
**ON APPROXIMATION TOOLS AND DECAY RATES FOR EIGENVALUES
SEQUENCES OF CERTAIN OPERATORS ON A GENERAL SETTING**

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On approximation tools and decay rates for eigenvalues sequences of certain operators on a general setting

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The present paper brings a sweeping generalization of very new results obtained on the spherical framework exchanging unit spheres by compact two-point homogeneous spaces. We first prove a convenient characterization of a K -functional on this framework given by the rate of approximation of mean operators. Later, we apply such result in order to show that an abstract Hölder condition or finite order of differentiability assumption on kernels generating positive integral operators implies a sharp polynomial decay rates for eigenvalues sequences of such operators.

1 Introduction

The basic framework here refers to a compact two-point homogeneous space \mathbb{M} of dimension $m \geq 1$. Such space is both a Riemannian m -manifold and a compact symmetric space of rank 1 for which there is a well-developed harmonic analysis structure. A very large class of problems in approximation theory, harmonic analysis and functional analysis (as it can be seen in the present paper) can be considered naturally on these spaces.

Two-point homogeneous spaces can be represented as the quotient \mathcal{L}/S_O where O is a fixed point in \mathbb{M} , \mathcal{L} is a compact Lie group related to the identity component of the isometry group of \mathbb{M} and S_O is the stationary subgroup of the point O . Let e be the identity of \mathcal{L} , $\pi : \mathcal{L} \rightarrow \mathcal{L}/S_O$ the natural mapping then the *pole of \mathbb{M}* , $o := \pi(e)$, is invariant under all motions of S_O . Each one of these manifolds \mathbb{M} has an invariant Riemannian metric $d(\cdot, \cdot)$ and a measure dx induced by the normalized left Haar measure on L which is invariant under the action of \mathcal{L} . Also, these spaces admit essentially one invariant second order differential operator called Laplace-Beltrami operator. We suggest [10, 5, 15, 18, 19, 24, 28] and references therein for more detailed information about these spaces. Important properties are described bellow in a summarised way and can be found in references above.

According to Wang [33], the spaces we are taking in account here are: the unit spheres \mathbb{S}^m , $m = 1, 2, \dots$; the real projective spaces $\mathbb{P}^m(\mathbb{R})$, $m = 2, 3, \dots$; the complex projective spaces $\mathbb{P}^m(\mathbb{C})$, $m = 4, 6, \dots$; the quaternion projective spaces $\mathbb{P}^m(\mathbb{H})$, $m = 8, 12, \dots$ and 16-dimensional Cayley's elliptic plane \mathbb{P}^{16} . These spaces have a very similar geometry and we shall assume here that $\mathbb{M} \neq \mathbb{P}^m(\mathbb{R})$. We do not have any loss assuming that because the problems of harmonic analysis on the real projective spaces $\mathbb{P}^m(\mathbb{R})$ can be reduced to the corresponding problems on the spheres \mathbb{S}^m ([29]), and the results we will present here already have their spherical version studied ([6, 7, 20]).

A function on \mathbb{M} , identified in \mathcal{L}/S_O , is invariant under the left action of S_O on \mathcal{L} if, and only if, it depends only upon the distance of its argument from the pole of \mathbb{M} . Let θ be the distance

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of a point from the pole. One can choose a geodesic polar coordinate system (θ, u) , where u is an angular parameter, in which the radial part of Δ can be written, up to a multiplicative constant, as

$$\Delta_\theta = \frac{1}{(\sin \lambda\theta)^\sigma (\sin 2\lambda\theta)^\rho} \frac{d}{d\theta} (\sin \lambda\theta)^\sigma (\sin 2\lambda\theta)^\rho \frac{d}{d\theta},$$

in which $\lambda = \pi/2l$, $l = \max \{d(x, y) : x, y \in \mathbb{M}\}$, and

\mathbb{S}^m	$\sigma = 0$	$\rho = m - 1$
$\mathbb{P}^m(\mathbb{R})$	$\sigma = m - 1$	$\rho = 0$
$\mathbb{P}^m(\mathbb{C})$	$\sigma = m - 2$	$\rho = 1$
$\mathbb{P}^m(\mathbb{H})$	$\sigma = m - 4$	$\rho = 3$
\mathbb{P}^{16}	$\sigma = 8$	$\rho = 7$

Furthermore, the change of variables $x = \cos 2\lambda\theta$ gives us

$$\Delta_x = (1-x)^{-\alpha} (1+x)^{-\beta} \frac{d}{dx} (1-x)^{1+\alpha} (1+x)^{1+\beta} \frac{d}{dx},$$

with $\alpha = (\sigma + \rho - 1)/2 = (m - 2)/2$ and $\beta = (\rho - 1)/2$. We define $\mathcal{B} = -\Delta_x$ and also call it *Laplace-Beltrami operator* on spaces of functions defined on \mathbb{M} .

Restricting ourselves to $m \geq 2$ and $1 \leq p \leq \infty$ we write $(L^p(\mathbb{M}), \|\cdot\|_p)$ the usual Banach spaces of p -integrable complex functions on \mathbb{M} . In particular, $L^2(\mathbb{M})$ is the Hilbert space of all square-integrable functions on \mathbb{M} endowed with the inner product

$$\langle f, g \rangle_2 := \frac{1}{\sigma_m} \int_{\mathbb{M}} f(x) \overline{g(x)} dx, \quad f, g \in L^2(\mathbb{M}),$$

where σ is a normalizing constant given by the volume of \mathbb{M} .

The Laplace-Beltrami operator on \mathbb{M} has a discrete spectrum given by real and non-negative numbers, which are arranged in an increasing order $\{k(k + \alpha + \beta + 1) : k = 0, 1, \dots\}$. For each k the eigenspace \mathcal{H}_k^m attached to $k(k + \alpha + \beta + 1)$ has finite dimension $\dim \mathcal{H}_k^m = d_m^k$. They are mutually orthogonal with respect to $\langle \cdot, \cdot \rangle_2$ and if we write $\{Y_{k,j} : j = 1, 2, \dots, d_m^k\}$ for an orthonormal basis of \mathcal{H}_k^m , then $\{Y_{k,j} : k = 0, 1, \dots, j = 1, 2, \dots, d_m^k\}$ is an orthonormal basis of $L^2(\mathbb{M})$. On the sphere all those objects are the well known space of spherical harmonics in $m + 1$ variables and degree k ([30, 34]).

The *shifting operator* on $L^2(\mathbb{M})$ is defined by

$$S_t(f)(x) := \frac{1}{\sigma_t^m} \int_{\sigma_t^x} f(y) d\sigma_x(y), \quad f \in L^2(\mathbb{M}), x \in \mathbb{M},$$

where σ_t^x is the sphere on \mathbb{M} of radius t , it means: $\sigma_t^x := \{y \in \mathbb{M} : d(x, y) = t\}$, $0 < t < l$, σ_t^m denotes its area (which does not depend upon x) and $d\sigma_x$ is the area element of σ_t^x . The shifting operator is bounded on $L^2(\mathbb{M})$, namely,

$$\|S_t(f)\|_2 \leq \|f\|_2, \quad f \in L^2(\mathbb{M}).$$

Additionally, the shifting operator can be seen through its Fourier series on $L^2(\mathbb{M})$ (see [5]) as

$$S_t(f) = \sum_{k=0}^{\infty} Q_k^{(\alpha,\beta)}(\cos t) \mathcal{Y}_k(f), \quad f \in L^2(\mathbb{M}); \quad (1.1)$$

where $Q_k^{(\alpha,\beta)}$ denotes the normalized Jacobi polynomial and \mathcal{Y}_k is the projection of $L^2(\mathbb{M})$ onto \mathcal{H}_k^m , $k = 0, 1, \dots$. All the tools mentioned above can be found constructed and/or explored in the following references [5, 24, 28, 29].

Within all this in mind we are able to treat of the main problem of this paper. We will deal with integral operators defined by

$$\mathcal{K}(f) = \int_{\mathbb{M}} K(\cdot, y) f(y) dy, \quad (1.2)$$

in which the generating kernel $K: \mathbb{M} \times \mathbb{M} \rightarrow \mathbb{C}$ is an element of $L^2(\mathbb{M} \times \mathbb{M})$. It is easy to see that (1.2) defines a compact operator on $L^2(\mathbb{M})$. Additional assumption of positivity on the operator above implies self-adjointness of it. Then the standard spectral theorem for compact and self-adjoint operators is applicable and we obtain a sequence of nonnegative real numbers (possibly finite) $\{\lambda_n(\mathcal{K})\}$ which is the eigenvalues sequence of \mathcal{K} .

We analyse the asymptotic behavior of $\{\lambda_n(\mathcal{K})\}$ under additional assumptions: an abstract Hölder condition on K , given by the shifting operator, and the smoothness of the kernel K , given by the Laplace-Beltrami operator. Results of this sort can give us a decay rates of Fourier coefficients of kernels having a Mercer-like series representation as we will see. For a historical review of related results on the spherical setting see [6, 20].

A converse way of related studies is given by the relation of smoothness and moduli of smoothness which can be connected with what we are proposing in here via Fourier coefficients ([12]) in the particular setting where \mathbb{M} is the unit sphere. New results of these very last observation on the same framework we are intended to consider (compact two-point homogeneous space) can be seen in [5, 24, 28] and references therein.

The organization of the necessary background and results of the paper is as follows. Section 2 contains basic material about harmonic analysis in two-point homogeneous spaces and the statement of two results of the paper. Under a Hölder condition assumption based on the shifting operator on the kernel we obtain polynomial decay rates for the eigenvalues sequence of the integral operator. A very new technique involving relations between the growth of Fourier coefficients and eigenvalues sequences of the operator is employed. Section 3 is divided into two main subsections. In both we make smoothness assumption on the kernel and techniques applied are different from the previous section. The first subsection is regarded for finite order of differentiability and it also gives us sharp polynomial decay rates for the eigenvalues sequence of the integral. While in the second one we analyse the impact of infinitely many times differentiability assumption on the kernel and as expected we get exponential decay rates for the eigenvalues sequence. Finally, in Section 4 we give some pertinent information related to examples and optimality of the results.

2 Kernels satisfying an abstract Hölder condition on \mathbb{M}

Our study in this paper concerns to kernels on $\mathbb{M} \times \mathbb{M}$ having a Mercer-like series expansion of the form:

$$K(x, y) = \sum_{k=0}^{\infty} \sum_{j=1}^{d_k^m} a_{k,j} Y_{k,j}(x) Y_{k,j}(y), \quad \sum_{k=0}^{\infty} \sum_{j=1}^{d_k^m} a_{k,j} < \infty, \quad x, y \in \mathbb{M}. \quad (2.3)$$

We make two basic assumptions on these kernels: the first one, called *positivity*, means that the expansion coefficients are non-negative, i.e., $a_{k,j} \geq 0$; and the second one, called *monotonicity* means that the expansion coefficients are monotone decreasing with respect to k , i.e., $a_{k+1,j} \leq a_{k,j'}$, $1 \leq j, j' \leq d_k^m$.

Schoenberg ([31]) characterized all the continuous zonal positive definite kernels on the sphere as series expansion given by formula (2.3) with coefficients do not depending on index j and satisfying the positivity definition above. Recently, Berg and collaborators ([4]) showed that a similar characterization for positive definite kernels in a general setting, namely on products of compact Gelfand pairs with locally compact groups. Therefore, assumptions made here on compact two-point homogeneous spaces are very natural and expected in most of the applications.

Positivity assures that the operator \mathcal{L}_K is positive and has a uniquely defined square root operator $\mathcal{L}_K^{1/2}$ whose kernel $K_{1/2}$ has the following series expansion

$$K_{1/2}(x, y) = \sum_{k=0}^{\infty} \sum_{j=1}^{d_k} a_{k,j}^{1/2} Y_{k,j}(x) Y_{k,j}(y), \quad x, y \in \mathbb{M}. \quad (2.4)$$

Both \mathcal{L}_K and $\mathcal{L}_K^{1/2}$ are self-joint positive operators. Referencing to (1.2) it is easy to see that the spherical harmonics $Y_{k,j}$, $j = 1, 2, \dots, d_k^m$ and $k = 0, 1, \dots$, are all eigenvectors of the operator \mathcal{L}_K associated to the eigenvalues $a_{k,j}$, respectively. Since we have made a monotonicity assumption on coefficients of K it gives us an eigenvalue sequence ordering that is suitable for our analysis.

The first goal in this paper is to continue the path designed by the authors in [20]. Compact two-point homogeneous spaces are rich in their symmetrical structures and let us to explore and utilize them. We say that a kernel K on \mathbb{M} satisfies the (B, β) -Hölder condition if there exist a fixed $\beta \in (0, 2]$ and a function B in $L^1(\mathbb{M})$ such that

$$|S_t(K(y, \cdot))(x) - K(y, x)| \leq B(y) t^\beta, \quad x, y \in \mathbb{M}, \quad t \in (0, l). \quad (2.5)$$

For the second goal we need to introduce some more notation. For a positive real number r , we write $\mathcal{B}^r(f)$ to denote the *fractional derivative of order r* of a function f in $L^2(\mathbb{M})$, it is also called *fractional Laplace-Beltrami operator of order r* . Since this notion of derivative is a generalization of the Laplace-Beltrami operator we take advantage of notation given previously \mathcal{B} . Exactly the same way as it is done on spheres, for $r = 1$ we recover \mathcal{B} from this definition ([5, 30]).

We define \mathcal{B}^r on \mathbb{M} in the distributional sense, through the Laplace-Beltrami operator and its spectrum, by

$$\mathcal{B}^r(f) \sim \sum_{k=0}^{\infty} (k(k + \alpha + \beta + 1))^{r/2} \mathcal{Y}_k(f), \quad (2.6)$$

where f is a distribution on \mathbb{M} . Then we have the Sobolev class defined

$$W_p^r(\mathbb{M}) := \{f \in L^p(\mathbb{M}) : \|f\|_p + \|\mathcal{B}^r(f)\|_p < \infty\}.$$

Here we clearly assume that $\mathcal{B}^r(f) \in L^p(\mathbb{M})$ and endow such space with the norm $\|\cdot\|_{W_p^r} := \|\cdot\|_p + \|\mathcal{B}^r(\cdot)\|_p$. See [30] for the equivalent definition on the spherical setting and [5, 24] for details in the context here.

Theorem bellow, which has its version already proved in the spherical setting, is an improvement and a generalization of previously-known results (see [21, 20, 25] for details) over compacto two-point homogeneous spaces.

Theorem 2.1. *Let \mathcal{L}_K be the integral operator induced by a kernel K as in (2.3) and under assumptions of positivity and monotonicity. If K satisfies the (B, β) -Hölder condition, then it holds*

$$\lambda_n(\mathcal{L}_K) = O(n^{-1-\beta/m}), \text{ as } n \rightarrow \infty.$$

The second result is a generalization of both Theorem 2.5 in [7] and Theorem 3 in [20], so that it will also work with Laplace-Beltrami derivatives of fractional orders on compact two-point homogeneous spaces. It can be seen as consequence of previous theorem since in both we apply similar techniques in order to prove it.

Corollary 2.2. *Let \mathcal{L}_K be the integral operator induced by a kernel K as in (2.3) and under assumptions of positivity, monotonicity and such that for a fixed $r > 0$, all K^y belong to $W_2^{2r}(\mathbb{M})$. If the integral operator generated by $\mathcal{B}^{2r,0}K$ is trace-class, then*

$$\lambda_n(\mathcal{L}_K) = O(n^{-1-2r/m}), \text{ as } n \rightarrow \infty.$$

2.1 Tools: Fourier coefficients, K -functionals and moduli of smoothness

In this section we present some background material in order to prove our results. They include realization theorem, moduli of smoothness and the associated K -functional as well. Relations between these were proved recently by Dai, Ditzian and Tikhonov on two-point homogenous spaces and play an important role here. References are [5, 13, 32]. Our main interest is on the relation of Fourier coefficients of a functions and the eigenvalues attached to the integral operator we are working with.

If r is a positive real number we introduce the K -functional associated to the space W_p^r . For $r > 0$ and $t > 0$, it is given by

$$K_r(f, t)_p := \inf \left\{ \|f - g\|_p + t^r \|g\|_{W_p^r} : g \in W_p^r(\mathbb{M}) \right\}. \quad (2.7)$$

An important property involving the K -functional is the Realization Theorem for $K_r(f, t)_p$ ([13]), which is given by the relation below. In its statement, the multiplier operator η_t depends upon a best approximation function $\eta \in C^\infty[0, \infty)$ such that $\eta = 1$ in $[0, 1]$, $\eta = 0$ in $[2, \infty)$ and $\eta(s) \leq 1$, $s \in (1, 2)$. The operator η_t is defined by the formula

$$\eta_t(f) = \sum_{k=1}^{\infty} \eta(tk) \mathcal{Y}_k(f), \quad f \in L^p(\mathbb{M}).$$

For $r > 0$ and $f \in L^p(\mathbb{M})$ Realization Theorem ([13]) assures that the K -functional $K_r(f, t)_p$ assumes its infimum via the operator η_t as bellow:

$${}^1\|f - \eta_t(f)\|_p + t^r \|\eta_t(f)\|_{W_p^r} \asymp K_r(f, t)_p, \quad t > 0. \quad (2.8)$$

The Fourier coefficients of a function $f \in L^p(\mathbb{M})$ are defined by

$$c_{k,l}(f) := \sigma_m^{-1} \int_{\mathbb{M}} f(y) \overline{Y_{k,j}(y)} dy, \quad j = 1, 2, \dots, d_k^m; \quad k = 0, 1, \dots,$$

where $\{Y_{k,j} : j = 1, 2, \dots, d_k^m; k = 0, 1, \dots\}$ is the basis of eigenfunctions of \mathcal{B} in $L^2(\mathbb{M})$. In the remainder of the section, we provide estimates for certain sums of Fourier coefficients

$$s_k(f) := \sum_{j=1}^{d_k^m} |c_{k,j}(f)|^2, \quad k = 0, 1, \dots \quad (2.9)$$

The following lemma is proved in [14] over the spherical setting. The same proof fits into compact two-point homogeneous spaces setting but we include it here for the sake of completeness.

Lemma 2.3. *($1 \leq p \leq 2$) If f belongs to $L^p(\mathbb{M})$ and q is the conjugate exponent of p , then*

$$\left\{ \sum_{k=1}^{\infty} (d_k^m)^{(2-q)/2q} [s_k(f)]^{q/2} \right\}^{1/q} \leq a(p, m) \|f\|_p,$$

in which $a(p, m)$ is a positive constant depending on p and m .

Proof. Observe that for $f \in L^p(\mathbb{M})$

$$s_k(f) = (s_k(f))^{1/2} \sigma_m^{-1} \int_{\mathbb{M}} f(x) \left(\sum_{j=1}^{d_k^m} \overline{c_{k,j} Y_{k,j}(x)} \right) (s_k(f))^{-1/2} dx, \quad k = 0, 1, \dots \quad (2.10)$$

We define

$$Z_k := \left(\sum_{j=1}^{d_k^m} \overline{c_{k,j} Y_{k,j}} \right) (s_k(f))^{-1/2},$$

which is an element of \mathcal{H}_k^m such that

$$\sigma_m^{-1} \int_{\mathbb{M}} Z_k(x) \overline{Z_k(x)} dx = 1, \quad k = 0, 1, \dots$$

Additionally, $\{Z_k\}_k$ is an orthonormal system in $L^2(\mathbb{M})$ and its elements can be identified to elements of the harmonic spherical basis already defined before.

The addition formula (see [5], for example) implies

$$|Z_k(x)| \leq d_k^m, \quad k = 0, 1, \dots,$$

¹ $A(t) \asymp B(t)$ means that there exist positive constantes c_1 and c_2 such that $c_1 A(t) \leq B(t) \leq c_2 A(t)$.

which leads us, by formula (2.10), to the inequalities

$$\begin{aligned} |s_k(f)| &\leq |s_k(f)|^{1/2} \sigma_m^{-1} \int_{\mathbb{M}} |f(x)| |Z_k(x)| dx \\ &\leq |s_k(f)|^{1/2} \sigma_m^{-1} d_k^m \|f\|_1. \end{aligned}$$

Consequently, $|s_k(f)|^{1/2} \leq \sigma_m^{-1} d_k^m \|f\|_1$, $k = 0, 1, \dots$, and the Riez-Thorin interpolation Theorem finishes the proof. ■

The following theorem relates the growth of the Fourier coefficients of a function to the K -functional defined in (2.7). Ditzian [13] proved this theorem for the special case in which r is a positive integer (making an observation that the same proof can be slightly modified to fit for r a real number) and the general case can be founded proved in [20]. We choose do not reproduce the proof here because it is exactly the same one on the sphere context and can be founded in [20].

Proposition 2.4. *If f belongs to $L^p(\mathbb{M})$ ($1 \leq p \leq 2$) and q is the conjugate exponent of p , then for each fixed $r > 0$, there exists a constant c_p for which*

$$\left\{ \sum_{k=1}^{\infty} (d_k^m)^{(2-q)/2q} (\min\{1, tk\})^{r/q} [s_k(f)]^{q/2} \right\}^{1/q} \leq c_p K_r(f, t)_p, \quad t > 0. \quad (2.11)$$

We need the following result which is a version for the compact two-point homogeneous space of famous Marcinkiewicz's Multiplier Theorem. The result below, gives us a sufficient condition so that a given operator constructed via sequences (multipliers) is limited.

Theorem 2.5 (Theorem 7.1 in [3]). *Let \mathbb{M} a compact two-point homogeneous space of dimension m and $\{\mu_k\}_k$ a sequence of real numbers satisfying the following conditions:*

- i) $\sup_k \{|\mu_k|\}_k \leq M < \infty$,
- ii) $\sup_j \left\{ 2^{j(n-1)} \sum_{l=2^j}^{2^{j+1}} |\Delta^j \mu_l| \right\} \leq M < \infty$,

with $n = (m + 1)/2$ if n is odd and $n = (m + 2)/2$ if n is even. Then, we have the inequality

$$\left\| \sum_{k=0}^{\infty} \mu_k \mathcal{Y}_k(f) \right\|_p \leq A_p M \|f\|_p, \quad f \in L_p(\mathbb{M}), \quad (2.12)$$

where A_p is a constant independent of f . Here Δ^s denotes the ordinary difference of order s , that is, $\Delta^1 \mu_k = \Delta \mu_k := \mu_{k+1} - \mu_k$ and $\Delta^s \mu_k := \Delta^{s-1} \mu_{k+1} - \Delta^{s-1} \mu_k$ for $s \geq 2$.

We conclude this section by bringing the shifting operator into the inequality presented in the above theorem, for $p = 2$. Its derivation requires two additional equivalences described bellow.

Lemma 2.6 (Theorem 1.2 in [28]). *If f belongs to $L^p(\mathbb{M})$ ($1 \leq p < \infty$) and r is a natural number, then it holds*

$$K_{2r}(f, t)_2 \asymp \omega_{2r}(f, t)_2, \quad t > 0 \quad (2.13)$$

where

$$\omega_r(f, t)_p := \sup\{\|(I - S_s)^{r/2}(f)\|_p : s \in (0, t]\}.$$

and I denotes the identity operator on $L^p(\mathbb{M})$.

The second necessary equivalence in order to bring the shifting operator into the inequality in Theorem 2.4 is proved bellow.

Theorem 2.7. *For $1 < p < \infty$, it holds*

$$\omega_2(f, t)_p \asymp \|f - S_t(f)\|_p, \quad f \in L^p(\mathbb{M}), t > 0. \quad (2.14)$$

Proof. We first note that from the definition of the moduli of smoothness we have

$$\|f - S_t(f)\|_p \leq \omega_2(f, t)_p, \quad t > 0, f \in L^p(\mathbb{M}).$$

Now, we just need to prove that there exists a constant c (depending on \mathbb{M}) such that

$$\omega_2(f, t)_p \leq c \|f - S_t(f)\|_p, \quad f \in L^p(\mathbb{M}), t > 0.$$

Properties of the moduli of smoothness ([28, p. 870]) assure us that

$$\begin{aligned} \omega_2(f, t)_p &\leq \omega_2(f - \eta_{2t}(f), t)_p + \omega_2(\eta_{2t}(f), t)_p \\ &\leq 2\|f - \eta_{2t}(f)\|_p + c_0 t^2 \|\mathcal{B}(\eta_{2t}(f))\|_p, \end{aligned}$$

for some constant c_0 only depending on \mathbb{M} . Proposition 4.4 in [28] leads us to

$$\|\mathcal{B}(\eta_{2t}(f))\|_p \leq c_1 h^{-2} \|(I - S_h)(f)\|_p, \quad h \in (0, t],$$

also, for a constant c_1 depending upon \mathbb{M} .

In particular,

$$\|\mathcal{B}(\eta_{2t}(f))\|_p \leq c_1 t^{-2} \|(I - S_t)(f)\|_p,$$

and then we obtain

$$\omega_2(f, t)_p \leq 2\|f - \eta_{2t}(f)\|_p + c_0 c_1 \|(I - S_t)(f)\|_p. \quad (2.15)$$

In order to finish the proof we need to verify that

$$\|f - \eta_{2t}(f)\|_p \leq c_2 \|(I - S_t)(f)\|_p,$$

for a constant c_2 not depending on t and f . To prove the inequality, it suffices to show that

$$\|f - \eta_{2t}(f) - (I + S_t + S_t^2 + S_t^3)(I - \eta_{2t})(f - S_t(f))\|_p \leq c_3 \|f - S_t(f)\|_p$$

that is, that

$$m_k = \frac{(1 - \eta(2tk)) \left(Q_k^{(\alpha, \beta)}(\cos t) \right)^4}{1 - Q_k^{(\alpha, \beta)}(\cos t)}$$

is a multiplier, applying the Theorem 2.5.

For $2tk \leq 1$, since $\eta(2tk) = 1$, we have $m_k = 0$. Now, for $2tk \geq \tau > 1$, Proposition 3.3 of [28] assures that there exists $c_4 > 0$ such that $1 - Q_k^{(\alpha, \beta)}(\cos t) \geq c_4$ and then $|1 - \eta(2tk)| \leq c_5$. Thus, the boundedness of the Jacobi polynomial, given by Lemma 3.2 in [2], implies

$$\begin{aligned} |\Delta^s m_k| &\leq \frac{c_5}{c_4} \Delta^s \left(Q_k^{(\alpha, \beta)}(\cos t) \right)^4 \\ &\leq \frac{c_5}{c_4} c(\alpha, \beta) \frac{t^s}{(kt)^{4\alpha+4/2}} \\ &= c_6 \left(\frac{1}{kt} \right)^{4\alpha+4/2-s} \frac{1}{k^s}, \end{aligned}$$

where all the constants do not depend on t or k .

We observe that for m odd or even, we have $4\alpha + 4/2 - s \geq 0$ and then, $\{(1/kt)^{4\alpha+4/2-s}\}_k$ is bounded, and there exists c_7 such that

$$|\Delta^s m_k| \leq c_7 \frac{1}{k^s}.$$

From the inequality above we have

$$\begin{aligned} \sup_j 2^{j(s-1)} \sum_{k=2^j}^{2^{j+1}} |\Delta^s m_k| &\leq \sup_j 2^{j(s-1)} \sum_{k=2^j}^{2^{j+1}} c_7 \frac{1}{k^s} \\ &\leq \sup_j 2^{j(s-1)} \sum_{k=2^j}^{2^{j+1}} c_7 \frac{1}{(2^j)^s} \\ &= \sup_j 2^{j(s-1)} 2^j c_7 2^{-js} \\ &= c_7 \end{aligned}$$

where the constant c_7 does not depend on t and f . Therefore the sequence $\{m_k\}_k$ fits in Marcinkiewicz type multiplier Theorem and the proof is complete. \blacksquare

2.2 Proof of Theorem 2.1 and Corollary 2.2

Our goal in this section is to prove both Theorem 2.1 and Corollary 2.2. To present them we will first derive some additional technical results as following. We remind readers that the kernels K we are dealing with satisfy assumptions made in the begging of Section 2.

Under assumptions we have made here it follows that for each $y \in \mathbb{M}$, the Fourier coefficients of the function $K^y := K(\cdot, y)$ are $c_{k,j}(K^y) = a_{k,j} \overline{Y_{k,j}(y)}$, $j = 1, 2, \dots, d_k^m$ and $k = 0, 1, \dots$. Considering the kernel $K_{1/2}$ (formula (2.4)) in a similar way we have its Fourier coefficients $c_{k,j}(K_{1/2}^y) = a_{k,j}^{1/2} \overline{Y_{k,j}(y)}$, $j = 1, 2, \dots, d_k^m$ and $k = 0, 1, \dots$, which implies that

$$\int_{\mathbb{M}} s_k(K_{1/2}^y) dy = \sum_{j=1}^{d_k^m} a_{k,j}, \quad k = 0, 1, \dots \quad (2.16)$$

The action of the fractional derivative (formula (2.6)) on $K_{1/2}^y$ gives us

$$\mathcal{B}^r(K_{1/2}^y) \sim \sum_{k=0}^{\infty} \sum_{j=1}^{d_k^m} a_{k,j}^{1/2} (k(k+\alpha+\beta+1))^{r/2} \overline{Y_{k,j}(y)} Y_{k,j}, \quad y \in \mathbb{M}.$$

It follows that $\left| \mathcal{B}^r(K_{1/2}^y) \right|^2$ has a convenient Fourier series expansion from which, under the assumption that $K^y \in W_2^{2r}(\mathbb{M})$, we can easily verify

$$\left\| \mathcal{B}^r(K_{1/2}^y) \right\|_2^2 = \sum_{k=0}^{\infty} \sum_{j=1}^{d_k^m} a_{k,j} (k(k+\alpha+\beta+1))^r |Y_{k,j}(y)|^2 = \mathcal{B}^{2r} K^y(y) = \mathcal{B}^{2r,0} K(y, y), \quad y \in \mathbb{M}.$$

We derive a convenient estimate based on the (B, β) -Hölder condition which will be used in the proof of our theorem. The proof is exactly the same one on the spherical setting and can be found in [20].

Lemma 2.8. *If K satisfies the (B, β) -Hölder condition, then*

$$\int_{\mathbb{M}} \|S_t(K_{1/2}^y) - K_{1/2}^y\|_2^2 dy \leq 2 \|B\|_1 t^\beta, \quad y \in \mathbb{M}, \quad t \in (0, l).$$

We are ready to prove the main result in this section.

Proof of Theorem 2.1 Applying Theorem 2.4 for $p = q = 2$ and $r = 2$ we have

$$\sum_{k=1}^{\infty} (\min\{1, tk\})^4 s_k(K_{1/2}^z) \leq c_2 \|S_t(K_{1/2}^z) - K_{1/2}^z\|_2^2, \quad z \in \mathbb{M}, \quad t \in (0, l).$$

Integrating both sides of inequality above we reach

$$\sum_{k=0}^{\infty} (\min\{1, tk\})^4 \sum_{j=1}^{d_k^m} a_{k,j} \leq c_2 \int_{S^m} \|S_t(K_{1/2}^z) - K_{1/2}^z\|_2^2 d\sigma_m(z), \quad t \in (0, \pi).$$

Since K satisfies the (B, β) -Hölder condition, Lemma 2.8 asserts that

$$\sum_{k=0}^{\infty} (\min\{1, tk\})^4 \sum_{j=1}^{d_k^m} a_{k,j} \leq 2 c_2 \|B\|_1 t^\beta, \quad t \in (0, \pi).$$

For $t = 1/n$, n a natural number bigger than one, the above inequality became

$$\sum_{k=0}^{\infty} (\min\{1, k/n\})^4 \sum_{j=1}^{d_k^m} a_{k,j} \leq C_2 n^{-\beta}, \quad n = 1, 2, \dots$$

Dropping those terms with index $k < n$, we derive the following inequality:

$$\sum_{k=n}^{\infty} \sum_{j=1}^{d_k^m} a_{k,j} \leq C_2 n^{-\beta}, \quad n = 1, 2, \dots,$$

which implies

$$d_n^m \sum_{k=n}^{\infty} a_k \leq \sum_{k=n}^{\infty} d_k^m a_k \leq C_2 n^{-\beta}, \quad n = 1, 2, \dots,$$

where $a_k := \min\{a_{k,j} : j = 1, 2, \dots, d_k^m\}$, $k = 0, 1, \dots$

The equivalence $d_n^m \asymp n^{m-1}$ ([5, p. 405]), which is a consequence of the addition formula, as $n \rightarrow \infty$, leads us to

$$n^{m-1} \sum_{k=n}^{\infty} a_k \leq C_3 C_2 n^{-\beta}, \quad n = 1, 2, \dots,$$

for some $C_3 > 0$, that is,

$$\sum_{k=n}^{\infty} a_k \leq C_3 n^{-\beta-m+1}, \quad n = 1, 2, \dots$$

Finally, we observe that

$$n^{\beta+m} a_n = n^{\beta+m-1} \sum_{k=n}^{2n-1} a_n \leq n^{\beta+m-1} \sum_{k=n}^{\infty} a_k \leq C_3, \quad n = 1, 2, \dots,$$

or, equivalently, $a_k = O(n^{-\beta-m})$, as $n \rightarrow \infty$. Returning to our original notation for the eigenvalues of \mathcal{L}_K and recalling that $\{\lambda_n(\mathcal{L}_K)\}$ decreases to 0, we have that $a_n = \lambda_{d_n^{m+1}}(\mathcal{L}_K)$, $n = 1, 2, \dots$. In particular,

$$\lambda_{d_n^{m+1}}(\mathcal{L}_K) = O(n^{-\beta-m}), \quad (n \rightarrow \infty).$$

Therefore, the decay in the statement of the theorem follows. \blacksquare

Proof of Corollary 2.2 Applying Proposition 2.4, in the particular case that $p = q = 2$, to the function $K_{1/2}^z$ we have

$$\sum_{k=0}^{\infty} (\min\{1, tk\})^{2r} s_k(K_{1/2}^z) \leq c_p \left[\omega_r(K_{1/2}^z, t)_2 \right]^2, \quad z \in \mathbb{M}, t \in (0, l).$$

Since $K_{1/2}^z \in W_2^{2r}$, Proposition 4.2 in [28] asserts the existence of a constant $C_1 > 0$ (independent of both $K_{1/2}^z$ and t) so that

$$\omega_r(K_{1/2}^z, t)_2 \leq C_1 t^r \|\mathcal{B}^r(K_{1/2}^z)\|_2, \quad z \in \mathbb{M}, t \in (0, l).$$

Hence, we have

$$\sum_{k=0}^{\infty} (\min\{1, tk\})^{2r} \left(\int_{\mathbb{M}} s_k(K_{1/2}^z) dz \right) \leq c_p C_1^2 t^{2r} \int_{\mathbb{M}} \|\mathcal{B}^r(K_{1/2}^z)\|_2^2 dz, \quad t \in (0, l).$$

Since $B^{2r,0}K$ is the kernel of a trace-class operator, the equality right above the statement of Lemma 2.8 asserts that $c_p C_1^2 \|\mathcal{B}^r(K_{1/2}^z)\|_2^2$ is a nonnegative constant. Denoting this constant by C_2 , 2.16 assures that

$$\sum_{k=0}^{\infty} (\min\{1, tk\})^{2r} \sum_{j=1}^{d_k^m} a_{k,j} \leq C_2 t^{2r}, \quad t \in (0, l).$$

For $t = 1/n$, n a natural number bigger than one, the above inequality turns out

$$\sum_{k=0}^{\infty} (\min\{1, k/n\})^{2r} \sum_{j=1}^{d_k^m} a_{k,j} \leq C_2 n^{-2r}, \quad n = 1, 2, \dots$$

Dropping those terms with index $k < n$, we derive the following inequality:

$$\sum_{k=n}^{\infty} \sum_{j=1}^{d_k^m} a_{k,j} \leq C_2 n^{-2r}, \quad n = 1, 2, \dots,$$

which implies

$$d_n^m \sum_{k=n}^{\infty} a_k \leq \sum_{k=n}^{\infty} d_k^m a_k \leq C_2 n^{-2r}, \quad n = 1, 2, \dots,$$

where $a_k := \min\{a_{k,j} : j = 1, 2, \dots, d_k^m\}$, $k = 0, 1, \dots$

The equivalence $d_n^m \asymp n^{m-1}$ as $n \rightarrow \infty$, leads us to

$$n^{m-1} \sum_{k=n}^{\infty} a_k \leq C_3 C_2 n^{-2r}, \quad n = 1, 2, \dots,$$

for some $C_3 > 0$, that is,

$$\sum_{k=n}^{\infty} a_k \leq C_3 n^{-2r-m+1}, \quad n = 1, 2, \dots$$

We observe that

$$n^{2r+m} a_n = n^{2r+m-1} \sum_{k=n}^{2n-1} a_n \leq n^{2r+m-1} \sum_{k=n}^{\infty} a_k \leq C_3, \quad n = 1, 2, \dots,$$

or, equivalently, $a_k = O(n^{-2r-m})$, as $n \rightarrow \infty$. The same way we made in the previous proof we return to our original notation for the eigenvalues of \mathcal{L}_K and recalling that $\{\lambda_n(\mathcal{L}_K)\}$ decreases to 0, we have that $a_n = \lambda_{d_n^{m+1}}(\mathcal{L}_K)$, $n = 1, 2, \dots$. In particular,

$$\lambda_{d_n^{m+1}}(\mathcal{L}_K) = O(n^{-2r-m}), \quad (n \rightarrow \infty).$$

The proof follows. ■

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