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GLOBAL HYPOELLIPTICITY BY LYAPUNOV FUNCTION

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Abstract

In this work we treat the global hypoellipticity, in the first degree, for a class of abstract differential operators complexes, the ones are given by the following differential operators

$$L_j = \frac{\partial}{\partial t_j} + \frac{\partial \phi}{\partial t_j}(t, A)A, \ \ j = 1, 2, \cdots, n$$

where $A : D(A) \subset H \to H$ is a self-adjoint linear operator, positive with $0 \in \rho(A)$, in a Hilbert space H and $\phi = \phi(t, A)$ is a series of nonnegative power of A^{-1} with coefficients in $C^\infty(\Omega)$, Ω being an open set of $\mathbb{R}^n$, for any $n \in \mathbb{N}$, different of what happens in [Hounie] who studys the problem only in the case $n = 1$.

We provide sufficient condition to get the local hypoellipticity for that complex in the elliptic region, using a Lyapunov function and the dynamics properties of solutions of the Cauchy problem

$$\begin{cases} t'(s) = -\nabla Re\ \phi_0(t(s)), \ s \geq 0, \\ t(0) = t_0 \in \Omega, \end{cases}$$

being $\phi_0 : \Omega \longrightarrow \mathbb{C}$ the first coefficient of $\phi(t, A)$.

Besides, to get over the problem out of the elliptic region, that is, in the points $t^* \in \Omega$ such that $\nabla Re\phi_0(t^*)) = 0$, we will use the techniques developed in [BCM] for the particular operator $A = 1 - \Delta : H^2(\mathbb{R}^N) \subset L^2(\mathbb{R}^N) \longrightarrow L^2(\mathbb{R}^N)$.

1 Introduction

In this work, we want to lay down sufficient condition for the global hypoellipticity, in the first degree, of the differential complex given by the following operators

$$L_j = \frac{\partial}{\partial t_j} + \frac{\partial \phi}{\partial t_j}(t, A)A, \ j = 1, 2, \cdots, n.$$

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Where $A : D(A) \subset H \to H$ is a self-adjoint linear operator, positive with $0 \in \rho(A)$, in a Hilbert space H and $\phi(t, A)$ is a series of nonnegative power of A^{-1} with coefficients in $C^\infty(\Omega)$, Ω being an open set of $\mathbb{R}^n$.

The map $\phi = \phi(t, A)$ is given by

$$\phi(t, A) = \sum_{k=0}^{\infty} \phi_k(t) A^{-k},$$

with convergence in $\mathcal{L}(H)$, uniform in compacts of Ω, and $\phi_k \in C^\infty(\Omega) = C^\infty(\Omega; \mathbb{C})$ for every $k \in \mathbb{N} \cup \{0\}$.

We will observe, using a method from [Treves 1, Treves 2, LY], that the global hypoellipticity of the differential complex generated by the operators above is equivalent to the global hypoellipticity of a simpler complex, namely, the one generated by the differential operators

$$L_{j,0} := \frac{\partial}{\partial t_j} + \frac{\partial Re\phi_0}{\partial t_j}(t)A, \ j = 1, 2, \cdots, n.$$

The local solvability of the transpose of this complex in top degree was firstly studied in [LY]. There, the authors consider a method, a result from [TOP] we might added, to get the local solvability and they assume that the leading coefficient is analytic. Here, we will just assume that the leading coefficient is C^∞ and use dynamic property to obtain the local hypoellipticity in the elliptic region and, after that, use some of the techniques developed in [BCM] to study the problem in the non elliptic one, only case we suppose the analyticity of ϕ_0.

To be more specific, we are going to explore the properties of the gradient system generated by the C^∞ function $Re\phi_0$, that is, the system

$$\begin{cases} t'(s) = -\nabla Re\ \phi_0(t(s)), \ s \geq 0, \\ t(0) = t_0 \in \Omega, \end{cases}$$

to get that for every point $t_0 \in \Omega \setminus \mathcal{E}$, where $\mathcal{E} := \{t^* \in \Omega : \nabla Re\phi_0(t^*) = 0\}$, there exists an open set $U \subset \Omega$ with $t_0 \in U$ and $U \cap \mathcal{E} = \varnothing$, such that for each $u \in C^\infty(U; H^{-\infty})$ which fulfill

$$\sum_{j=1}^{n} L_{j,0} u dt_j = f \text{ in } U,$$

with $f \in \Lambda^1 C^\infty(U; H^\infty)$, then u is actually in $C^\infty(U; H^\infty)$.

In order to do that, we need to clarify every concept in the set above and which we will work with in this paper.

We begin the work introducing, in a precise way, the complex of differential operators which we want to study and talking about its local hypoellipcity in the "elliptic region" and after that its hypoellipcity out of it.

2 The complex in study

Let $A : D(A) \subset H \to H$ be a self-adjoint linear operator, positive with $0 \in \rho(A)$, in a Hilbert space H with inner product $(\cdot, \cdot)_H$ and norm $\| \cdot \|_H$. Therefore, A is a sectorial operator with $Re\sigma(A) > 0$ (see [Henry], for a definition) and, for each real s, let H^s be its fractional power space associated, that is, for $s \geq 0$, $H^s := \{A^{-s}f : f \in H\}$ with inner product $(u, v)_s := (A^s u, A^s v)_H$, for $u, v \in H^s$, where the operador A^{-s} is given by

$$A^{-s} := \frac{1}{\Gamma(s)} \int_0^\infty \theta^{s-1} e^{-A\theta} d\theta,$$

the one which is injective whose inverse is denoted by $A^s : H^s \longrightarrow H$, being $\{e^{-A\theta} : \theta \geq 0\}$ the analytic semigroup generated by $-A$ and, for $s < 0$, H^s is the topological dual space of H^{-s}, that is, $H^s := (H^{-s})^*$.

That way, as the spaces H^s are Hilbert spaces, we obtain that for every real s, H^s and H^{-s} are the topological dual space one each other.

Now, we put $H^\infty := \bigcap_{s\in\mathbb{R}} H^s$, with the topology projective limit, we mean, the topology generated by the family of norms $(\| \cdot \|_s)_{s>0}$, and $H^{-\infty} := \bigcup_{s\in\mathbb{R}} H^s$, with the topology weak star, namely, the one such that: "a net $(x_\lambda)_{\lambda\in\Lambda}$ in $H^{-\infty}$ converges to $x \in H^{-\infty}$ if, and only if, the net $(\langle x_\lambda - x, u\rangle)_{\lambda\in\Lambda}$ converges to zero, in $\mathbb{C}$, when λ runs in directed set Λ, for every $u \in H^\infty$ ". That is, $H^{-\infty}$ is the topological dual space of H^∞.

When we have $A = 1 - \Delta : H^2(\mathbb{R}^N) \subset L^2(\mathbb{R}^N) \longrightarrow L^2(\mathbb{R}^N)$, A fulfill the properties above, the fractional power space are the usual Sobolev spaces in $\mathbb{R}^N$ and, as we well know, in this case, holds

$$H^\infty \subset C^\infty(\mathbb{R}^N) \cap L^2(\mathbb{R}^N) \text{ and } H^{-\infty} \subset \mathcal{D}'_{(F)}(\mathbb{R}^N) \cap \mathcal{S}'(\mathbb{R}^N),$$

where $\mathcal{D}'_{(F)}(\mathbb{R}^N)$ stands for the finite order distribution on $\mathbb{R}^N$ and $\mathcal{S}'(\mathbb{R}^N)$ for the tempered distribution on $\mathbb{R}^N$ (go to [Hormander, Treves] for a proof).

On the other hand, let

$$\phi(t, A) = \sum_{k=0}^{\infty} \phi_k(t) A^{-k},$$

with convergence in $\mathcal{L}(H)$, uniform in compacts of Ω, where Ω is an open set of $\mathbb{R}^n$, and $\phi_k \in C^\infty(\Omega)$ for every $k \in \mathbb{N} \cup \{0\}$.

We define, for $j = 1, 2, \cdots, n$, the differential operators $L_j : C^\infty(\Omega; H^\infty) \longrightarrow C^\infty(\Omega; H^\infty)$, by

$$L_j u := \frac{\partial u}{\partial t_j} + \frac{\partial \phi}{\partial t_j}(t, A) A u. \tag{1}$$

Taken the leading coefficient of $\phi(t, A)$, that is, $\phi_0 \in C^\infty(\Omega)$, we also define, for each $j = 1, 2, \cdots, n$, the differential operator $L_{j,0} : C^\infty(\Omega; H^\infty) \longrightarrow C^\infty(\Omega; H^\infty)$, by

$$L_{j,0} u := \frac{\partial u}{\partial t_j} + \frac{\partial Re\phi_0}{\partial t_j}(t) A u.$$

It is easy to see that, for each $j = 1, 2, \cdots, n$, the operator given by

$$L_{j,0}^{*}u := -\frac{\partial u}{\partial t_j} + \frac{\partial Re\phi_0}{\partial t_j}(t)Au,$$

is the adjoint of $L_{j,0}$.

Indeed, for if $\varphi, \psi \in C_c^\infty(\Omega; H^\infty)$, by the fact that A is self-adjoint, integrating by parts, we see

$$\langle L_{j,0}\varphi, \psi\rangle = \int_\Omega \left(\frac{\partial \varphi(t)}{\partial t_j} + \frac{\partial Re\phi_0}{\partial t_j}(t)A\varphi(t), \psi(t)\right)_H dt =$$

$$-\int_\Omega \left(\varphi(t), \frac{\partial \psi(t)}{\partial t_j}\right)_H dt + \int_\Omega \left(\varphi(t), \frac{\partial Re\phi_0}{\partial t_j}(t)A\psi(t)\right)_H dt = \langle \varphi, L_{j,0}^{*}\psi\rangle.$$

Observe that $supp(L_{j,0}u) \subset supp(u)$, for every $u \in C^\infty(\Omega)$.

The same way we can see that, for each $j = 1, 2, \cdots, n$, the operador

$$L_j^{*} = -\frac{\partial}{\partial t_j} + \frac{\partial \bar{\phi}}{\partial t_j}(t, A)A$$

is the adjoint of L_j, where $\bar{\phi}(t, A)$ is the series $\sum_{k=0}^{\infty} \bar{\phi}_k(t)A^{-k}$, whose the coefficients are the complex conjugated of the ones from $\phi(t, A)$.

That observation allows us to define L_j and $L_{j,0}$ on distributions, $L_j : \mathcal{D}'(\Omega; H^{-\infty}) \longrightarrow \mathcal{D}'(\Omega; H^{-\infty})$ and $L_{j,0} : \mathcal{D}'(\Omega; H^{-\infty}) \longrightarrow \mathcal{D}'(\Omega; H^{-\infty})$, putting

$$\langle L_j u, \varphi\rangle := \langle u, L_j^{*}\varphi\rangle, \text{ for } u \in \mathcal{D}'(\Omega; H^{-\infty}) \text{ and } \varphi \in C_c^\infty(\Omega; H^\infty)$$

and

$$\langle L_{j,0} u, \varphi\rangle := \langle u, L_{j,0}^{*}\varphi\rangle, \text{ for } u \in \mathcal{D}'(\Omega; H^{-\infty}) \text{ and } \varphi \in C_c^\infty(\Omega; H^\infty),$$

just recalling that $\mathcal{D}'(\Omega; H^{-\infty})$ is the topological dual space of $C_c^\infty(\Omega; H^\infty)$, where the last one is equipped with the inductive limit.

The operators L_j and $L_{j,0}$, defined above, can be used to define complexes of differential operator,

$$\mathbb{L} : \Lambda^p C^\infty(\Omega; H^\infty) \longrightarrow \Lambda^{p+1} C^\infty(\Omega; H^\infty),$$
$$\mathbb{L} : \Lambda^p \mathcal{D}'(\Omega; H^{-\infty}) \longrightarrow \Lambda^{p+1} \mathcal{D}'(\Omega; H^{-\infty})$$

$0 \le p \le n$, and

$$\mathbb{L}_0 : \Lambda^p C^\infty(\Omega; H^\infty) \longrightarrow \Lambda^{p+1} C^\infty(\Omega; H^\infty),$$
$$\mathbb{L}_0 : \Lambda^p \mathcal{D}'(\Omega; H^{-\infty}) \longrightarrow \Lambda^{p+1} \mathcal{D}'(\Omega; H^{-\infty})$$

also with $0 \le p \le n$, by

$$\mathbb{L}u := \sum_{|J|=p} \sum_{j=1}^{n} L_j u_J dt_j \wedge dt_J, \text{ for } u = \sum_{|J|=p} u_J dt_J$$

and

$$\mathbb{L}_0 u := \sum_{|J|=p} \sum_{j=1}^{n} L_{j,0} u_J dt_j \wedge dt_J, \text{ for } u = \sum_{|J|=p} u_J dt_J,$$

where $dt_J = dt_{j_1} \wedge \cdots \wedge dt_{j_p}$, $J = \{j_1 < \cdots < j_p\} \subset I_n = \{1, 2, \cdots, n\}$, are the basic elements from the canonical basis of the $C^\infty(\Omega)$-módulo $\Lambda^p C^\infty(\Omega)$.

Thus, we get the global form of these complexes

$$\mathbb{L}u := d_t u + \omega(t, A) \wedge Au, \tag{2}$$

and

$$\mathbb{L}_0 u := d_t u + Re\omega_0(t) \wedge Au, \tag{3}$$

with

$$\omega(t, A) := \sum_{k=0}^{\infty} \omega_k(t) A^{-k} \in \Lambda^p C^\infty(\Omega; \mathcal{L}(H))$$

where

$$\omega_k(t) := \sum_{j=1}^{n} \frac{\partial \phi_k}{\partial t_j}(t) dt_j,$$

for every non negative integer k, d_t stands for the exterior derivative in the t variable in Ω, being $u \in \Lambda^p C^\infty(\Omega; H^\infty)$ or $u \in \Lambda^p \mathcal{D}'(\Omega; H^{-\infty})$ and $Au := \sum_{|J|=p} Au_J dt_J$.

Consequently, $\mathbb{L} \circ \mathbb{L} = 0$ and $\mathbb{L}_0 \circ \mathbb{L}_0 = 0$, condition which defines the concept of a differential complex.

Of corse, just by restriction, we see that $\mathbb{L}$ and $\mathbb{L}_0$ define complexes on currents with coefficients in $C^\infty(\Omega; H^{-\infty})$ (see [Treves 2]), that is, we can look at

$$\mathbb{L} : \Lambda^p C^\infty(\Omega; H^{-\infty}) \longrightarrow \Lambda^{p+1} C^\infty(\Omega; H^{-\infty}), \text{ for } 0 \leq p \leq n$$

and

$$\mathbb{L}_0 : \Lambda^p C^\infty(\Omega; H^{-\infty}) \longrightarrow \Lambda^{p+1} C^\infty(\Omega; H^{-\infty}), \text{ for } 0 \leq p \leq n.$$

In these conditions, we can introduce the kind of hypoellipcity that we are going to work with

Definition 2.1. Let Ω be an open set of $\mathbb{R}^n$. Given U an open set of Ω, we say that an operator

$$\mathbb{M} : C^\infty(\Omega; H^{-\infty}) \longrightarrow \Lambda^1 C^\infty(\Omega; H^{-\infty})$$

is hypoelliptic in U, in the first degree, when for every distribution $u \in C^\infty(U; H^{-\infty})$ such that $\mathbb{M}u \in \Lambda^1 C^\infty(U; H^\infty)$, we actually have $u \in C^\infty(U; H^\infty)$.

When $\mathbb{M}$ is hypoelliptic in U, where $U = \Omega$, we say that $\mathbb{M} : C^\infty(\Omega; H^{-\infty}) \longrightarrow \Lambda^1 C^\infty(\Omega; H^{-\infty})$ is globally hypoelliptic (in Ω) and when $\mathbb{M} : C^\infty(\Omega; H^{-\infty}) \longrightarrow \Lambda^1 C^\infty(\Omega; H^{-\infty})$ is hypoelliptic in U, for every open set $U \subset \Omega$, we say that $\mathbb{M}$ is locally hypoelliptic in Ω.

We should say that, in this work, our concern is the regularity of the distributions $u \in C^\infty(\Omega; H^{-\infty})$ in the "x variable", we mean, the regularity relatively to the scale of spaces H^s where the distributions have theirs range.

To be more precise, in this work, we are not able, yet, to show in the more general framework that $\mathbb{L} : C^\infty(\Omega; H^{-\infty}) \longrightarrow \Lambda^1 C^\infty(\Omega; H^{-\infty})$ is globally hypoelliptic in the whole Ω. What we actually are going to do is to show that $\mathbb{L}$ is locally hypoelliptic in $\Omega_0 := \Omega \backslash \mathcal{E}$, where $\mathcal{E} := \{t^* \in \Omega : \nabla Re\phi_0(t^*) = 0\}$, set we will call the *elliptic region* of $\mathbb{L}$ and $\mathbb{L}_0$, and after that, using the techniques we have learned from [BCM], we will consider $A := 1 - \Delta$ and get the global hypoellipcity for the $\mathbb{L}$ associated.

In other words, in the general case, we do not have the total information about $\mathbb{L}$ which allows us to obtain its global hypoellipcity in Ω, but our knowledge of the dynamics properties of the solution of the Cauchy problem

$$\begin{cases} t' = -\nabla Re\phi_0(t), \ s \geq 0, \\ \quad t(0) = t_0 \in \Omega, \end{cases}$$

will give us the local hypoellipticity in Ω_0 and the nature, or noble structure, of the operator $1 - \Delta$ will be used to solve the problem out of Ω_0, that is, in some neighborhood of $\mathcal{E}$.

The analysis we will do bellow in Ω_0 will be strongly inspired in the study made in [Hounie], where the author considers the same kind of problem as us, but only in one dimension, getting complete characterization of the global hypoellipticity, in the abstract framework, by the conditions (ψ) and (τ). Such conditions, however, we will not assume, explicitly, here.

Before we start to study the hypoellipticity of the operator $\mathbb{L}$ let us point out that, as it was done in [Treves 1, Treves 2, LY], we can isolate the "principal part" of $\mathbb{L}$ and conclude that, to study its hypoellipticity is equivalent to study the hypoellipticity of the simpler operator $\mathbb{L}_0$.

Lemma 2.2. *For each $0 \leq p \leq n$ and each open set $U \subset \Omega$,*

$$\mathbb{L} : \Lambda^p C^\infty(U; H^{-\infty}) \longrightarrow \Lambda^{p+1} C^\infty(U; H^{-\infty})$$

is hypoelliptic in U if and only if

$$\mathbb{L}_0 : \Lambda^p C^\infty(U; H^{-\infty}) \longrightarrow \Lambda^{p+1} C^\infty(U; H^{-\infty})$$

is hypoelliptic in U.

Proof. We just have to define, for each $t \in \Omega$, the operator

$$\alpha(t, A) := Re\phi_0(t) - \phi(t, A) = [\phi_0(t) - \phi(t, A)] - iIm\phi_0(t)$$

and to observe that the composition $\alpha(t, A)A$ is the sum of an operador type Schrödinger (hence, infinitesimal generator of a group of linear operators, see [Pazy]) with a bounded.

Therefore, we can define the operator $U(t) := e^{\alpha(t,A)A}$, $t \in \Omega$.

Thus, this one can be used to generate an automorphism of $\Lambda^p C^\infty(U; H^\infty)$ and $\Lambda^p C^\infty(U; H^{-\infty})$, for each $0 \leq p \leq n$, putting

$$(\mathcal{U}u)(t) := U(t)u(t) = e^{\alpha(t,A)A}u(t), \text{ for } u \in C^\infty(U; H^\infty) \text{ and } t \in U.$$

It is not hard to see that $\mathcal{U} : C^\infty(U; H^\infty) \longrightarrow C^\infty(U; H^\infty)$ defines an automorphism, because $e^{\alpha(t,A)A}$ is invertible for every $t \in \Omega$, which extends to an other $\mathcal{U} : C^\infty(U; H^{-\infty}) \longrightarrow C^\infty(U; H^{-\infty})$, just by taking its adjoint.

From the definition of $\mathcal{U}$ it is just a calculation to get, for $j = 1, 2, \cdots, n$, the equality

$$[L_j(\mathcal{U}u)](t) = [\mathcal{U}(L_{j,0}u)](t), \text{ for } u \in C^\infty(U; H^\infty) \text{ and } t \in U. \tag{4}$$

If we define, for $u = \sum_{|J|=p} u_J dt_J$,

$$\mathcal{U}u := \sum_{|J|=p} (\mathcal{U}u_J)\, dt_J$$

the equality (4) tells us that

$$\mathbb{L}(\mathcal{U}u) = (\mathcal{U}\mathbb{L}_0)\, u, \text{ for } u \in C^\infty(U; H^\infty).$$

As the same equality above it is true for $u \in C^\infty(U; H^{-\infty})$, our claim holds. □

3 The main Theorems

We begin our contribution introducing a very simple result, from the ordinary differential equations theory, which the proof will be left to the reader.

Lemma 3.1. *Let $\phi_0 \in C^\infty(\Omega)$, consider the Cauchy problem*

$$\begin{cases} t'(s) = -\nabla Re\phi_0(t(s)), \; s \geq 0, \\ t(0) = t_0 \in \Omega, \end{cases} \tag{5}$$

and let $\mathcal{E} := \{t^* \in \Omega : \nabla Re\phi_0(t^*) = 0\}$ be the set of all equilibrium points of it.

If, for each $t_0 \in \Omega$, $\omega(t_0) > 0$ indicates the maximal time of existence of the solution $T(s)t_0$, $s > 0$, of this problem, then for each $t_0 \in \Omega_0 := \Omega \setminus \mathcal{E}$ and $\delta > 0$ with $d(t_0, \mathcal{E}) > 2\delta$, there exist an open set $U \subset \Omega$ with $t_0 \in U$ and $\tau > 0$, such that:

(i) $\omega(t) \geq \tau$ for every $t \in U$,

(ii) $T(s)U \subset \mathcal{O}_\delta(\mathcal{E} \cup \partial\Omega)$ whenever $s \geq \tau$ [1],

[1]When $X \subset \Omega$, the symbol $\mathcal{O}_\delta(X)$ stands for the union of all open balls with radius $\delta > 0$ and center in some point of X.

(iii) $T(s)U \subset \Omega_0$ *when* $0 \le s \le \tau$ *and*

(iv) $U \cap \mathcal{O}_\delta(\mathcal{E} \cup \partial\Omega) = \varnothing$.

As we have seen in the Lemma 2.2, we just need to study the complex generated by $\mathbb{L}_0$. That fact will be implicit in the results we establish below.

Theorem 3.2. *In the conditions above, given* $t_0 \in \Omega \backslash \mathcal{E}$*, there exists an open set* $U \subset \Omega\backslash\mathcal{E}$*, with* $t_0 \in U$*, such that* $\mathbb{L}$ *is hypoelliptic in* U*.*

Proof. Indeed, given $t_0 \in \Omega_0 = \Omega \setminus \mathcal{E}$ and $\delta > 0$ with $d(t_0, \mathcal{E}) > 2\delta$, let U and $\tau > 0$ be the ones given by the lemma above.

Also, let $\{e^{-sA} : s \ge 0\}$ be the analytic semigroup generated by the minus sectorial operator $-A$. As we well known, $e^{-As}u \in H^\infty$ for every $u \in H^{-\infty}$ whenever $s > 0$ (see [Henry]).

Now, for $\omega \in \Lambda^1 C^\infty(U; H^\infty)$ (or $\omega \in \Lambda^1 C^\infty(U; H^{-\infty})$) and for $t \in U$, inspired in the work [Hounie], we define the linear operator

$$(K\omega)(t) := -\int_{\gamma_t} e^{Re(\phi_0(z)-\phi_0(t))A}\omega(z)dz, \tag{6}$$

where the integration path is $\gamma_t(s) := T(s)t$, $s \in [0, \tau]$.

The same way, we can define K in each open subset W of U.

We have to say that, the value $(K\omega)(t)$ is well defined because the function $Re\phi_0$ is a Lyapunov function for the Cauchy problem (5), so $Re\phi_0(T(s)t) \le Re\phi_0(t)$ for every $s \in [0, \tau]$ and $t \in U$, hence we may apply the semigroup $\{e^{-As} : s \ge 0\}$ in $s = -Re\,(\phi_0(T(s)t) - \phi_0(t)) \ge 0$ and, for the case when $\omega \in \Lambda^1 C^\infty(U; H^{-\infty})$, $H^{-\infty}$, endowed with the weak star topology, is complete.

Besides, it is not hard to see that K maps $\Lambda^1 C^\infty(U'; H^\infty)$ into $C^\infty(U'; H^\infty)$ and $\Lambda^1 C^\infty(U'; H^{-\infty})$ into $C^\infty(U'; H^{-\infty})$, for every open subset $U' \subset U$.

On the other hand, let $g \in C_c^\infty(U; H^{-\infty})$, consider $\mathbb{L}_0 g \in \Lambda^1 C^\infty(U; H^{-\infty})$ and define $K(\mathbb{L}_0 g)$.

From this, for every $t \in U$ we have, by Lemma 3.1, that $T(\tau)t \notin U$ hence $T(\tau)t \notin$ *supp*(g), so integrating by parts and using the fact that $T(s)t$ is the solution of (5), we see that for $t \in U$

$$[K(\mathbb{L}_0 g)](t) = -\int_{\gamma_t} e^{Re(\phi_0(z)-\phi_0(t))A}\left(\mathbb{L}_0 g\right)(z)dz =$$

$$-\int_{\gamma_t} e^{Re(\phi_0(z)-\phi_0(t))A}(d_t g)(z)dz - \int_{\gamma_t} e^{Re(\phi_0(z)-\phi_0(t))A}\omega_0(z) \wedge Ag(z)dz =$$

$$-\left[e^{Re(\phi_0(T(s)t)-\phi_0(t))A}g(z)\right]\Big|_{s=0}^{\tau} +$$

$$\int_{\gamma_t} e^{Re(\phi_0(z)-\phi_0(t))A}\omega_0(z)\wedge Ag(z)dz - \int_{\gamma_t} e^{Re(\phi_0(z)-\phi_0(t))A}\omega_0(z)\wedge Ag(z)dz =$$
$$-\left[e^{Re(\phi_0(T(\tau)t)-\phi_0(t))A}g(T(\tau)t) - e^{Re(\phi_0(t)-\phi_0(t))A}g(t)\right] = g(t).$$

In resume

$$[K(\mathbb{L}_0 g)](t) = g(t), \text{ for every } t \in U. \tag{7}$$

Thus, if $u \in C^\infty(U; H^{-\infty})$ has $\mathbb{L}_0 u = f \in \Lambda^1 C^\infty(U; H^\infty)$, for each $t^{'} \in U$ we may choose $\varphi \in C_c^\infty(U;\mathbb{R})$, with $\varphi = 1$ in some neighborhood of $U^{'}$ of $t^{'}$. Then, $g := \varphi u \in C_c^\infty(U; H^{-\infty})$ and we have

$$\mathbb{L}_0(\varphi u) = \varphi \mathbb{L}_0 u + \sum_{j=1}^n \frac{\partial \varphi}{\partial t_j}(t)u(t)dt_j = \varphi f + \sum_{j=1}^n \frac{\partial \varphi}{\partial t_j}(t)u(t)dt_j.$$

So, by (7), we have

$$\left[K\left(\varphi f\right)\right](t) + \left[K\left(\sum_{j=1}^n \frac{\partial \varphi}{\partial t_j}udt_j\right)\right](t) = \left[K\mathbb{L}_0(\varphi u)\right](t) = (\varphi u)(t), \text{ for all } t \in U.$$

Since $\varphi f \in \Lambda^1 C^\infty(U; H^\infty)$, we have $K(\varphi f) \in C^\infty(U; H^\infty)$. So if we show that

$$K\left(\sum_{j=1}^n \frac{\partial \varphi}{\partial t_j}udt_j\right)$$

is in $C^\infty(U^{'}; H^\infty)$, then the theorem follows, once $U^{'}$ was arbitrary.

Indeed for, on one hand, since φ is constant in $U^{'}$ we have $\sum_{j=1}^n \frac{\partial\varphi}{\partial t_j}(r)u(r)dt_j = 0$ as long as $r \in U^{'}$.

On the other, for each $t^{'} \in U^{'}$ there exist an neighborhood $V^{'}$ in $U^{'}$, for it, and $\tau_1 > 0$ such that $T(s)t \in U^{'}$ whenever $s \in [0, \tau_1]$ and $t \in V^{'}$.

So, for $t \in V^{'}$

$$K\left(\sum_{j=1}^n \frac{\partial \varphi}{\partial t_j}u\right)(t) = -\int_{\tau_1}^{\tau} e^{Re(\phi_0(T(s)t)-\phi_0(t)+\eta)A}\left[e^{-\eta A}\left(\sum_{j=1}^n \frac{\partial \varphi}{\partial t_j}(T(s)t)u(T(s)t)\frac{dT(s)t}{ds}\right)\right]ds,$$

where $\eta := Re(\phi_0(t) - \phi_0(T(s)\tau_1)) > 0$.

Observe that $\eta > 0$, because for $t \in U$ fixed, we only have $Re\phi_0(T(s)t) = Re\phi_0(t)$ to a finite number of s in $[0, \tau]$. Otherwise, there exists a sequence $(s_j)_{j\in\mathbb{N}}$ in $[0,\tau]$ with $s_j \longrightarrow s_0 \in [0,\tau]$, so $\nabla Re\phi_0(T(s_0)t) = 0$, that is, $T(s_0)t \in \mathcal{E}$, but it cannot be true, because $T(s)U \subset \Omega_0$ when $0 \le s \le \tau$.

Finally, it is not hard to see that, if $\alpha \in \mathbb{R}$ is fixed, for every $h \in C^\infty([\tau_1,\tau]; H^{-\infty})$ we have that $e^{-\eta A}h \in C^\infty([\tau_1,\tau]; H^\alpha)$ and, by that,

$$e^{Re(\phi_0(T(s)t)-\phi_0(t)+\eta)A}[e^{-\eta A}h] \in C^\infty([\tau_1,\tau]; H^\infty).$$

Putting all this results together we get that, for every $t \in U'$ holds

$$(\varphi u)(t) = K(\varphi f)(t) - \int_{\tau_1}^{\tau} e^{Re(\phi_0(T(s)t)-\phi_0(t)+\eta)A}[e^{-\eta A}h(T(s)t)]\frac{dT(s)t}{ds}ds,$$

where $h(s) = \sum_{j=1}^{n} \frac{\partial \varphi}{\partial t_j}(T(s)t)u(T(s)t)$, so the second term in the sum above defines also an element of $C^\infty(U'; H^\infty)$, therefore $\varphi u \in C^\infty(U'; H^\infty)$. But $\varphi u = u$ in U' and the proof is complete. □

As we saw in the theorem above, we have not gave the answer to our problem for points in the set $\mathcal{E}$, yet. However, the next result shows us that, it might exists points in $\mathcal{E}$ where we can not obtain the hypoellipcity.

Proposition 3.3. If $t^* \in \mathcal{E}$ is a local minimal point for $Re\phi_0$, then t^* possess a neighborhood V in Ω where $\mathbb{L}$ is not hypoelliptic.

Proof. Indeed, let V be an open set of Ω where $Re\phi_0(t^*) \leq Re\phi_0(t)$ for all $t \in V$.

Take $u_0 \in H \setminus H^\infty$ and define $u : V \longrightarrow H^{-\infty}$ by

$$u(t) := e^{Re(\phi_0(t^*)-\phi_0(t))A}u_0, \ t \in V.$$

It follows that u is well defined and $u \in C^\infty(V; H^{-\infty})$.

Now, it is pretty easy to see that $\mathbb{L}_0 u = 0$ in V, so $\mathbb{L}_0 u \in \Lambda^1 C^\infty(V; H^\infty)$. However, since $u(t^*) = u_0 \notin H^\infty$, we do not have $u \in C^\infty(V; H^\infty)$, and the claim is true. □

Remark 3.4. It is easy to see that, when $t^* \in \mathcal{E}$ is an isolated saddle point, then $Re\phi_0$ is an open map in same neighborhood of t^*.

We finish this section restricting us to the case where the operator $A : \mathcal{D}(A) \subset H \longrightarrow H$ and the Hilbert space H are $A = 1 - \Delta$, $\mathcal{D}(A) = H^2(\mathbb{R}^N)$ and $H = L^2(\mathbb{R}^N)$, the ones which have the properties we have consider in the abstract framework above.

The reason that leads us to do this hypothesis is the fact that, the nature of this operator in the L^2 situation, allows us to use the Fourier transform to get the regularity of the solutions of the equation $\mathbb{L}u = f$ by studing its Fourier transform decay rate in the infinity, the same way the authors do to lay down the work [BCM].

Just by completeness of this paper, we write bellow the technical lemma showed in [BCM] and which we are also going to need here, with a little alteration, which does not change its proof.

Lemma 3.5 (Lemma 4.4 in [BCM])**.** *Suppose that $Re\phi_0$ is an analytic function.*

Let $t^* \in \mathcal{E}$ and B an open ball contained in Ω such that $B \cap \mathcal{E}$ is **connected by piecewise smooth paths** *and take $t_0 \in B \cap \mathcal{E}$. Then there exist*

(a) An open neighborhood $B^* \subset B$ of t^*;

(b) A constant $K > 0$ and $\varepsilon > 0$;

(c) A family $(\gamma_t)_{t\in B^*}$ of piecewise smooth path $\gamma_t : [0,1] \longrightarrow B$, such that:

(I) $\gamma_t(0) = t$, for every $t \in B^*$;

(II) $Re\phi_0(\gamma_t(s)) \leq Re\phi_0(t)$, for all $s \in [0,1]$ and all $t \in B^*$;

(III) The length $l(\gamma_t)$ of γ_t is such that $l(\gamma_t) \leq K$ for all $t \in B^*$;

(IV) If $t \in B^*$, then one of the following properties holds:

$(IV)_1$ $\gamma_t(1) = t_0$,

$(IV)_2$ $Re\phi_0(\gamma_t(1)) \leq Re\phi_0(t) - \varepsilon$.

The reader must to observe that we have made a little alteration in the statement of the Lemma 3.5, more precisely, we have introduce the hypotheses "$B\cap\mathcal{E}$ is **connected by piecewise smooth paths**" instead "$B\cap\mathcal{E}$ is **connected**", only, as the authors consider there. We made this because our data $Re\phi_0$ not need to be constant equals to zero on $\mathcal{E}$, as they have there, but the fact that "$B\cap\mathcal{E}$ is connected by piecewise smooth paths" allows us to get that $Re\phi_0$ is constant on $B\cap\mathcal{E}$, alteration which does not change the proof that we meet in [BCM].

Other thing, the hypotheses "$B\cap\mathcal{E}$ is connected by piecewise smooth paths" is always fulfill when $\mathcal{E}$ is discrete, just taking B with radius as small as it needs to be $B\cap\mathcal{E}$ a singleton.

Finally, the proof of the Lemma 3.5 lies on the Lojasiewicz-Simon Inequality, which can be obtained without the hypothesis of analyticity of $Re\phi_0$ if we suppose, for example, that second derivative of $Re\phi_0$ in $t^* \in \mathcal{E}$ is a isomorphism, as we can see in [AAPG].

We are now in position to proof our final theorem.

Theorem 3.6. *Suppose that $Re\phi_0$ is an analytic function.*

Let $A = 1 - \Delta : H^2(\mathbb{R}^N) \subset L^2(\mathbb{R}^N) \longrightarrow L^2(\mathbb{R}^N)$, $u \in C^\infty(\Omega; H^{-\infty})$ with $\mathbb{L}_0 u = f \in \Lambda^1 C^\infty(\Omega; H^\infty)$, $t^* \in \mathcal{E}$ and suppose that one of the following properties holds:

(i) $Re\phi_0$ is an open map at t^*.

(ii) There is $t_0 \in B\cap\mathcal{E}$ such that $u(t_0,\cdot) \in H^\infty$, where B is taken from Lemma 3.5.

Then, $u \in C^\infty(B^* \times \mathbb{R}^N)$ for some neighborhood $B^* \subset B$ of t^*.

Proof. Well, applying the Fourier transform in variable $x \in \mathbb{R}^N$ to the equality $\mathbb{L}_0 u = f$ we get

$$d_t\hat{u} + \omega_0(t) \wedge a(\xi)\hat{u} = \hat{f}, \text{ for } t \in B, \tag{8}$$

where the "hat" stands for the Fourier transform in the variable x, $a(\xi) = 1 + 4\pi^2|\xi|^2$ is the symbol of the operator $1 - \Delta$ and B^* is the one gave in the last lemma.

Multiplying the equality (8) by $e^{a(\xi)Re\phi_0(t)}$ and using the product rule we may write

$$d_t\left(e^{a(\xi)Re\phi_0(t)}\hat{u}(t,\xi)\right) = e^{a(\xi)Re\phi_0(t)}\hat{f}(t,\xi), \text{ for all } t \in B \text{ and all } \xi \in \mathbb{R}^N.$$

Also by Lemma 3.5, considering the family of paths $(\gamma_t)_{t\in B^*}$ and integrating the equality above along γ_t, for $t \in B^*$ and $\xi \in \mathbb{R}^N$, we get

$$e^{a(\xi)Re\phi_0(\gamma_t(1))}\hat{u}(\gamma_t(1),\xi) - e^{a(\xi)Re\phi_0(t)}\hat{u}(t,\xi) = \int_{\gamma_t} d_t\left(e^{a(\xi)Re\phi_0(z)}\hat{u}(z,\xi)\right) = \int_{\gamma_t} e^{a(\xi)Re\phi_0(z)}\hat{f}(z,\xi),$$

so, for all $t \in B^*$ and $\xi \in \mathbb{R}^N$, holds

$$\hat{u}(t,\xi) = e^{a(\xi)[Re\phi_0(\gamma_t(1)) - Re\phi_0(t)]}\hat{u}(\gamma_t(1),\xi) - \int_{\gamma_t} e^{a(\xi)[Re\phi_0(z) - Re\phi_0(t)]}\hat{f}(z,\xi)dz,$$

hence

$$|\hat{u}(t,\xi)| \leq e^{a(\xi)[Re\phi_0(\gamma_t(1)) - Re\phi_0(t)]}|\hat{u}(\gamma_t(1),\xi)| + \left|\int_{\gamma_t} e^{a(\xi)[Re\phi_0(z) - Re\phi_0(t)]}\hat{f}(z,\xi)dz\right|. \tag{9}$$

In this point, we divide the proof in two cases.

Case one. The conclusion $(IV)_1$ of the Lemma 3.5 holds:

In this case, we use the hypothesis (ii), therefore for every $s \in \mathbb{R}$ we have that

$$(1+|\xi|^2)^{s/2}\hat{u}(t_0,\cdot) \in L^2(\mathbb{R}^N) \tag{10}$$

Thanks to the fact that $f \in \Lambda^1 C^\infty(\Omega; H^\infty)$, for every $s \in \mathbb{R}$ we also have

$$(1+|\xi|^2)^{s/2}\hat{f}_j(t,\cdot) \in L^2(\mathbb{R}^N)$$

for all $t \in \Omega$ (in particular, for $t \in B^*$) and the map $\Omega \ni t \mapsto f_j(t,\cdot) \in H^\infty$ is C^∞, for all j, where we have writen $f = \sum_{j=1}^n f_j dt_j$.

Thus, using theses facts and the conclusion (III) from Lemma 3.5 in the inequality (9) we obtain, for each real s, all $\xi \in \mathbb{R}^N$ and $t \in B^*$

$$(1+|\xi|^2)^{s/2}|\hat{u}(t,\xi)| \leq (1+|\xi|^2)^{s/2}|\hat{u}(t_0,\xi)| + \left|\int_{\gamma_t}(1+|\xi|^2)^{s/2}\hat{f}(z,\xi)dz\right|. \tag{11}$$

Now, observe that, by the Minköwski inequality for integrals we have

$$\left(\int_{\mathbb{R}^N}\left|\int_{\gamma_t}(1+|\xi|^2)^{s/2}\hat{f}(z,\xi)dz\right|^2 d\xi\right)^{1/2} \leq$$

$$\int_{\gamma_t}\left(\int_{\mathbb{R}^N}(1+|\xi|^2)^s|\hat{f}(z,\xi)|^2 d\xi\right)^{1/2}|dz| \leq K\sup_{z\in B}\|f(z,\cdot)\|_{H^s} < \infty.$$

This and (10) give us that $(1+|\xi|^2)^{s/2}|\hat{u}(t,\cdot)| \in L^2(\mathbb{R}^N)$ for all real s.

Case two. The conclusion $(IV)_2$ of the Lemma 3.5 holds:

In this situation, by Lemma 3.5, we are actually using the hypothesis (i) so, the estimate (9) gives us, for each real s,

$$(1+|\xi|^2)^{s/2}|\hat{u}(t,\xi)| \leq (1+|\xi|^2)^{s/2}e^{-\varepsilon a(\xi)}|\hat{u}(\gamma_t(1),\xi)| + \left|\int_{\gamma_t}(1+|\xi|^2)^{s/2}\hat{f}(z,\xi)dz\right|. \quad (12)$$

From where we see that, to take care of $\left|\int_{\gamma_t}(1+|\xi|^2)^{s/2}\hat{f}(z,\xi)dz\right|$ we may use the same method we have used in the Case one and, since

$$(1+|\xi|^2)^{\alpha/2}|\hat{u}(\gamma_t(1),\cdot)| \in L^2(\mathbb{R}^N)$$

for same real α, the exponencial decay of $e^{-\varepsilon a(\xi)}$ gives us that, for every real s,

$$(1+|\xi|^2)^{s/2}e^{-\varepsilon a(\xi)}|\hat{u}(\gamma_t(1),\cdot)| \in L^2(\mathbb{R}^N),$$

hence

$$\|u(t,\cdot)\|_{H^s} \leq \|(1+|\xi|^2)^{s/2}e^{-\varepsilon a(\xi)}\hat{u}(\gamma_t(1),\cdot)\|_{L^2} + K\sup_{z\in B}\|f(z,\cdot)\|_{H^s} < \infty$$

for all $t \in B^*$ and $s \in \mathbb{R}$, completing the proof of this case.

From the cases we have studied above, we conclude that $u(t,\cdot) \in H^\infty \subset C^\infty(\mathbb{R}^N)$ for all $t \in B^*$.

Finally, differentiating with respect to t_k the equation $\mathbb{L}_0 u = f$ we get

$$\frac{\partial}{\partial t_k}\left(\frac{\partial u}{\partial t_j}\right)(t) + \frac{\partial Re\phi_0}{\partial t_k}(t)A\left(\frac{\partial u}{\partial t_j}(t)\right) = \frac{\partial f_j}{\partial t_k}(t) - \frac{\partial^2 Re\phi_0}{\partial t_k \partial t_j}(t)Au(t),$$

so we can repeat the procedure we have made above to conclude that

$$\frac{\partial u}{\partial t_j}(t,\cdot) \in H^\infty \subset C^\infty(\mathbb{R}^N)$$

for all $t \in B^*$, thus the induction will shows us the $u \in C^\infty(B^* \times \mathbb{R}^N)$, and the proof is done.

□

4 Final Comments

We must do some comments to ensure to the reader that the question we have treat here was not done in [Treves 2] because, even the kind of problem treated there is similar to that we study here, the structure of the operator we consider is different to that was seem there.

For example, our operador A is an abstract one in the Hilbert space framework, abstract as well, whereas in [Treves 2] he considers a different class of operators in the specific space $\mathcal{F}^2_{loc}(\mathbb{R}^n)$, topological dual space of the space $\mathcal{F}^2_c(\mathbb{R}^n)$, the one which is a inductive limite of Hilbert spaces.

There, the author does a systematic study of the problem $d_t u + b(t, D_x) \wedge u = f$, for $u \in \mathcal{F}^2_{loc}(\mathbb{R}^n)$, where $b(t, D_x) : \mathcal{F}^2_{loc}(\mathbb{R}^n) \longrightarrow \mathcal{F}^2_{loc}(\mathbb{R}^n)$, $t \in \Omega$, is a pseudo-differential operator which has no need to be in the same class as our operator $Re\phi_0(t)A : D(A) : H \longrightarrow H$, $t \in \Omega$.

Other situation we must point out is that, if the operator $A : D(A) \subset H \to H$ fulfill all the properties we have made above to proof the Theorem 3.2 and, beside these, H is separable and A^{-1} is compact, as we well known, in this case, the operator A admits the spectral resolution

$$Au = \sum_{j=1}^{\infty} \lambda_j P_j u, \;\; u \in D(A),$$

where the λ_j's are the auto-values of A and $P_j : H \longrightarrow E_j$ are the sequence of projections into the auto-spaces E_j corresponding and the semigroup analytic writs like this

$$e^{-As}u = \sum_{j=1}^{\infty} e^{-\lambda_j s} P_j u, \;\; u \in H.$$

In this situation, for $s \geq 0$, the spaces H^s admits the following characterization

$$H^s = \left\{ u \in H : \left(\lambda_j^s \|P_j u\|_H \right)_{j \in \mathbb{N}} \in l^2(\mathbb{N}) \right\}, \tag{13}$$

and is equipped with the norm

$$H^s \ni u \mapsto \|u\|_s := \left(\sum_{j=1}^{\infty} \lambda_j^{2s} \|P_j u\|_H^2 \right)^{1/2}. \tag{14}$$

Also, for $s < 0$ the space H^s is the topological dual space of H^{-s} or even the completion of the set H_s defined the same way as (13) with respect to the norm $\|\cdot\|_s$ defined just as (14).

For each $j \in \mathbb{N}$, it is possible to extend the projection $P_j : H \longrightarrow E_j$ to a new projection $\tilde{P}_j : H^{-\infty} \longrightarrow E_j$. Therefore, in these conditions, considering the differential operator $\mathbb{L}$ associated to the operator A, we see that to get the regularity of the solutions of the equation $\mathbb{L}u = f$ we just have to study the decay behavior of the sequences $\left(\lambda_j^s \|\tilde{P}_j u(t)\|_H \right)_{j \in \mathbb{N}}$ the same way as we have done in the Theorem 3.6, that is, to proof that this sequence is in $l^2(\mathbb{N})$ for every real s. This way, the same proof we gave for the Theorem 3.6 applies to this case and we can state:

Theorem 4.1. *Beside the hypothesis we have made for the operator $A : D(A) \subset H \to H$, suppose also that H is separable and A^{-1} is compact.*

If $u \in C^\infty(\Omega; H^{-\infty})$ verify $\mathbb{L}u = f$ with $f \in \Lambda^1 C^\infty(\Omega; H^\infty)$, for $t^* \in \mathcal{E}$ suppose that one of the following properties holds:

(i) $Re\phi_0$ is an open map at t^*

(ii) There is $t_0 \in B \cap \mathcal{E}$ such that $u(t_0, \cdot) \in H^\infty$.

Then, $u \in C^\infty(\Omega; H^\infty)$.

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