

The polynomial $(f, g)^{(k)} \in \mathbb{R}[\tilde{a}, x, y]$ is called *the transvectant of index k of (f, g)* (cf. [9], [15])).

Theorem 3 (see [32])**.** *Any GL–comitant of systems* (3) *can be constructed from the elements* (5) *by using the operations: $+$, $-$, $\times$, and by applying the differential operation $(*, *)^{(k)}$.*

Remark 6. *We point out that the elements* (5) *generate the whole set of GL–comitants and hence also the set of affine comitants as well as the set of T-comitants and CT-comitants (see [22] for detailed definitions).*

We construct the following GL–comitants of the second degree with respect to the coefficients of the initial systems

$$\begin{array}{lll} T_1=(C_0,C_1)^{(1)}\,, & T_2=(C_0,C_2)^{(1)}\,, & T_3=(C_0,D_2)^{(1)}\,,\\ T_4=(C_1,C_1)^{(2)}\,, & T_5=(C_1,C_2)^{(1)}\,, & T_6=(C_1,C_2)^{(2)}\,,\\ T_7=(C_1,D_2)^{(1)}\,, & T_8=(C_2,C_2)^{(2)}\,, & T_9=(C_2,D_2)^{(1)}\,. \end{array} \tag{6}$$

Using these GL–comitants as well as the polynomials (5) we construct additional invariant polynomials. In order to be able to directly calculate the values of the invariant polynomials we need, for every canonical system we define here a family of T–comitants expressed through C_i $(i=0,1,2)$ and D_j $(j=1,2)$:

$$\begin{aligned}
\hat{A}=&\big(C_1,T_8-2T_9+D_2^2\big)^{(2)}/144,\\
\widehat{D}=&\Big[2C_0(T_8-8T_9-2D_2^2)+C_1(6T_7-T_6-(C_1,T_5)^{(1)}+6D_1(C_1D_2-T_5)-9D_1^2C_2\Big]/36,\\
\widehat{E}=&\Big[D_1(2T_9-T_8)-3\,(C_1,T_9)^{(1)}-D_2(3T_7+D_1D_2)\Big]/72,\\
\widehat{F}=&\big[6D_1^2(D_2^2-4T_9)+4D_1D_2(T_6+6T_7)+48C_0\,(D_2,T_9)^{(1)}-9D_2^2T_4+288D_1\widehat{E}\\
&-24\left(C_2,\widehat{D}\right)^{(2)}+120\left(D_2,\widehat{D}\right)^{(1)}-36C_1\,(D_2,T_7)^{(1)}+8D_1\,(D_2,T_5)^{(1)}\Big]/144,\\
\widehat{B}=&\big\{16D_1\,(D_2,T_8)^{(1)}\,(3C_1D_1-2C_0D_2+4T_2)+32C_0\,(D_2,T_9)^{(1)}\,(3D_1D_2-5T_6+9T_7)\\
&+2\,(D_2,T_9)^{(1)}\left(27C_1T_4-18C_1D_1^2-32D_1T_2+32\,(C_0,T_5)^{(1)}\right)\\
&+6\,(D_2,T_7)^{(1)}\left[8C_0(T_8-12T_9)-12C_1(D_1D_2+T_7)+D_1(26C_2D_1+32T_5)+C_2(9T_4+96T_3)\right]\\
&+6\,(D_2,T_6)^{(1)}\left[32C_0T_9-C_1(12T_7+52D_1D_2)-32C_2D_1^2\right]+48D_2\,(D_2,T_1)^{(1)}\left(2D_2^2-T_8\right)\\
&-32D_1T_8\,(D_2,T_2)^{(1)}+9D_2^2T_4\,(T_6-2T_7)-16D_1\,(C_2,T_8)^{(1)}\left(D_1^2+4T_3\right)\\
&+12D_1\,(C_1,T_8)^{(2)}\,(C_1D_2-2C_2D_1)+6D_1D_2T_4\left(T_8-7D_2^2-42T_9\right)\\
&+12D_1\,(C_1,T_8)^{(1)}\,(T_7+2D_1D_2)+96D_2^2\Big[D_1\,(C_1,T_6)^{(1)}+D_2\,(C_0,T_6)^{(1)}\Big]-\\
&-16D_1D_2T_3\left(2D_2^2+3T_8\right)-4D_1^3D_2\left(D_2^2+3T_8+6T_9\right)+6D_1^2D_2^2\,(7T_6+2T_7)\\
&-252D_1D_2T_4T_9\big\}/(2^83^3),\\
\widehat{K}=&(T_8+4T_9+4D_2^2)/72,\quad \widehat{H}=(8T_9-T_8+2D_2^2)/72,\quad \widehat{N}=4\widehat{K}-4\widehat{H}.
\end{aligned}$$

These polynomials in addition to (5) and (6) will serve as bricks in constructing affine invariant polynomials for systems (3).

Using the above bricks, the following 42 affine invariants $A_1,\ldots,A_{42}$ are constructed from the minimal polynomial basis of affine invariants up to degree 12. This fact was proved in [5].

$$
\begin{aligned}
A_1 &= \hat{A}, & A_{22} &= \tfrac{1}{1152}\big[C_2,\widehat{D})^{(1)},D_2\big)^{(1)},D_2\big)^{(1)},D_2\big)^{(1)}D_2\big)^{(1)},\\
A_2 &= (C_2,\widehat{D})^{(3)}/12, & A_{23} &= \big[\widehat{F},\widehat{H})^{(1)},\widehat{K}\big)^{(2)}/8,\\
A_3 &= \big[C_2,D_2)^{(1)},D_2\big)^{(1)},D_2\big)^{(1)}/48, & A_{24} &= \big[C_2,\widehat{D})^{(2)},\widehat{K}\big)^{(1)},\widehat{H}\big)^{(2)}/32,\\
A_4 &= (\widehat{H},\widehat{H})^{(2)}, & A_{25} &= \big[\widehat{D},\widehat{D})^{(2)},\widehat{E}\big)^{(2)}/16,\\
A_5 &= (\widehat{H},\widehat{K})^{(2)}/2, & A_{26} &= /36,\\
A_6 &= (\widehat{E},\widehat{H})^{(2)}/2, & A_{27} &= \big[\widehat{B},D_2)^{(1)},\widehat{H}\big)^{(2)}/24,\\
A_7 &= \big[C_2,\widehat{E})^{(2)},D_2\big)^{(1)}/8, & A_{28} &= \big[C_2,\widehat{K})^{(2)},\widehat{D}\big)^{(1)},\widehat{E}\big)^{(2)}/16,\\
A_8 &= \big[\widehat{D},\widehat{H})^{(2)},D_2\big)^{(1)}/8, & A_{29} &= \big[\widehat{D},\widehat{F})^{(1)},\widehat{D}\big)^{(3)}/96,\\
A_9 &= \big[\widehat{D},D_2)^{(1)},D_2\big)^{(1)},D_2\big)^{(1)}/48, & A_{30} &= \big[C_2,\widehat{D})^{(2)},\widehat{D}\big)^{(1)},\widehat{D}\big)^{(3)}/288,\\
A_{10} &= \big[\widehat{D},\widehat{K})^{(2)},D_2\big)^{(1)}/8, & A_{31} &= \big[\widehat{D},\widehat{D})^{(2)},\widehat{K}\big)^{(1)},\widehat{H}\big)^{(2)}/64,\\
A_{11} &= (\widehat{F},\widehat{K})^{(2)}/4, & A_{32} &= \big[\widehat{D},\widehat{D})^{(2)},D_2\big)^{(1)},\widehat{H}\big)^{(1)},D_2\big)^{(1)}/64,\\
A_{12} &= (\widehat{F},\widehat{H})^{(2)}/4, & A_{33} &= \big[\widehat{D},D_2)^{(1)},\widehat{F}\big)^{(1)},D_2\big)^{(1)},D_2\big)^{(1)}/128,\\
A_{13} &= \big[C_2,\widehat{H})^{(1)},\widehat{H}\big)^{(2)},D_2\big)^{(1)}/24, & A_{34} &= \big[\widehat{D},\widehat{D})^{(2)},D_2\big)^{(1)},\widehat{K}\big)^{(1)},D_2\big)^{(1)}/64,\\
A_{14} &= (\widehat{B},C_2)^{(3)}/36, & A_{35} &= \big[\widehat{D},\widehat{D})^{(2)},\widehat{E}\big)^{(1)},D_2\big)^{(1)},D_2\big)^{(1)}/128,
\end{aligned}
$$

$$
\begin{aligned}
A_{15} &= (\widehat{E},\widehat{F})^{(2)}/4, & A_{36} &= \big[\widehat{D},\widehat{E})^{(2)},\widehat{D}\big)^{(1)},\widehat{H}\big)^{(2)}/16,\\
A_{16} &= \big[\widehat{E},D_2)^{(1)},C_2\big)^{(1)},\widehat{K}\big)^{(2)}/16, & A_{37} &= \big[\widehat{D},\widehat{D})^{(2)},\widehat{D}\big)^{(1)},\widehat{D}\big)^{(3)}/576,\\
A_{17} &= \big[\widehat{D},\widehat{D})^{(2)},D_2\big)^{(1)},D_2\big)^{(1)}/64, & A_{38} &= \big[C_2,\widehat{D})^{(2)},\widehat{D}\big)^{(2)},\widehat{D}\big)^{(1)},\widehat{H}\big)^{(2)}/64,\\
A_{18} &= \big[\widehat{D},\widehat{F})^{(2)},D_2\big)^{(1)}/16, & A_{39} &= \big[\widehat{D},\widehat{D})^{(2)},\widehat{F}\big)^{(1)},\widehat{H}\big)^{(2)}/64,\\
A_{19} &= \big[\widehat{D},\widehat{D})^{(2)},\widehat{H}\big)^{(2)}/16, & A_{40} &= \big[\widehat{D},\widehat{D})^{(2)},\widehat{F}\big)^{(1)},\widehat{K}\big)^{(2)}/64,\\
A_{20} &= \big[C_2,\widehat{D})^{(2)},\widehat{F}\big)^{(2)}/16, & A_{41} &= \big[C_2,\widehat{D})^{(2)},\widehat{D}\big)^{(2)},\widehat{F}\big)^{(1)},D_2\big)^{(1)}/64,\\
A_{21} &= \big[\widehat{D},\widehat{D})^{(2)},\widehat{K}\big)^{(2)}/16, & A_{42} &= \big[\widehat{D},\widehat{F})^{(2)},\widehat{F}\big)^{(1)},D_2\big)^{(1)}/16.
\end{aligned}
$$

In the above list, the bracket "[" is used in order to avoid placing the otherwise necessary up to five parentheses "(".

Using the elements of the minimal polynomial basis given above the following affine invariant polynomials were constructed in [14].

$$\gamma_1(\tilde{a}) = A_1^2(3A_6 + 2A_7) - 2A_6(A_8 + A_{12}),$$

$$\gamma_2(\tilde{a}) = 9A_1^2A_2(23252A_3 + 23689A_4) - 1440A_2A_5(3A_{10} + 13A_{11}) - 1280A_{13}(2A_{17} + A_{18} + 23A_{19} - 4A_{20}) - 320A_{24}(50A_8 + 3A_{10} + 45A_{11} - 18A_{12}) + 120A_1A_6(6718A_8 + 4033A_9 + 3542A_{11} + 2786A_{12}) + 30A_1A_{15}(14980A_3 - 2029A_4 - 48266A_5) - 30A_1A_7(76626A_1^2 - 15173A_8 + 11797A_{10} + 16427A_{11} - 30153A_{12}) + 8A_2A_7(75515A_6 - 32954A_7) + 2A_2A_3(33057A_8 - 98759A_{12}) - 60480A_1^2A_{24} + A_2A_4(68605A_8 - 131816A_9 + 131073A_{10} + 129953A_{11}) - 2A_2(141267A_6^2 - 208741A_5A_{12} + 3200A_2A_{13}),$$

$$\gamma_3(\tilde{a}) = 843696A_5A_6A_{10} + A_1(-27(689078A_8 + 419172A_9 - 2907149A_{10} - 2621619A_{11})A_{13} - 26(21057A_3A_{23} + 49005A_4A_{23} - 166774A_3A_{24} + 115641A_4A_{24})).$$

$$\gamma_4(\tilde{a}) = -9A_4^2(14A_{17} + A_{21}) + A_5^2(-560A_{17} - 518A_{18} + 881A_{19} - 28A_{20} + 509A_{21}) - A_4(171A_8^2 + 3A_8(367A_9 - 107A_{10}) + 4(99A_9^2 + 93A_9A_{11} + A_5(-63A_{18} - 69A_{19} + 7A_{20} + 24A_{21}))) + 72A_{23}A_{24},$$

$$\gamma_5(\tilde{a}) = -488A_2^3A_4 + A_2(12(4468A_8^2 + 32A_9^2 - 915A_{10}^2 + 320A_9A_{11} - 3898A_{10}A_{11} - 3331A_{11}^2 + 2A_8(78A_9 + 199A_{10} + 2433A_{11})) + 2A_5(25488A_{18} - 60259A_{19} - 16824A_{21}) + 779A_4A_{21}) + 4(7380A_{10}A_{31} - 24(A_{10} + 41A_{11})A_{33} + A_8(33453A_{31} + 19588A_{32} - 468A_{33} - 19120A_{34}) + 96A_9(-A_{33} + A_{34}) + 556A_4A_{41} - A_5(27773A_{38} + 41538A_{39} - 2304A_{41} + 5544A_{42})),$$

$$\gamma_6(\tilde{a}) = 2A_{20} - 33A_{21},$$

$$\gamma_7(\tilde{a}) = A_1(64A_3 - 541A_4)A_7 + 86A_8A_{13} + 128A_9A_{13} - 54A_{10}A_{13} - 128A_3A_{22} + 256A_5A_{22} + 101A_3A_{24} - 27A_4A_{24},$$

$$\gamma_8(\tilde{a}) = 3063A_4A_9^2 - 42A_7^2(304A_8 + 43(A_9 - 11A_{10})) - 6A_3A_9(159A_8 + 28A_9 + 409A_{10}) + 2100A_2A_9A_{13} + 3150A_2A_7A_{16} + 24A_3^2(34A_{19} - 11A_{20}) + 840A_5^2A_{21} - 932A_2A_3A_{22} + 525A_2A_4A_{22} + 844A_{22}^2 - 630A_{13}A_{33},$$

$$\gamma_9(\tilde{a}) = 2A_8 - 6A_9 + A_{10},$$

$$
\begin{aligned}
\gamma_{10}(\tilde{a}) =& 3A_8 + A_{11},\\
\gamma_{11}(\tilde{a}) =& -5A_7A_8 + A_7A_9 + 10A_3A_{14},\\
\gamma_{12}(\tilde{a}) =& 25A_2^2A_3 + 18A_{12}^2,\\
\gamma_{13}(\tilde{a}) =& A_2,\\
\gamma_{14}(\tilde{a}) =& A_2A_4 + 18A_2A_5 - 236A_{23} + 188A_{24},\\
\gamma_{15}(\tilde{a},x,y) =& 144T_1T_7^2 - T_1^3(T_{12}+2T_{13}) - 4(T_9T_{11} + 4T_7T_{15} + 50T_3T_{23} + 2T_4T_{23} + 2T_3T_{24} + 4T_4T_{24}),\\
\gamma_{16}(\tilde{a},x,y) =& T_{15},\\
\gamma_{17}(\tilde{a},x,y) =& T_{11} + 12T_{13},\\
\tilde{\gamma}_{18}(\tilde{a},x,y) =& C_1(C_2,C_2)^{(2)} - 2C_2(C_1,C_2)^{(2)},\\
\tilde{\gamma}_{19}(\tilde{a},x,y) =& D_1(C_1,C_2)^{(2)} - ((C_2,C_2)^{(2)},C_0)^{(1)},\\
\delta_1(\tilde{a}) =& 9A_8 + 31A_9 + 6A_{10},\\
\delta_2(\tilde{a}) =& 41A_8 + 44A_9 + 32A_{10},\\
\delta_3(\tilde{a}) =& 3A_{19} - 4A_{17},\\
\delta_4(\tilde{a}) =& -5A_2A_3 + 3A_2A_4 + A_{22},\\
\delta_5(\tilde{a}) =& 62A_8 + 102A_9 - 125A_{10},\\
\delta_6(\tilde{a}) =& 2T_3 + 3T_4,\\
\beta_1(\tilde{a}) =& 3A_1^2 - 2A_8 - 2A_{12},\\
\beta_2(\tilde{a}) =& 2A_7 - 9A_6,\\
\beta_3(\tilde{a}) =& A_6,\\
\beta_4(\tilde{a}) =& -5A_4 + 8A_5,\\
\beta_5(\tilde{a}) =& A_4,\\
\beta_6(\tilde{a}) =& A_1,\\
\beta_7(\tilde{a}) =& 8A_3 - 3A_4 - 4A_5,\\
\beta_8(\tilde{a}) =& 24A_3 + 11A_4 + 20A_5,\\
\beta_9(\tilde{a}) =& -8A_3 + 11A_4 + 4A_5,\\
\beta_{10}(\tilde{a}) =& 8A_3 + 27A_4 - 54A_5,\\
\beta_{11}(\tilde{a},x,y) =& T_1^2 - 20T_3 - 8T_4,\\
\beta_{12}(\tilde{a},x,y) =& T_1,\\
\beta_{13}(\tilde{a},x,y) =& T_3,\\
\mathcal{R}_1(\tilde{a}) =& -2A_7(12A_1^2 + A_8 + A_{12}) + 5A_6(A_{10} + A_{11}) - 2A_1(A_{23} - A_{24}) + 2A_5(A_{14} + A_{15})\\
& + A_6(9A_8 + 7A_{12}),\\
\mathcal{R}_2(\tilde{a}) =& A_8 + A_9 - 2A_{10},\\
\mathcal{R}_3(\tilde{a}) =& A_9,\\
\mathcal{R}_4(\tilde{a}) =& -3A_1^2A_{11} + 4A_4A_{19},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{R}_5(\tilde{a},x,y) =&(2C_0(T_8-8T_9-2D_2^2)+C_1(6T_7-T_6)-(C_1,T_5)^{(1)}+6D_1(C_1D_2-T_5)-9D_1^2C_2),\\
\mathcal{R}_6(\tilde{a}) =&-213A_2A_6+A_1(2057A_8-1264A_9+677A_{10}+1107A_{12})+746(A_{27}-A_{28}),\\
\mathcal{R}_7(\tilde{a}) =&-6A_7^2-A_4A_8+2A_3A_9-5A_4A_9+4A_4A_{10}-2A_2A_{13},\\
\mathcal{R}_8(\tilde{a}) =&A_{10},\\
\mathcal{R}_9(\tilde{a}) =&-5A_8+3A_9,\\
\mathcal{R}_{10}(\tilde{a}) =&7A_8+5A_{10}+11A_{11},\\
\mathcal{R}_{11}(\tilde{a},x,y) =&T_{16}.\\
\chi_A^{(1)}(\tilde{a}) =&A_6(A_1A_2-2A_{15})(3A_1^2-2A_8-2A_{12}),\\
\chi_C^{(1)}(\tilde{a}) =&\ \theta\beta_1\beta_3\big[8A_1(42A_{23}-24A_2A_3+59A_2A_5)+A_6(2196A_1^2+384A_9+24A_{10}+360A_{11}\\
&-432A_{12})+4A_7(123A_8-61A_{10}-23A_{11}+123A_{12})+8(2A_4A_{14}-34A_5A_{15}-19A_2A_{16})\big],\\
\widetilde{\chi}_D^{(1)}(\tilde{a}) =&\ -378A_1^2+213A_8+40A_9-187A_{10}-205A_{11}+317A_{12},\\
\chi_A^{(2)}(\tilde{a}) =&\ A_4(5A_8-18A_1^2-A_{10}-3A_{11}+9A12),\\
\chi_E^{(3)}(\tilde{a}) =&\ 54A_1^2A_2+611A_2A_9-104A_2A_{11}-140A_2A_{12}+732A_1A_{14}-243A_{31}-234A_{33}+245A_{34},\\
\chi_A^{(7)}(\tilde{a}) =&\ (A_3-A_4)(A_8-A_{10}),\\
\chi_F^{(7)}(\tilde{a}) =&\ 24A_8-23A_{10},\\
\chi_A^{(8)}(\tilde{a}) =&\ 5A_8-A_9,
\end{aligned}
$$

We also need here the following additional affine invariant polynomials, constructed in [26]:

$$
\begin{aligned}
&H_2 =-\big[(C_1,8\widehat{H}+\widehat{N})^{(1)}+2D_1\widehat{N}\big],\quad H_9=-\big[(\widehat{D},\widehat{D})^{(2)},\widehat{D})^{(1)},\widehat{D})^{(3)}\equiv 12\mathbf{D},\\
&H_{10} =-\big[\widehat{D},\widehat{N})^{(2)},D_2)^{(1)},\quad H_{11}=3\big[(C_1,8\widehat{H}+\widehat{N})^{(1)}+2D_1\widehat{N}\big]^2-32\widehat{H}\big[(C_2,\widehat{D})^{(2)}+(\widehat{D},D_2)^{(1)}\big],\\
&H_{12} =(\widehat{D},\widehat{D})^{(2)},\quad N_7=12D_1(C_0,D_2)^{(1)}+2D_1^3+9D_1(C_1,C_2)^{(2)}+36\big[C_0,C_1)^{(1)},D_2)^{(1)},
\end{aligned}
$$

Next we construct the following T-comitants (for the definition of T-comitants see [23]) which are responsible for the existence of invariant straight lines of systems (3):

Notation 1.

$$
\begin{aligned}
B_3(a,x,y) &= (C_2,D)^{(1)} = Jacob\,(C_2,D)\,,\\
B_2(a,x,y) &= (B_3,B_3)^{(2)} - 6B_3(C_2,D)^{(3)},\\
B_1(a) &= \operatorname{Res}_x\,(C_2,D)\,/y^9 = -2^{-9}3^{-8}\,(B_2,B_3)^{(4)}\,.
\end{aligned}
\tag{7}
$$

Lemma 2 (see [22])**.** *For the existence of invariant straight lines in one (respectively 2; 3 distinct) directions in the affine plane it is necessary that $B_1=0$ (respectively $B_2=0$; $B_3=0$).*

At the moment we only have necessary and not necessary and sufficient conditions for the existence of an invariant straight line or for invariant lines in two or three directions.

Let us apply a translation $x=x'+x_0$, $y=y'+y_0$ to the polynomials $p(\tilde{a},x,y)$ and $q(\tilde{a},x,y)$. We obtain $\hat{p}(\hat{a}(a,x_0,y_0),x',y')=p(\tilde{a},x'+x_0,y'+y_0)$, $\hat{q}(\hat{a}(a,x_0,y_0),x',y')=q(\tilde{a},x'+x_0,y'+y_0)$. Let us construct the following polynomials

$$\Gamma_i(\tilde{a}, x_0, y_0) \equiv \mathrm{Res}_{\,x'}\Big(C_i\big(\hat{a}(\tilde{a}, x_0, y_0), x', y'\big), C_0\big(\hat{a}(\tilde{a}, x_0, y_0), x', y'\big)\Big)/(y')^{i+1},$$
$$\Gamma_i(\tilde{a}, x_0, y_0) \in \mathbb{R}[\tilde{a}, x_0, y_0],\ \ i = 1, 2.$$

Notation 2. *We denote by*

$$\tilde{\mathcal{E}}_i(\tilde{a}, x, y) = \Gamma_i(\tilde{a}, x_0, y_0)|_{\{x_0=x,\ y_0=y\}} \in \mathbb{R}[\tilde{a}, x, y] \quad (i = 1, 2).$$

Observation 1. *We note that the polynomials $\tilde{\mathcal{E}}_1(a, x, y)$ and $\tilde{\mathcal{E}}_2(a, x, y)$ are affine comitants of systems* (3) *and are homogeneous polynomials in the coefficients $a, b, c, d, e, f, g, h, k, l, m, n$ and non-homogeneous in x, y and $deg_{\tilde{a}}\tilde{\mathcal{E}}_1 = 3$, $deg_{(x,y)}\tilde{\mathcal{E}}_1 = 5$, $deg_{\tilde{a}}\tilde{\mathcal{E}}_2 = 4$, $deg_{(x,y)}\tilde{\mathcal{E}}_2 = 6$.*

Notation 3. *Let $\mathcal{E}_i(\tilde{a}, X, Y, Z)$, $i = 1, 2$, be the homogenization of $\tilde{\mathcal{E}}_i(\tilde{a}, x, y)$, i.e.*

$$\mathcal{E}_1(\tilde{a}, X, Y, Z) = Z^5\tilde{\mathcal{E}}_1(\tilde{a}, X/Z, Y/Z), \qquad \mathcal{E}_2(\tilde{a}, X, Y, Z) = Z^6\tilde{\mathcal{E}}_1(\tilde{a}, X/Z, Y/Z)$$

The geometrical meaning of these affine comitants is given by the following lemma (see [22]):

Lemma 3 (see [22])**.** *1) The straight line $\mathcal{L}(x, y) \equiv ux + vy + w = 0$, $u, v, w \in \mathbb{C}$, $(u, v) \neq (0, 0)$ is an invariant line for a quadratic system* (3) *if and only if the polynomial $\mathcal{L}(x, y)$ is a common factor of the polynomials $\tilde{\mathcal{E}}_1(\tilde{a}, x, y)$ and $\tilde{\mathcal{E}}_2(\tilde{a}, x, y)$ over $\mathbb{C}$, i.e.*

$$\tilde{\mathcal{E}}_i(\tilde{a}, x, y) = (ux + vy + w)\widetilde{W}_i(x, y),\ \ i = 1, 2,$$

where $\widetilde{W}_i(x, y) \in \mathbb{C}[x, y]$.

2) If $\mathcal{L}(x, y) = 0$ is an invariant straight line of multiplicity λ for a quadratic system (3)*, then $[\mathcal{L}(x, y)]^\lambda \mid \gcd(\tilde{\mathcal{E}}_1, \tilde{\mathcal{E}}_2)$ in $\mathbb{C}[x, y]$, i.e. there exist $W_i(\tilde{a}, x, y) \in \mathbb{C}[x, y]$, $i = 1, 2$, such that*

$$\tilde{\mathcal{E}}_i(\tilde{a}, x, y) = (ux + vy + w)^\lambda W_i(\boldsymbol{a}, x, y),\ \ i = 1, 2.$$

3) If the line $l_\infty : Z = 0$ is of multiplicity $\lambda > 1$, then $Z^{\lambda-1} \mid \gcd(\mathcal{E}_1, \mathcal{E}_2)$.

In order to detect the parallel invariant lines we need the following invariant polynomials:

$$N(\tilde{a}, x, y) = D_2^2 + T_8 - 2T_9 = 9\widehat{N}, \quad \theta(\tilde{a}) = 2A_5 - A_4\ (\equiv \mathrm{Discriminant}\big(N(a, x, y)\big)/1296).$$

Lemma 4 (see [22])**.** *A necessary condition for the existence of one couple (respectively two couples) of parallel invariant straight lines of a system* (3) *corresponding to $\tilde{a} \in \mathbb{R}^{12}$ is the condition $\theta(\tilde{a}) = 0$ (respectively $N(\tilde{a}, x, y) = 0$).*

Now we introduce some important GL–comitant in the study of the invariant conics. Considering $C_2(\tilde{a}, x, y) = yp_2(\tilde{a}, x, y) - xq_2(\tilde{a}, x, y)$ as a cubic binary form of x and y we calculate

$$\eta(\tilde{a}) = \mathrm{Discrim}[C_2, \xi], \quad M(\tilde{a}, x, y) = \mathrm{Hessian}[C_2],$$

where $\xi = y/x$ or $\xi = x/y$. According to [28] we have the next result.

Lemma 5 ([28]). *The number of infinite singularities (real and imaginary) of a quadratic system in* **QS** *is determined by the following conditions:*

(*i*) 3 *real if* $\eta > 0$;

(*ii*) 1 *real and* 2 *imaginary if* $\eta < 0$;

(*iii*) 2 *real if* $\eta = 0$ *and* $M \neq 0$;

(*iv*) 1 *real if* $\eta = M = 0$ *and* $C_2 \neq 0$;

(*v*) ∞ *if* $\eta = M = C_2 = 0$.

Moreover, for each one of these cases the quadratic systems (3) *can be brought via a linear transformation to one of the following canonical systems* $(\mathbf{S}_I) - (\mathbf{S}_V)$*:*

$$\begin{cases} \dot{x} &= a + cx + dy + gx^2 + (h-1)xy, \\ \dot{y} &= b + ex + fy + (g-1)xy + hy^2; \end{cases} \qquad (\mathbf{S}_I)$$

$$\begin{cases} \dot{x} &= a + cx + dy + gx^2 + (h+1)xy, \\ \dot{y} &= b + ex + fy - x^2 + gxy + hy^2; \end{cases} \qquad (\mathbf{S}_{II})$$

$$\begin{cases} \dot{x} &= a + cx + dy + gx^2 + hxy, \\ \dot{y} &= b + ex + fy + (g-1)xy + hy^2; \end{cases} \qquad (\mathbf{S}_{III})$$

$$\begin{cases} \dot{x} &= a + cx + dy + gx^2 + hxy, \\ \dot{y} &= b + ex + fy - x^2 + gxy + hy^2; \end{cases} \qquad (\mathbf{S}_{IV})$$

$$\begin{cases} \dot{x} &= a + cx + dy + x^2, \\ \dot{y} &= b + ex + fy + xy. \end{cases} \qquad (\mathbf{S}_V)$$

Finally, in order to detect if an invariant conic

$$\Phi(x,y) \equiv p + qx + ry + sx^2 + 2txy + uy^2 = 0 \qquad (8)$$

(or an invariant line) of a system (3) has the multiplicity greater than one, we use the notion of *k-th extactic curve* $\mathscr{E}_k(X)$ of the vector field X (see (2)), associated to systems (3). This curve is defined in the paper [7, Definition 5.1] as follows:

$$\mathscr{E}_k(X) = \det \begin{pmatrix} v_1 & v_2 & \dots & v_l \\ X(v_1) & X(v_2) & \dots & X(v_l) \\ \vdots & \vdots & \dots & \vdots \\ X^{l-1}(v_1) & X^{l-1}(v_2) & \dots & X^{l-1}(v_l) \end{pmatrix},$$

where $v_1, v_2, \dots, v_l$ is the basis of the $\mathbb{C}$-vector space $\mathbb{C}_n[x,y]$ which is the set of all polynomials in x, y of degree n , of polynomials in $\mathbb{C}_n[x,y]$ and $l = (k+1)(k+2)/2$. Here $X^0(v_i) = v_i$ and $X^j(v_1) = X(X^{j-1}(v_1))$.

According to [7] the following statement holds:

Lemma 6. *Assume that an algebraic curve* $\Phi(x,y) = 0$ *of degree* k *is an invariant curve for systems* (3)*. Then this curve has multiplicity* m *if and only if* $\Phi(x,y)^m$ *divides* $\mathscr{E}_k(X)$*.*

3 Proof of statement *(B)* of Main Theorem

In this section we provide the proof of statement *(B)* of our Main Theorem, following the conditions given by DIAGRAM 5 (the case $\eta = 0$).

So in what follows we assume $\eta = 0$ and we consider two possibilities: $M(\tilde{a}, x, y) \neq 0$ (i.e. at infinity we have two distinct real singularities) and $M = 0 = C_2$ (when we have an infinite number of singularities at infinity).

3.1 The possibility $M(\tilde{a}, x, y) \neq 0$

According to Lemma 5 there exists a linear transformation and time rescaling which brings systems (3) to the systems

$$\frac{dx}{dt} = a + cx + dy + gx^2 + hxy, \quad \frac{dy}{dt} = b + ex + fy + (g-1)xy + hy^2. \tag{9}$$

For this systems we calculate

$$C_2(x, y) = x^2y, \quad \theta = -h^2(g-1)/2. \tag{10}$$

3.1.1 The case $\theta \neq 0$

In this case $h(g-1) \neq 0$ and due to a translation we may assume $d = e = 0$. So in what follows we consider the family of systems

$$\begin{aligned} \frac{dx}{dt} &= a + cx + gx^2 + hxy, \\ \frac{dy}{dt} &= b + fy + (g-1)xy + hy^2 \end{aligned} \tag{11}$$

for which calculations yield:

$$\gamma_1 = (2c - f)(c + f)^2h^4(g-1)^2/32, \quad \beta_2 = h^2(2c - f)/2.$$

According to Theorem 2 for the existence of an invariant hyperbola of the above systems the condition $\gamma_1 = 0$ is necessary. So we consider two subcases: $\beta_2 \neq 0$ and $\beta_2 = 0$.

3.1.1.1 The subcase $\beta_2 \neq 0$ Then $2c - f \neq 0$ and the condition $\gamma_1 = 0$ implies $f = -c$. Then we calculate

$$\gamma_2 = -14175c^2h^5(g-1)^2(3g-1)[a(2g-1) - 2bh], \quad \beta_1 = -3c^2h^2(g-1)(3g-1)/4$$

and following Diagram 5 (see Theorem 2) we examine two possibilities: $\beta_1 \neq 0$ and $\beta_1 = 0$.

3.1.1.1.1 The possibility $\beta_1 \neq 0$. Then the necessary condition $\gamma_2 = 0$ (for the existence of a hyperbola) gives $a(2g-1) - 2bh = 0$ and setting $a = 2a_1h$ (since $h \neq 0$) we get $b = a_1(2g-1)$. Therefore keeping the old parameter a (instead of a_1) we arrive at the following family of systems

$$\frac{dx}{dt} = 2ah + cx + gx^2 + hxy, \quad \frac{dy}{dt} = a(2g-1) - cy + (g-1)xy + hy^2.$$

We observe that since $ch \neq 0$, we may assume $c = h = 1$ due to the rescaling $(x, y, t) \mapsto (cx, cy/h, t/c)$ and the additional parametrization $ah/c^2 \to a$. So we get the following 2-parameter family of systems

$$\frac{dx}{dt} = 2a + x + gx^2 + xy, \quad \frac{dy}{dt} = a(2g - 1) - y + (g - 1)xy + y^2, \tag{12}$$

which possess the following invariant hyperbola (with cofactor $(2g - 1)x + 2y$):

$$\Phi(x, y) = a + xy = 0 \tag{13}$$

and for which the following coefficient conditions (defined by $\theta\beta_2\beta_1\mathcal{R}_1 \neq 0$) must be satisfied:

$$a(g - 1)(3g - 1) \neq 0. \tag{14}$$

For systems (12) we calculate

$$B_1 = 4a^3(g - 1)^2(1 - 2g). \tag{15}$$

1) *The case $B_1 \neq 0$.* In this case by Lemma 2 we have no invariant lines. For systems (12) we calculate $\mu_0 = g$ and we consider two subcases: $\mu_0 \neq 0$ and $\mu_0 = 0$.

***a*)** *The subcase $\mu_0 \neq 0$.* Then by Lemma 1 the systems have finite singularities of total multiplicity four. More exactly, systems (12) possess the singular points $M_{1,2}\big(x_{1,2}, y_{1,2}\big)$ and $M_{3,4}\big(x_{3,4}, y_{3,4}\big)$, where

$$x_{1,2} = \frac{-1 \pm \sqrt{1 - 4ag}}{2g}, \quad y_{1,2} = \frac{1 \pm \sqrt{1 - 4ag}}{2},$$
$$x_{3,4} = -1 \pm \sqrt{1 - 4a}, \quad y_{3,4} = (2g - 1)(1 \mp \sqrt{1 - 4a}\,)/2.$$

We detect that the singularities $M_{1,2}\big(x_{1,2}, y_{1,2}\big)$ are located on the hyperbola. On the other hand for systems (12) we calculate the invariant polynomials

$$\chi_A^{(1)} = 9(g - 1)^2(3g - 1)^2(1 - 4ag)/64$$

and by (14) we conclude that $\operatorname{sign}(\chi_A^{(1)}) = \operatorname{sign}(1 - 4ag)$ (if $1 - 4ag \neq 0$) and we consider three possibilities: $\chi_A^{(1)} < 0$, $\chi_A^{(1)} > 0$ and $\chi_A^{(1)} = 0$.

***a*$_1$)** *The possibility $\chi_A^{(1)} < 0$.* So we have no real singularities located on the invariant hyperbola and we arrive at the configurations of invariant curves given by *Config.* $\widetilde{H}$.1.

***a*$_2$)** *The possibility $\chi_A^{(1)} > 0$.* In this case the singularities $M_{1,2}\big(x_{1,2}, y_{1,2}\big)$ located on the hyperbola are real and we have the next result.

Lemma 7. *Assume that the singularities $M_{1,2}\big(x_{1,2}, y_{1,2}\big)$ (located on the hyperbola) are finite. Then these singularities are located on different branches of the hyperbola if $\chi_C^{(1)} < 0$ and they are located on the same branch if $\chi_C^{(1)} > 0$, where $\chi_C^{(1)} = 315ag(g - 1)^4(3g - 1)^2/32$.*

Proof: Since the asymptotes of the hyperbola (13) are the lines $x = 0$ and $y = 0$ it is clear that the singularities $M_{1,2}$ are located on different branches of the hyperbola if and only if $x_1x_2 < 0$. We calculate

$$x_1x_2 = \left[\frac{-1 + \sqrt{1 - 4ag}}{2g}\right]\left[\frac{-1 - \sqrt{1 - 4ag}}{2g}\right] = \frac{a}{g} \tag{16}$$

and due to the condition (14) we obtain that $\text{sign}\,(x_1x_2) = \text{sign}\,(\chi_C^{(1)})$. This completes the proof of the lemma. ∎

Other two singular points $M_{3,4}\big(x_{3,4}, y_{3,4}\big)$ of systems (12) are generically located outside the hyperbola. We need to determine the conditions when some singular points of the system become singular points lying on the hyperbola. Considering (13) we calculate

$$\Phi(x,y)|_{\{x=x_{3,4},\ y=y_{3,4}\}} = (2g-1)\big(-1\pm\sqrt{1-4a}\big) + a(4g-1) \equiv \Omega_\pm(a,g).$$

Put $\Omega_3(a,g) = \Omega_+(a,g)$ and $\Omega_4(a,g) = \Omega_-(a,g)$. It is clear that at least one of the singular points $M_3(x_3,y_3)$ or $M_4(x_4,y_4)$ belongs to the hyperbola (13) if and only if

$$\Omega_3\Omega_4 = a\big[2(1-2g) + a(1-4g)^2\big] \equiv aZ_1 = 0.$$

On the other hand for systems (12) we have $\widetilde{\chi}_D^{(1)} = 54Z_1$ and clearly due to (14) the condition $\widetilde{\chi}_D^{(1)} = 0$ is equivalent to $Z_1 = 0$. We examine two cases: $\widetilde{\chi}_D^{(1)} \neq 0$ and $\widetilde{\chi}_D^{(1)} = 0$.

α) *The case* $\widetilde{\chi}_D^{(1)} \neq 0$. Then $Z_1 \neq 0$ and on the hyperbola there are two simple real singularities (namely $M_{1,2}(x_{1,2}, y_{1,2})$). By Lemma 7 their position is defined by the invariant polynomial $\chi_C^{(1)}$ and we arrive at the configuration given by *Config.* $\widetilde{H}.2$ if $\chi_C^{(1)} < 0$ and by *Config.* $\widetilde{H}.3$ if $\chi_C^{(1)} > 0$.

β) *The case* $\widetilde{\chi}_D^{(1)} = 0$. In this case the condition $Z_1 = 0$ implies $4g-1 \neq 0$ (otherwise for $g = 1/4$ we get $Z_1 = 1 \neq 0$. So we obtain $a = 2(2g-1)/(4g-1)^2$. In this case the coordinates of the finite singularities $M_i(x_i, y_i)$ (i=1,2,3,4) are as follows

$$x_1 = \frac{1-2g}{g(4g-1)},\quad y_1 = \frac{2g}{4g-1};\qquad x_2 = x_3 = \frac{2}{1-4g},\quad y_2 = y_3 = \frac{2g-1}{4g-1};$$
$$x_4 = \frac{4(1-2g)}{(4g-1},\quad y_4 = \frac{2(g-1)^2}{4g-1},$$

i.e. all the singularities are real. Then considering Proposition 1 we calculate

$$\mathbf{D} = 0,\quad \mathbf{T} = -3\big[2g(g-1)x + (2g-1)y\big]^2\mathbf{P},$$
$$\mathbf{P} = \frac{(4g-3)^2(gx-y)^2(2gx-x+2y)^2}{(4g-1)^4}.$$

β_1) *The subcase* $\mathbf{T} \neq 0$. Then $\mathbf{T} < 0$ and according to Proposition 1 systems (12) possess one double and two simple real finite singularities. As it is mentioned above, the singular point $M_3(x_3,y_3)$ coalesces with the singular point $M_2(x_2,y_2)$ located on the hyperbola, whereas $M_4(x_4,y_4)$ remains outside the hyperbola.

Considering the coordinates of the singular points we calculate

$$\text{sign}\,(x_1x_2) = \text{sign}\,(g(2g-1)),\quad \chi_C^{(1)} = \frac{315g(2g-1)(g-1)^4(3g-1)^2}{16(4g-1)^2}.$$

Therefore in the case $\chi_C^{(1)} < 0$ the singular points M_1 and $M_2 = M_3$ are located on different branches of the hyperbola and we arrive at the configuration *Config.* $\widetilde{H}.4$.

Assume now that the condition $\chi_C^{(1)} > 0$ holds, i.e. the two singular points (one double and one simple) are located on the same branch of the hyperbola. Since on this branch are also located two infinite singular points (one double and one simple), it is clear that the reciprocal position of singular points M_1 and M_2 (double) on the branch leads do different configurations. So we need to determine the conditions to distinguish these two situations.

We calculate

$$x_1 - x_2 = \frac{1-2g}{g(4g-1)} - \frac{2}{1-4g} = \frac{1}{g(4g-1)}$$

and hence the reciprocal position of M_1 and M_2 depends on the sign of the expression $g(4g-1)$. On the other hand, the condition $\chi_C^{(1)} > 0$ implies $g(2g-1) > 0$, i.e. we have either $g < 0$ or $g > 1/2$. Since $\mu_0 = g$ we deduce that these two possibilities are governs by the invariant polynomial μ_0.

It is easy to detect that we arrive at *Config.* $\widetilde{H}$.5 if $\mu_0 < 0$ (i.e. $g < 0$) and we get *Config.* $\widetilde{H}$.6 if $\mu_0 > 0$ (i.e. $g > 1/2$).

***β*₂)** *The subcase* $\mathbf{T} = 0$. In this case due to the condition $B_1 \neq 0$ (i.e. $2g - 1 \neq 0$) the equality $\mathbf{T} = 0$ holds if and only if $\mathbf{P} = 0$ which is equivalent to $4g - 3 = 0$, i.e. $g = 3/4$. In this case we obtain

$$\mathbf{D} = \mathbf{T} = \mathbf{P} = 0, \quad \mathbf{R} = 3(3x - 4y)^2/64$$

and since $\mathbf{R} \neq 0$, by Proposition 1 we obtain one triple and one simple singularities. More precisely the singular points M_2, M_3 and M_4 coalesce and since all the parameters of systems (12) are fixed we get the unique configuration given by *Config.* $\widetilde{H}$.7.

***a*₃)** *The possibility* $\chi_A^{(1)} = 0$. In this case we get $g = 1/(4a)$ and the singularities $M_{1,2}(x_{1,2}, y_{1,2})$ located on the hyperbola coincide. On the other hand we have $Z_1 = a \neq 0$ and hence none of the singular points $M_{3,4}$ could belong to the hyperbola. So we arrive at the unique configuration presented by *Config.* $\widetilde{H}$.8.

***b*)** *The subcase* $\mu_0 = 0$. Then we have $\mu_1 = -y$ and by Lemma 1 one finite singular point has gone to infinity and coalesced with the infinite singular point $[1, 0, 0]$. In this case we arrive at the 1-parameter family of systems

$$\frac{dx}{dt} = 2a + x + xy, \quad \frac{dy}{dt} = -a - y - xy + y^2 \tag{17}$$

possessing the singular points $M_1'(x_1', y_1')$ and $M_{2,3}(x_{2,3}, y_{2,3})$ (the same points for the particular case $g = 0$) with the coordinates

$$x_1' = -a, \quad y_1' = 1; \qquad x_{3,4} = -1 \pm \sqrt{1-4a}, \quad y_{3,4} = (-1 \pm \sqrt{1-4a}\,)/2.$$

We observe that only the singular point M_1' is located on the hyperbola. On the other hand it was shown earlier that one of the points $M_{2,3}(x_{2,3}, y_{2,3})$ belongs to the hyperbola if and only if $Z_1 = 0$ which in this case gets the value $Z_1 = a + 2$. For systems (17) we calculate

$$\widetilde{\chi}_D^{(1)} = 54(a + 2)$$

and it is not too difficult to detect that in the case $\widetilde{\chi}_D^{(1)} \neq 0$ (i.e. $a + 2 \neq 0$) we arrive at the unique configuration given by *Config.* $\widetilde{H}$.9.

Assume now $\widetilde{\chi}_D^{(1)} = 0$. Then $a = -2$ and we get a system with constant coefficients for which the singular point M_2 has coalesced with M_1'. As a result we obtain *Config.* $\widetilde{H}$.10.

2) *The case* $B_1 = 0$. Considering (15) and the condition (14) this implies $g = 1/2$ and we obtain the following 1-parameter family of systems

$$\frac{dx}{dt} = 2a + x + x^2/2 + xy, \quad \frac{dy}{dt} = -y(1 + x/2 - y). \tag{18}$$

These systems besides the hyperbola (13) possess the invariant line $y = 0$ and four singular points $M_i(x_i, y_i)$ with the coordinates

$$x_{1,2} = -1 \pm \sqrt{1-2a}, \quad y_{1,2} = \frac{1 \pm \sqrt{1-2a}}{2},$$
$$x_{3,4} = -1 \pm \sqrt{1-4a}, \quad y_{3,4} = 0.$$

We observe that the singular point M_1 and M_2 are located on the hyperbola, whereas M_3 and M_4 are situated on the invariant line $y = 0$, which is one of the asymptotes of the hyperbola (13). For the above systems we calculate

$$\mathbf{D} = 48a^2(1-2a)(4a-1), \quad \chi_A^{(1)} = 9(1-2a)/1024$$

and it is clear that due to the condition (14) (i.e. $a \neq 0$) two of the finite singular point could coalesce if and only if $\mathbf{D} = 0$. So we examine three subcases: $\mathbf{D} < 0$, $\mathbf{D} > 0$ and $\mathbf{D} = 0$.

a) *The subcase* $\mathbf{D} < 0$. Then $(1-2a)(4a-1) < 0$ and we observe that if $\chi_A^{(1)} < 0$ (i.e. $a > 1/2$) all the singular points are complex and we get the unique configuration given by *Config.* $\widetilde{H}$.11.

Assume now $\chi_A^{(1)} > 0$ (i.e. $a < 1/2$). Then the condition $\mathbf{D} < 0$ implies $a < 1/4$ and all singular points are real. We calculate $x_1x_2 = 2a$ and $\chi_C^{(1)} = 316a/4096$ and hence this invariant polynomials governs the position of the singular points located on the hyperbola (on the same branch or not). Thus we get *Config.* $\widetilde{H}$.12 when $\chi_C^{(1)} < 0$ and *Config.* $\widetilde{H}$.13 when $\chi_C^{(1)} > 0$.

b) *The subcase* $\mathbf{D} > 0$. In this case we have $1/4 < a < 1/2$ and therefore the singular points located on the hyperbola are real, whereas the singularities from the invariant line are complex. As $a > 0$ we deduce that the real singularities are located on the same branch of the hyperbola. As a result, we get the unique configuration *Config.* $\widetilde{H}$.14.

c) *The subcase* $\mathbf{D} = 0$. Then either $a = 1/4$ or $a = 1/2$ and these possibilities are distinguished by $\chi_A^{(1)}$. Therefore we get the configuration *Config.* $\widetilde{H}$.15 if $\chi_A^{(1)} \neq 0$ and *Config.* $\widetilde{H}$.16 if $\chi_A^{(1)} = 0$.

3.1.1.1.2 The possibility $\beta_1 = 0$. Then due to $\theta \neq 0$ (i.e. $h(g-1) \neq 0$) and to the condition $\beta_2 = 3ch^2/2 \neq 0$, the condition $\beta_1 = 0$ implies $g = 1/3$ and $\gamma_2 = 0$. So we arrive at the following family of systems

$$\frac{dx}{dt} = a + cx + x^2/3 + hxy, \quad \frac{dy}{dt} = b - cy - 2xy/3 + hy^2.$$

We observe that since $ch \neq 0$ we may assume $c = h = 1$ due to the rescaling $(x, y, t) \mapsto (cx, cy/h, t/c)$. According to Theorem 2 (see Diagram 2) the above systems possess an invariant hyperbola if and only if $\gamma_4 = 0$ and $\mathcal{R}_3 \neq 0$. Considering the condition $c = h = 1$ for these systems we calculate

$$\gamma_4 = 16(a+6b)^2/3, \quad \mathcal{R}_3 = 3b/2$$

and hence the condition $\gamma_4 = 0$ gives $b = -a/6 \neq 0$. So we get the following 1-parameter family of systems

$$\frac{dx}{dt} = a + x + x^2/3 + xy, \quad \frac{dy}{dt} = -a/6 - y - 2xy/3 + y^2 \tag{19}$$

with $a \neq 0$ which possess the following invariant hyperbola

$$\Phi(x, y) = a + 2xy = 0 \tag{20}$$

and singular points $M_i(x_i, y_i)$ (i=1,2,3,4) with the coordinates

$$x_{1,2} = (-3 \pm \sqrt{3(3 - 2a)})/2, \quad y_{1,2} = (3 \pm \sqrt{3(3 - 2a)})/6,$$
$$x_{3,4} = -1 \pm \sqrt{1 - 2a}, \quad y_{3,4} = (-1 \pm \sqrt{1 - 2a}\,)/6.$$

We observe that the singularities $M_{1,2}\big(x_{1,2}, y_{1,2}\big)$ are located on the hyperbola and since $\chi_A^{(2)} = 2(3 - 2a)/9$ we deduce that these points are complex (respectively, real) if $\chi_A^{(2)} < 0$ (respectively $\chi_A^{(2)} > 0$) and they coincide if $\chi_A^{(2)} = 0$.

On the other hand we have $x_1x_2 = 3a/2$ and $\chi_A^{(8)} = 23a/12$ and therefore we conclude that the singular points $M_{1,2}$ are located on different branches of the hyperbola if $\chi_A^{(8)} < 0$ and on the same branch if $\chi_A^{(8)} > 0$.

Other two singular points $M_{3,4}\big(x_{3,4}, y_{3,4}\big)$ of systems (19) generically are located outside the hyperbola. In order to determine the conditions when at least one of these singular points is located on the hyperbola we calculate

$$\Phi(x, y)|_{\{x=x_{3,4},\ y=y_{3,4}\}} = (a + 2 \mp 2\sqrt{1 - 2a})/3 \equiv \Omega_{3,4}(a),$$
$$\Omega_3\Omega_4 = a(12 + a)/9, \quad \chi_E^{(3)} = -9a(12 + a)/8.$$

It is clear that at least one of the singular points M_3 or M_4 belongs to the hyperbola (20) if and only if $\chi_E^{(3)} = 0$.

Since for systems (19) we have $B_1 = 2a^3/27 \neq 0$ and $\mu_0 = 1/3 \neq 0$, by Lemmas 1 and 2 we have no invariant lines and none of the finite singularities could go to infinity. So we arrive at the following conditions and configurations:

- $\chi_A^{(2)} < 0 \ \Rightarrow$ *Config.* $\widetilde{H}$.1;
- $\chi_A^{(2)} > 0,\ \chi_A^{(8)} < 0$ and $\chi_E^{(3)} \neq 0 \ \Rightarrow$ *Config.* $\widetilde{H}$.2;
- $\chi_A^{(2)} > 0,\ \chi_A^{(8)} < 0$ and $\chi_E^{(3)} = 0 \ \Rightarrow$ *Config.* $\widetilde{H}$.4;
- $\chi_A^{(2)} > 0$ and $\chi_A^{(8)} > 0 \ \Rightarrow$ *Config.* $\widetilde{H}$.3;
- $\chi_A^{(2)} = 0 \ \Rightarrow$ *Config.* $\widetilde{H}$.8.

3.1.1.2 The subcase $\beta_2 = 0$ Then $f = 2c$ and this implies $\gamma_1 = 0$. By Theorem 2 (see Diagram 5) in this case we have an invariant hyperbola if and only if $\gamma_2 = \beta_1 = \gamma_{14} = 0$ and $\mathcal{R}_{10} \neq 0$. Moreover, this hypebola is simple if $\beta_7\beta_8 \neq 0$ and it is double if $\beta_7\beta_8 = 0$. So we calculate

$$\gamma_2 = -\,14175ac^2h^5(g - 1)^3(1 + 3g), \quad \beta_1 = -9c^2(g - 1)^2h^2/16$$

and evidently the condition $\gamma_2 = \beta_1 = 0$ implies $c = 0$. Then we obtain

$$\gamma_{14} = -80h^3\big[a(2g-1) - 2bh\big], \quad \mathcal{R}_{10} = -4ah^2 \neq 0$$

and as $h \neq 0$ the condition $\gamma_{14} = 0$ gives $a(2g-1) - 2bh = 0$. Then setting $a = 2a_1h$ we get $b = a_1(2g-1)$ and keeping the old parameter a (instead of a_1) after the additional rescaling $y \to y/h$ we arrive and at the following 2-parameter family of systems

$$\frac{dx}{dt} = 2a + gx^2 + xy, \quad \frac{dy}{dt} = a(2g-1) + (g-1)xy + y^2. \tag{21}$$

These systems possess the invariant hyperbola (13) and we calculate

$$\beta_7 = 8(1-2g), \quad \beta_8 = 32(1-4g), \quad B_1 = 4a^3(g-1)^2(1-2g), \quad \mu_0 = g$$

and following Diagram 5 (see Theorem 2) we examine two possibilities: $\beta_7\beta_8 \neq 0$ and $\beta_7\beta_8 = 0$.

3.1.1.2.1 The possibility $\beta_7\beta_8 \neq 0$. In this case for systems (21) the condition

$$a(g-1)(2g-1)(4g-1) \neq 0 \tag{22}$$

is satisfied and this implies $B_1 \neq 0$. Therefore according to Lemma 2 these systems could not have invariant lines and as earlier we consider two cases: $\mu_0 \neq 0$ and $\mu_0 = 0$.

1) *The case* $\mu_0 \neq 0$. Then systems (21) possess four finite singular points $M_i(x_i, y_i)$ (i=1,2,3,4) with the coordinates

$$x_{1,2} = \pm\sqrt{-a/g}, \quad y_{1,2} = \pm\sqrt{-ag},$$
$$x_{3,4} = \pm 2\sqrt{-a}, \quad y_{3,4} = \pm\sqrt{-a}(1-2g).$$

We detect that the singularities $M_{1,2}(x_{1,2}, y_{1,2})$ are located on the hyperbola and they are complex (respectively, real) if $ag > 0$ (respectively $ag < 0$). Moreover since $x_1x_2 = a/g$ then in the case when they are real (i.e. $ag < 0$) these points are located on different branches of the hyperbola (13).

On the other hand considering singular points $M_{3,4}(x_{3,4}, y_{3,4})$ we calculate

$$\Phi(x,y)|_{\{x=x_3,\ y=y_3\}} = \Phi(x,y)|_{\{x=x_4,\ y=y_4\}} = a(4g-1) \neq 0,$$

i.e. for any values of the parameters a and g satisfying the condition (22) these singularities could not belong to the hyperbola (13).

For systems (21) we calculate $\mu_0\mathcal{R}_{10} = -8ag \neq 0$ and hence $\text{sign}\,(\mu_0\mathcal{R}_{10}) = -\text{sign}\,(ag)$. So we arrive at the configuration given by *Config.* $\widetilde{H}.1$ if $\mu_0\mathcal{R}_{10} < 0$ and by *Config.* $\widetilde{H}.2$ if $\mu_0\mathcal{R}_{10} > 0$.

2) *The case* $\mu_0 = 0$. Then $g = 0$ and we calculate

$$\mu_0 = \mu_1 = 0, \quad \mu_2 = ay^2 \neq 0$$

and by Lemma 1 two finite singular points have gone to infinity and both coalesced with the infinite singular point $[1,0,0]$. As a result we get the unique configuration *Config.* $\widetilde{H}.17$.

3.1.1.2.2 The possibility $\beta_7\beta_8 = 0$. Assume first $\beta_7 = 0$, i.e. $g = 1/2$ which implies $B_1 = 0$ and systems (21) possess the invariant line $y = 0$. Since $\mathcal{R}_{10} = -8a$, considering the coordinates of the singularities we arrive at *Config.* $\widetilde{H}$.11 if $\mathcal{R}_{10} < 0$ and at *Config.* $\widetilde{H}$.12 if $\mathcal{R}_{10} > 0$.

Suppose now $\beta_8 = 0$ which gives $g = 1/4$. Then the singularities M_3 and M_4 coalesce with M_1 and M_2, respectively. So in this case systems (21) have two double singular points located on the hyperbola which are complex if $a > 0$ and real if $a < 0$. So we obtain *Config.* $\widetilde{H}$.1 if $\mathcal{R}_{10} < 0$ and *Config.* $\widetilde{H}$.18 if $\mathcal{R}_{10} > 0$.

3.1.2 The case $\theta = 0$

According to (10) we get $h(g-1) = 0$ and since for systems (9) we have $\mu_0 = gh^2$ we consider two subcases: $\mu_0 \neq 0$ and $\mu_0 = 0$.

3.1.2.1 The subcase $\mu_0 \neq 0$ Then $h \neq 0$ and the condition $\theta = 0$ yields $g = 1$. Since $h \neq 0$ via the affine transformation

$$x_1 = x + d/h, \quad y_1 = hy + c - 2d/h$$

we may assume $d = f = 0$, $h = 1$ and systems (9) become as systems

$$\frac{dx}{dt} = a + cx + x^2 + xy, \quad \frac{dy}{dt} = b + ex + y^2 \tag{23}$$

for which we calculate

$$N = 9y^2, \quad \beta_4 = 2, \quad \beta_3 = -e/4, \quad \gamma_1 = 9ce^2/16.$$

Since $N\beta_4 \neq 0$ following Diagram 5 (see Theorem 2) for the existence of an invariant hyperbola the conditions $\gamma_1 = \gamma_2 = \beta_3 = 0$ are necessary. Therefore we have $e = 0$ and this implies $\gamma_1 = \gamma_2 = 0$ and

$$\gamma_8 = 42(9a - 18b - 2c^2)^2.$$

So setting for simplicity $c = 3c_1$ and $a = 2a_1$ the condition $\gamma_8 = 0$ yields $b = a_1 - c_1^2$ and keeping the notation for the parameters c and a we arrive at the 2-parameter family of systems

$$\frac{dx}{dt} = 2a + 3cx + x^2 + xy, \quad \frac{dy}{dt} = a - c^2 + y^2. \tag{24}$$

These systems possess the following invariant hyperbola and two invariant lines:

$$\Phi(x, y) = a + cx + xy = 0, \quad L_{1,2} = y \pm \sqrt{c^2 - a} = 0 \tag{25}$$

and singular points $M_i(x_i, y_i)$ (i=1,2,3,4) with the coordinates

$$x_{1,2} = -c \pm \sqrt{c^2 - a}, \quad y_{1,2} = \pm\sqrt{c^2 - a},$$
$$x_{3,4} = -2(c \pm \sqrt{c^2 - a}), \quad y_{3,4} = \pm\sqrt{c^2 - a}.$$

The singularities $M_{1,2}(x_{1,2}, y_{1,2})$ are located at the intersection points of the hyperbola with invariant lines, whereas the singularities $M_{3,4}$ are located only on the invariant lines. More precisely, the singular point M_3 (respectively, M_4) is located on the same invariant line as the singularity M_1 (respectively, M_2). Since $\chi_A^{(7)} = (c^2 - a)/4$ we deduce that all these finite singular points as well as the invariant lines $L_{1,2}$ are complex if $\chi_A^{(7)} < 0$ and real if $\chi_A^{(7)} > 0$. In the case $\chi_A^{(7)} = 0$ (then $a = c^2 \neq 0$) we obtain that the singular point M_1 (respectively, M_3) coincides with M_2 (respectively, M_4) and moreover, in this case invariant lines coincide, too. So we consider three possibilities: $\chi_A^{(7)} < 0$, $\chi_A^{(7)} > 0$ and $\chi_A^{(7)} = 0$.

3.1.2.1.1 The possibility $\chi_A^{(7)} < 0$. Then $c^2 - a < 0$ (this implies $a > 0$) and all the singularities and the invariant lines are complex. As a result we arrive at the unique configuration given by *Config.* $\widetilde{H}$.19.

3.1.2.1.2 The possibility $\chi_A^{(7)} > 0$. In this case the finite singularities $M_1 \neq M_2$ and $M_3 \neq M_4$ are real and we observe that the singular points $M_{3,4}$ of systems (24) generically are located outside the hyperbola. We calculate

$$\Phi(x,y)|_{\{x=x_{3,4},\ y=y_{3,4}\}} = 3a + 4c(-c \pm 2\sqrt{c^2 - a}) \equiv \Omega_{3,4}(a,c), \quad \Omega_3\Omega_4 = a(9a - 8c^2).$$

On the other hand by Theorem 2 (see Diagram 5) the hyperbola (25) is simple if $\delta_4 = 3(9a - 8c^2) \neq 0$ and it is double if $\delta_4 = 0$. So we conclude that at least one of the singularities $M_{3,4}$ belongs to the hyperbola if and only if the hyperbola is double (i.e. when $\delta_4 = 0$). So we consider two cases: $\delta_4 \neq 0$ and $\delta_4 = 0$.

1) *The case* $\delta_4 \neq 0$. Then all four finite singularities are real and distinct. In this case in order to detect the different configurations we need to distinguish the position of the branches of the hyperbola (which depends on the sign of the parameter a) as well as the position of the singular point M_3 on the line $y = \sqrt{c^2 - a}$ with respect to M_1 and the position of M_4 on the line $y = -\sqrt{c^2 - a}$ with respect to M_2. So considering the coordinates of the finite singularities we calculate

$$x_1x_2 = a, \quad (x_1 - x_3)(x_2 - x_4) = 9a - 8c^2, \quad \mathcal{R}_7 = -3a/4, \quad \chi_F^{(7)} = 9a - 8c^2.$$

So the singularities M_1 and M_2 are located on the same branch of the hyperbola if $\mathcal{R}_7 < 0$ and on different branches if $\mathcal{R}_7 > 0$. To determine exactly the position of M_1 and M_3 as well as of M_2 and M_4 we observe, that due to the rescaling $(x,y,t) \mapsto (-x,-y,-t)$ we may assume that the parameter $c \geq 0$. This means that $x_1 - x_3 = c + 3\sqrt{c^2 - a} > 0$ (due to $c \geq 0$ and $c^2 - a > 0$) and hence the sign of $x_2 - x_4$ is governed by the invariant polynomial $\chi_F^{(7)}$.

Thus in the case $\chi_A^{(7)} > 0$ and $\delta_4 \neq 0$ (then $\chi_F^{(7)} \neq 0$) we arrive at the following conditions and configurations:

- $\mathcal{R}_7 < 0 \ \Rightarrow$ *Config.* $\widetilde{H}$.20;
- $\mathcal{R}_7 > 0$ and $\chi_F^{(7)} < 0 \ \Rightarrow$ *Config.* $\widetilde{H}$.21;
- $\mathcal{R}_7 > 0$ and $\chi_F^{(7)} > 0 \ \Rightarrow$ *Config.* $\widetilde{H}$.22;

2) *The case* $\delta_4 = 0$. Then $a = 8c^2/9 \neq 0$ and by Theorem 2 (see Diagram 5) the hyperbola (25) is double. Moreover in this case the singular point M_4 coincides with M_2, located on the hyperbola. Since $c \neq 0$ (i.e. no other singularities could coincide) we get the unique configuration *Config.* $\widetilde{H}$.23.

3.1.2.1.3 The possibility $\chi_A^{(7)} = 0$. Then $a = c^2 \neq 0$ and this implies the coalescense of the singularity M_2 with M_1 and of M_4 with M_3. Clearly in this case we get the double line $y^2 = 0$ and since $c \neq 0$ we obtain *Config.* $\widetilde{H}.24$.

3.1.2.2 The subcase $\mu_0 = 0$ Then the condition $\theta = \mu_0 = 0$ gives $h = 0$ and for systems (9) in this case we calculate

$$N = 9(g-1)(1+g)x^2, \quad \gamma_1 = \gamma_2 = \beta_4 = 0, \quad \beta_6 = d(g-1)(1+g)/4.$$

We next consider two possibilities: $N \neq 0$ and $N = 0$.

3.1.2.2.1 The possibility $N \neq 0$. In this case by Theorem 2 (see Diagram 5) for the existence of at least one hyperbola the condition $(\mathfrak{C}_1)$ are necessary and sufficient, where

$$(\mathfrak{C}_1): \ (\beta_6 = 0, \beta_{11}\mathcal{R}_{11} \neq 0) \cap \big((\beta_{12} \neq 0, \gamma_{15} = 0) \cup (\beta_{12} = \gamma_{16} = 0)\big).$$

So the condition $\beta_6 = 0$ is necessary. Since $N \neq 0$ we get $d = 0$ and moreover as $g - 1 \neq 0$, due a translation, we may assume $e = f = 0$. Therefore we arrive at the family of systems

$$\frac{dx}{dt} = a + cx + gx^2, \quad \frac{dy}{dt} = b + (g-1)xy,$$

for which following Diagram 2 we calculate:

$$\beta_{11} = 4(2g-1)x^2, \quad \mathcal{R}_{11} = -3b(g-1)^2x^4, \quad \beta_{12} = (3g-1)x,$$
$$\gamma_{15} = 4(g-1)^2(3g-1)\big[a(3g-1)^2 + c^2(1-2g)\big]x^5.$$

So according to Theorem 2 the condition $\beta_{11}\mathcal{R}_{11} \neq 0$ is necessary for the existence of a hyperbola and considering Diagram 2 we have to consider the two cases: $\beta_{12} \neq 0$ and $\beta_{12} = 0$.

1) *The case* $\beta_{12} \neq 0$. By Theorem 2 in this case there exists one hyperbola if and only if $\gamma_{15} = 0$. We observe that due to $b \neq 0$ (since $\mathcal{R}_{11} \neq 0$) we may assume $b = 1$ due to the rescaling $(x, y, t) \mapsto (bx, y, t/b)$. Since $(3g-1) \neq 0$, setting $c = (3g-1)c_1$ the condition $\gamma_{15} = 0$ yields $a = c_1^2(2g-1)$ and renaming the parameter c_1 as c again we arrive at the 2-parameter family of systems

$$\frac{dx}{dt} = (c+x)\big[c(2g-1) + gx\big], \quad \frac{dy}{dt} = 1 + (g-1)xy \tag{26}$$

for which the condition $N\beta_{11}\beta_{12}\mathcal{R}_{11}$ implies

$$(g-1)(g+1)(2g-1)(3g-1) \neq 0. \tag{27}$$

These systems possesses the following invariant hyperbola and invariant lines:

$$\Phi(x, y) = \frac{1}{2g-1} + cy + xy = 0, \quad L_1 = gx + c(2g-1) = 0, \quad L_2 = x + c = 0. \tag{28}$$

On the other hand for systems (26) we calculate

$$\mu_0 = \mu_1 = 0, \quad \mu_2 = c^2g(g-1)^2(2g-1)x^2, \quad \gamma_{16} = c(g-1)^2(1-3g)x^3/2 \tag{29}$$

and by Lemma 1 in the case $\mu_2 \neq 0$ these systems possess finite singular points of total multiplicity two. Other two points have gone to infinity and coalesced with the singularity $[0,1,0]$. So we consider two cases: $\mu_2 \neq 0$ and $\mu_2 = 0$.

a) *The subcase* $\mu_2 \neq 0$. Then $c \neq 0$ and due to the rescaling $(x, y, t) \mapsto (cx, y/c, t/c)$ we may assume $c = 1$. In this case the 1-parameter family of systems (26) possess the finite singular points $M_i(x_i, y_i)$ (i=1,2) with the coordinates

$$x_1 = \frac{(1-2g)}{g}, \quad y_1 = \frac{g}{(g-1)(2g-1)}, \quad x_2 = -1, \quad y_2 = \frac{1}{g-1}.$$

We detect that the singular point M_1 is located at the intersection point of the hyperbola with invariant line $L_1 = 0$ (see (28)) whereas M_2 is located on the line $L_2 = 0$ outside the hyperbola.

On the other hand taking into account (29) for systems (26) with $c = 1$ we have $\gamma_{16} \neq 0$ (due to (27)) and hence by Theorem 2 (see Diagram 5) the hyperbola (28) is a simple one. So considering the condition (27) and looking at all the intervals given by this condition we arrive at the unique configuration presented by *Config.* $\widetilde{H}$.25.

b) *The subcase* $\mu_2 = 0$. Then considering (29) and condition (27) we get $cg = 0$ and we consider two possibilities: $\gamma_{16} \neq 0$ and $\gamma_{16} = 0$.

***b*₁**) *The possibility* $\gamma_{16} \neq 0$. Then $c \neq 0$ (and we may assume $c = 1$) and this implies $g = 0$. So we arrive at the system with constant coefficients

$$\frac{dx}{dt} = -(1+x), \quad \frac{dy}{dt} = 1 - xy$$

possessing one finite singular point $M_1(-1,-1)$, the invariant hyperbola $xy + y - 1 = 0$ and the invariant line $x + 1 = 0$. On the other hand following Lemma 3 we detect that the line at infinity $Z = 0$ is double for these systems because Z is a common factor of degree one of the polynomials $\mathcal{E}_1(X,Y,Z)$ and $\mathcal{E}_2(X,Y,Z)$. Moreover, since $\mu_0 = \mu_1 = \mu_2 = 0$ and $\mu_3 = -x^2y$, according to Lemma 1 we deduce that another finite singular point has gone to infinity and coalesced with $[1,0,0]$. We observe that M_1 belongs to the invariant line and it is outside the hyperbola, i.e. we get *Config.* $\widetilde{H}$.26.

***b*₂**) *The subcase* $\gamma_{16} = 0$. In this case $c = 0$ and we get the systems

$$\frac{dx}{dt} = gx^2, \quad \frac{dy}{dt} = 1 + (g-1)xy,$$

for which $g \neq 0$ (otherwise we obtain a degenerate system). For these systems we calculate

$$\mu_0 = \mu_1 = \mu_2 = \mu_3 = \gamma_{16} = 0, \quad \mu_4 = g^2x^4, \quad \delta_6 = (g-1)(4g-1)(x^2)/2$$

and by Lemma 1 we deduce that all four finite singular points have gone to infinity and coalesced with $[0,1,0]$. Moreover, for the above systems we calculate

$$\mathscr{E}_k(X) = gx^3(1 + g - xy + gxy)$$

and by Lemma 6 the invariant line $x = 0$ is a triple one.

According to Diagram 2 the hyperbola is simple if $\delta_6 \neq 0$ (i.e. $4g-1 \neq 0$) and it is double if $\delta_6 = 0$ (i.e. $4g-1 = 0$). So we arrive at *Config.* $\widetilde{H}$.27 if $\delta_6 \neq 0$ and at *Config.* $\widetilde{H}$.28 if $\delta_6 = 0$.

2) *The case* $\beta_{12}=0$. Then $g=1/3$ and we calculate $\gamma_{16}=-2cx^3/9$. Since by Theorem 2 in the case under consideration the condition $\gamma_{16}=0$ is necessary for the existence of an invariant hyperbola, we obtain $c=0$ and we arrive at the 1-parameter family of systems

$$\frac{dx}{dt}=a+x^2/3,\quad \frac{dy}{dt}=1-2xy/3.$$

For these systems we calculate $\gamma_{17}=32ax^2/9$ and following Theorem 2 we conclude that for $\gamma_{17}<0$ or $\gamma_{17}>0$ or $\gamma_{17}=0$ we obtain three different configurations due to the number and types of hyperbolas. Since $\operatorname{sign}(a)=\operatorname{sign}(\gamma_{17})$ setting a new parameter k as follows: $a=\operatorname{sign}(a)k^2/3$ after the rescaling $(x,y,t)\mapsto(kx,3y/k,3t/k)$ (in the case $k\neq 0$) or the rescaling $x\to 3x$ if $a=0$, the above systems become

$$\frac{dx}{dt}=x^2+\varepsilon,\quad \frac{dy}{dt}=1-2xy, \tag{30}$$

where $\varepsilon=\operatorname{sign}(\gamma_{17})$ if $\gamma_{17}\neq 0$ and $\varepsilon=0$ if $\gamma_{17}=0$, i.e. $\varepsilon\in\{-1,0,1\}$.

These systems possess the following invariant hyperbolas and invariant lines:

$$\Phi_{1,2}(x,y)=3\pm\sqrt{-\varepsilon}y-xy=0,\quad L_{1,2}=x\pm\sqrt{-\varepsilon}=0. \tag{31}$$

We detect that these systems possess the finite singularities $M_{1,2}\big(\pm\sqrt{\varepsilon},3\pm 1/(2\sqrt{\varepsilon})\big)$ (if $\varepsilon\neq 0$) and each one of the lines intersect only one of the hyperbolas.

On the other hand for systems (30) we calculate

$$\mu_0=\mu_1=0,\quad \mu_2=4\varepsilon x^2,\quad \mu_3=0,\quad \mu_4=x^2(x+2\varepsilon y)^2.$$

Therefore by Lemma 1 we conclude that in the case $\varepsilon\neq 0$ only two finite singularities of these systems have gone to infinity and coalesced with $[0,1,0]$ and we get *Config.* $\widetilde{H}$.29 if $\gamma_{17}<0$ and *Config.* $\widetilde{H}$.30 if $\gamma_{17}>0$.

Assume now $\gamma_{17}=0$ (i.e. $\varepsilon=0$). Then $\mu_i=0$ for $i=0,1,2,3$ and $\mu_4=x^4$ and by Lemma 1 all the finite singularities of this system have gone to infinity and coalesced with $[0,1,0]$.

We observe that the two lines coincide and we get the invariant multiple line $x=0$. Considering Lemma 6 for systems (30) with $\varepsilon=0$ we calculate

$$\mathscr{E}_k(X)=2x^3(2-3xy)$$

and by this lemma in the case under consideration the invariant line $x=0$ is a triple one. Since by Theorem 2 (see Diagram 5) the hyperbola (31) in the case $\gamma_{17}=0$ (i.e. $\varepsilon=0$) is double, we arrive at the same configuration given by *Config.* $\widetilde{H}$.28.

3.1.2.2.2 The possibility $N=0$.

Then $(g-1)(g+1)=0$ and as $\beta_{13}=(g-1)^2x^2/4$ we consider two cases: $\beta_{13}\neq 0$ and $\beta_{13}=0$

1) *The case* $\beta_{13}\neq 0$. Therefore the condition $N=0$ gives $g=-1$ and we can assume $e=f=0$ due to a translation. So we get the systems

$$\frac{dx}{dt}=a+cx+dy-x^2,\quad \frac{dy}{dt}=b-2xy,$$

which by Theorem 2 (see Diagram 5) possess an invariant hyperbola if and only if $\gamma_{10} = \gamma_{17} = 0$ and $\mathcal{R}_{11} \neq 0$. Calculations yield

$$\gamma_{10} = 14d^2 = 0, \quad \gamma_{17} = -8(16a + 3c^2)x^2 + 4dy(14cx + 9dy) = 0,$$
$$\mathcal{R}_{11} = -6x(2bx^3 - cdxy^2 - d^2y^3) \neq 0$$

and therefore we obtain $d = 0$, $a = -3c^2/16$ and $b \neq 0$ and we may assume $b = 1$ due to the rescaling $y \to by$. So we arrive at the 1-parameter of systems

$$\frac{dx}{dt} = -3c^2/16 + cx - x^2, \quad \frac{dy}{dt} = 1 - 2xy$$

possessing the invariant hyperbolas and the invariant lines

$$\Phi_{1,2}(x,y) = 4 + 3cy - 12xy = 0, \quad L_1 = 4x - c = 0, \quad L_2 = 4x - 3c = 0. \qquad (32)$$

We observe that for $c = 0$ the lines coincide and this phenomenon is governed by the invariant polynomial $\gamma_{16} = -2cx^3$. So we consider two subcases: $\gamma_{16} \neq 0$ and $\gamma_{16} = 0$.

a) *The subcase* $\gamma_{16} \neq 0$. Then $c \neq 0$ and we may assume $c = 4$ due to the rescaling $(x, y, t) \mapsto (cx/4, 4y/c, 4t/c)$. So we obtain the system

$$\frac{dx}{dt} = (x-1)(3-x), \quad \frac{dy}{dt} = 1 - 2xy \qquad (33)$$

which possesses the following invariant hyperbolas and invariant lines:

$$\Phi_{1,2}(x,y) = 1/3 + y - xy = 0, \quad L_1 = x - 1 = 0, \quad L_2 = x - 3 = 0 \qquad (34)$$

and two finite singularities: $M_1(1, 1/2)$ and $M_2(3, 1/6)$. Since $\mu_0 = \mu_1 = 0$ and $\mu_2 = 12x^2$ by Lemma 1 we conclude that two finite singularities of this system have gone to infinity and coalesced with $[0,1,0]$. So considering the position of the hyperbola, invariant lines and of the finite singularities we arrive at *Config.* $\widetilde{H}.25$.

b) *The subcase* $\gamma_{16} = 0$. Then $c = 0$ and we get the system

$$\frac{dx}{dt} = -x^2, \quad \frac{dy}{dt} = 1 - 2xy,$$

for which

$$\mu_0 = \mu_1 = \mu_2 = \mu_3 = 0, \quad \mu_4 = x^4.$$

So by Lemma 1 all the finite singularities of this system have gone to infinity and coalesced with $[0,1,0]$.

On the other hand we observe that the invariant line $x = 0$ is a multiple one. For the above system we calculate $\mathscr{E}_k(X) = 2x^4y$ and by Lemma 6 we deduce that the invariant line $x = 0$ has multiplicity four. So considering the invariant hyperbola (34) (for $c = 0$)we arrive at the configuration given by *Config.* $\widetilde{H}.31$.

2) *The case* $\beta_{13} = 0$. Then we have $g = 1$ and we can assume $c = 0$ due to a translation. So we get the systems

$$\frac{dx}{dt} = a + dy + x^2, \quad \frac{dy}{dt} = b + ex + fy,$$

and by Theorem 2 (see Diagram 5) these systems possess an invariant hyperbola if and only if $\gamma_9 = \tilde{\gamma}_{18} = \tilde{\gamma}_{19} = 0$. Calculations yield

$$\gamma_9 = -6d^2 = 0, \quad \tilde{\gamma}_{18} = 8x^2(ex^2 - 2dy^2) = 0, \quad \tilde{\gamma}_{19} = 4(4a + f^2)x + 4dfy = 0$$

and evidently this implies $d = e = 0$ and $a = -f^2/4$ which leads to the 2-parameter family of systems

$$\frac{dx}{dt} = -f^2/4 + x^2, \quad \frac{dy}{dt} = b + fy.$$

For these systems we calculate $\mu_0 = \mu_1 = 0$, $\mu_2 = f^2x^2$ and we consider two subcases: $\mu_2 \neq 0$ and $\mu_2 = 0$.

a) *The subcase* $\mu_2 \neq 0$. Then $f \neq 0$ and we may assume $f = 1$ and $b = 0$ due to the transformation $(x, y, t) \mapsto (fx, y - b/f, t/f)$. So we obtain the system

$$\frac{dx}{dt} = (2x - 1)(2x + 1)/4, \quad \frac{dy}{dt} = y \tag{35}$$

which possesses the 1-parameter family of hyperbola:

$$\Phi(x, y) = -q/2 + qx + y + 2xy = 0, \quad q \in \mathbb{C} \setminus \{0\}$$

as for $q = 0$ we get a reducible conic.

On the other hand system (35) possesses the following invariant lines and finite singularities:

$$L_1 = 2x - 1 = 0, \quad L_2 = 2x + 1 = 0, \quad L_3 = y = 0, \quad M_{1,2}(\pm 1/2, 0).$$

Following Lemmas 3 and 6 for this system we calculate

$$\gcd\big(\mathcal{E}_1(X, Y, Z), \mathcal{E}_2(X, Y, Z)\big) = YZ(2X - Z)^2(2X + Z), \quad \mathscr{E}_k(X) = (1 - 2x)^2(1 + 2x)y/4$$

and we deduce that the invariant lines $L_2 = 0$ and $L_3 = 0$ are simple, whereas the line $L_1 = 0$ as well as the infinite line $Z = 0$ are double ones.

So considering the fact that other two finite singular points have gone to infinity and coalesced with $[1, 0, 0]$ we arrive at *Config.* $\widetilde{H}$.32.

b) *The subcase* $\mu_2 = 0$. In this case we have $f = 0$ and as $b \neq 0$ (otherwise we get degenerate system) we may assume $b = 1$ due to the change $y \to by$ and we get the system

$$\frac{dx}{dt} = x^2, \quad \frac{dy}{dt} = 1 \tag{36}$$

which possesses the 1-parameter family of hyperbola:

$$\Phi(x, y) = 1 + rx + xy = 0, \quad r \in \mathbb{C}$$

and has no finite singularities. Calculations yield

$$\mu_0 = \mu_1 = \mu_2 = \mu_3 = 0, \quad \mu_4 = x^4, \quad \gcd\big(\mathcal{E}_1(X, Y, Z), \mathcal{E}_2(X, Y, Z)\big) = X^3Z^2, \quad \mathscr{E}_1(X) = 2X^3$$

and considering Lemma 1 we conclude that all the finite singularities of these systems have gone to infinity and coalesced with $[0, 1, 0]$. Moreover by Lemmas 3 and 6 the invariant line $x = 0$ as well as the infinite line $Z = 0$ are of multiplicity 3. As a result we arrive at the configuration given by *Config.* $\widetilde{H}$.33.

3.2 The possibility $M(\tilde{a},x,y)=0=C_2(\tilde{a},x,y)$

In this section we consider the configurations of invariant hyperbolas and invariant lines of quadratic systems with $C_2=0$, taking into account Theorem 2 (see Diagram 5). Then the line at infinity is filled up with singularities and according to [26] in this case via an affine transformation and time rescaling quadratic systems could be brought to the following systems

$$\dot{x}=k+cx+dy+x^2,\quad \dot{y}=l+xy. \tag{37}$$

Following [26] we consider the stratification of the parameter space of the above systems given by invariant polynomials H_9-H_{12} in [26, Table 1 on page 754] according to possible configurations of invariant lines. So for systems (37) we calculate $H_{10}=36d^2$ and we consider two cases: $H_{10}\neq 0$ and $H_{10}=0$.

3.2.1 The case $H_{10}\neq 0$

Then $d\neq 0$ and as it was shown in [26, pages 748,749], in this case via some parametrization and using an additional affine transformation and time rescaling we arrive at the following 2-parameter family of systems

$$\dot{x}=a+y+(x+c)^2,\quad \dot{y}=xy. \tag{38}$$

for which we calculate

$$N_7=16c(9a+c^2),\quad H_9=2304a(a+c^2)^2$$

and by Theorem 2 (see Diagram 5) for the existence of invariant hyperbola the condition $N_7=0$ is necessary and sufficient. So we have either $c=0$ or $9a+c^2=0$. However in the second case the condition $a\leq 0$ must hold and in the case $a=0$ we get again $c=0$. In the case $a<0$ we may assume $a=-1$ and $c>0$ due to the rescaling $\big(x,y,t\big)\mapsto(\operatorname{sign}(c)\sqrt{-a}\,x,-ay,t/(\operatorname{sign}(c)\sqrt{-a})\big)$, therefore we set $c=3$. Moreover the transformation

$$(x,y,t)\mapsto\big(2(x-1),\ 4(y-x-1),\ t/2\big).$$

sends the system (38) for $a=-1$, $c=3$ to the system (38) with $a=-1$ and $c=0$. Thus we assume $c=0$ and we get the systems

$$\dot{x}=a+y+x^2,\quad \dot{y}=xy \tag{39}$$

which possess the following 1-parameter family of hyperbolas

$$\Phi(s,x,y)=a+2y+x^2-m^2y^2=0 \tag{40}$$

as well as the following invariant lines and finite singularities:

$$L_1=y=0,\quad L_{2,3}=ax^2+(a+y)^2=0;\quad M_1(0,-a),\quad M_{2,3}(\pm\sqrt{-a},\,0).$$

We observe that the two lines $L_{2,3}=0$ as well as the singular points $M_{2,3}$ are real if $a<0$; they are complex if $a>0$ and they coincide if $a=0$. Moreover these three possibilities are distinguished by the invariant polynomial $H_9=2304a^3$.

So, considering that all the hyperbolas from the family (40) intersect invariant line $y=0$ at the singular points $M_{2,3}$ we arrive at the configuration *Config.* $\widetilde{H}$.34 if $H_9<0$; *Config.* $\widetilde{H}$.35 if $H_9>0$ and *Config.* $\widetilde{H}$.36 if $H_9=0$.

3.2.2 The case $H_{10} = 0$

In this case we have $d = 0$ and we distinguish two subcases: $k \neq 0$ and $k = 0$. Since for systems (37) with $d = 0$ we have $H_{12} = -8k^2x^2$ it is clear that this invariant polynomial governs these two subcases.

3.2.2.1 The subcase $H_{12} \neq 0$. Then $k \neq 0$ and as it was shown in [26, page 750] in this case via an affine transformation and time rescaling after some additional parametrization we arrive at the following 2-parameter family of systems

$$\dot{x} = a + (x + c)^2, \quad \dot{y} = xy. \tag{41}$$

For these systems the condition $H_{12} = -8(a + c^2)^2x^2 \neq 0$ must hold and according to Diagram 5 the condition $N_7 = 16c(9a + c^2) = 0$ must be satisfied for the existence of invariant hyperbolas. On the other hand for these systems we have $H_2 = 8cx^2$ and we consider two possibilities: $H_2 = 0$ and $H_2 \neq 0$.

3.2.2.1.1 The possibility $H_2 \neq 0$. Then $c \neq 0$ and in this case we get $9a + c^2 = 0$, i.e. $a = -c^2/9 \neq 0$. Therefore due to the rescaling $(x, y, t) \mapsto (2cx, y, t/(2c))$ systems (41) could be brought to the system

$$\dot{x} = (1 + 3x)(2 + 3x)/9, \quad \dot{y} = xy. \tag{42}$$

This system possesses the 1-parameter family of the hyperbolas and three invariant lines

$$\Phi(x, y) = 4 + 12x + 9x^2 + my + 3mxy = 0; \quad y = 0, \;\; 3x + 1 = 0, \;\; 3x + 2 = 0, \tag{43}$$

as well as the singularities $M_1(-1/3, 0)$ and $M_2(-2/3, 0)$. It is not too difficult to convince ourselves that in this case we get the configuration given by *Config.* $\widetilde{H}$.37.

3.2.2.1.2 The possibility $H_2 = 0$. Then $c = 0$ and we get the systems

$$\dot{x} = a + x^2, \quad \dot{y} = xy, \quad a \neq 0, \tag{44}$$

which possess the following family of conics and the invariant lines:

$$\Phi(x, y) = a + x^2 - m^2y^2 = 0; \quad L_1 = y = 0, \quad L_{2,3} = x^2 + a = 0 \tag{45}$$

as well as two finite singularities: $M_{1,2}(\pm\sqrt{-a},\, y)$.

On the other hand we calculate $H_{11} = -192ax^4$ and therefore $\text{sign}\,(a) = -\text{sign}\,(H_{11})$. So considering the position of the invariant lines and of the hyperbolas given in (45) we obtain the configuration *Config.* $\widetilde{H}$.38 if $H_{11} < 0$ and *Config.* $\widetilde{H}$.39 if $H_{11} > 0$.

3.2.2.2 The subcase $H_{12} = 0$. Then $k = 0$ and we arrive at the family of systems (37) with $d = k = 0$ for which we have $N_7 = -16c^3$ and by Theorem 2 (see Diagram 5) we have to force the condition $c = 0$. Since $l \neq 0$ (otherwise we get a degenerate system) due to the change $y \to ly$ we may assume $l = 1$ and we arrive at the system

$$\dot{x} = x^2, \quad \dot{y} = 1 + xy, \tag{46}$$

which possesses the following family of hyperbolas

$$\Phi(x, y) = 1 + mx^2 + 2xy = 0$$

and the invariant line $x = 0$. We remark that by Lemma 6 this line is triple since for this system we have $\mathscr{E}_1(X) = X^3$. So considering the absence of finite singularities of system (46) we obtain the configuration given by *Config.* $\widetilde{H}.40$.

This completes the proof of statement (B) of Main Theorem.

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