
**GLOBAL EULER OBSTRUCTION, GLOBAL BRASSELET
NUMBERS AND CRITICAL POINTS**

NICOLAS DUTERTRE
NIVALDO G. GRULHA JR.

Nº 438

NOTAS DO ICMC
SÉRIE MATEMÁTICA



São Carlos – SP
Jan./2018

GLOBAL EULER OBSTRUCTION, GLOBAL BRASSELET NUMBERS AND CRITICAL POINTS

NICOLAS DUTERTRE AND NIVALDO G. GRULHA JR.

ABSTRACT. Let $X \subset \mathbb{C}^n$ be an equidimensional complex algebraic set and let $f : X \rightarrow \mathbb{C}$ be a polynomial function. For each $c \in \mathbb{C}$, we define the global Brasselet number of f at c , a global counterpart of the Brasselet number defined by the authors in a previous work, and the Brasselet number at infinity of f at c . Then we establish several formulas relating these numbers to the topology of X and the critical points of f .

1. INTRODUCTION

The local Euler obstruction is an invariant defined by MacPherson in [27] as one of the main ingredients in his proof of the Deligne-Grothendieck conjecture on the existence of Chern classes for singular varieties. The local Euler obstruction at $0 \in X$, where X is a sufficiently small representative of the equidimensional analytic germ $(X, 0)$, is denoted by $\text{Eu}_X(0)$. After MacPherson's pioneer work, the Euler obstruction was studied by many authors. Let us mention briefly some of the most important results on this subject. If $\mathcal{V} = \{V_i\}_{i=1}^t$ is a Whitney stratification of X , then Brylinski, Dubson and Kashiwara [6] proved a famous formula that relates the $\text{Eu}_{\overline{V}_i}$'s to the Euler characteristic of the normal links of the strata. In [25], Lê and Teissier showed that $\text{Eu}_X(0)$ is equal to an alterned sum of multiplicities of generic polar varieties of X at 0. In [3], Brasselet, Lê and Seade proved a Lefschetz type formula for $\text{Eu}_X(0)$, i.e. they relate $\text{Eu}_X(0)$ to the topology of the real Milnor fibre on X of a generic linear function. There are also integral formulas for $\text{Eu}_X(0)$ in [26] and [15].

In [4], Brasselet, Massey, Parameswaran and Seade defined a relative version of the local Euler obstruction, introducing information for a function f defined on the variety X , called the Euler obstruction of a function and denoted by $\text{Eu}_{f,X}(0)$. They prove a Lefschetz type formula for this invariant. The Euler obstruction of a function can be seen as a generalization of the Milnor number ([4, 33, 21]). For instance, in [33], Seade, Tibăr and Verjovsky showed that $\text{Eu}_{f,X}(0)$ is equal up to sign to the number of critical points of a Morsefication of f lying on the regular part of X .

In [16], we study topological properties of functions defined on analytic complex varieties. In order to do it, we define an invariant called the Brasselet number, denoted by $B_{f,X}(0)$. This number is well defined even when f has arbitrary singularity. When f has isolated singularity we have $B_{f,X}(0) = \text{Eu}_X(0) - \text{Eu}_{f,X}(0)$.

Mathematics Subject Classification (2010) : 14B05, 32C18, 58K45.

Keywords: Euler obstruction, Global Euler obstruction, Constructible functions, Morsefications.

We established several formulas for $B_{f,X}$, among them a relative version of the multiplicity formula of Lê and Teissier, a relative version of the Brylinski-Dubson-Kashiwara formula and an integral formula.

In a manner similar to the local case, in [34], working with an affine equidimensional singular variety $X \subset \mathbb{C}^N$, Seade, Tibăr and Verjovsky defined the global Euler obstruction, denoted by $\text{Eu}(X)$. When X is smooth the global Euler obstruction of X coincides with the Euler characteristic of X . They prove a global version of the Lê-Teissier polar multiplicities formula. Later, this formula was generalized in [32] to an index formula for MacPherson cycles.

As the Euler obstruction of a function and the Brasselet number are useful to study the singularities of f in the local case, we introduce in this work the global Brasselet numbers and the Brasselet numbers at infinity, in order to investigate the topological behavior of the singularities, globally and at the infinity, of a given polynomial function f defined on an algebraic variety $X \subset \mathbb{C}^N$. The main references we use in this paper about the study of singularities at infinity are [9, 10, 41].

In Section 2 we give prerequisites on the topology of complex algebraic sets: stratified Morse functions, the complex link and the normal Morse datum, constructible functions, the local Euler obstruction and the Brasselet number, the global Euler obstruction. In Section 3 we recall the notions of t -regularity at infinity and ρ -regularity at infinity, some basic results and we adapt them to the stratified setting.

In Section 4 we define the global Brasselet numbers and the Brasselet numbers at infinity. We compare the global Brasselet number of f with the global Euler obstruction of the fibres of f . The relation presented in Corollary 4.8 can be seen as a global relative version of the local index formula of Brylinski, Dubson and Kashiwara [6].

In Section 5 we prove several formulas that relate the number of critical points of a Morsefication of a polynomial function f on an algebraic set X , to the global Brasselet numbers and the Brasselet numbers at infinity of f . The main result in this section is Theorem 5.2. From this result we obtain many interesting corollaries. Corollary 5.6, for instance, is a Brylinski–Dubson–Kashiwara type formula for the total Brasselet number at infinity. We also prove a relative version of the polar multiplicity formula of Seade, Tibăr and Verjovsky (Corollary 5.13).

We finish the paper at Section 6, relating the global Euler obstruction of an equidimensional algebraic set $X \subset \mathbb{C}^n$ to the Gauss-Bonnet curvature of its regular part and the Gauss-Bonnet curvature of the regular part of its link at infinity. The result is a global counterpart of the formula that the first author established for analytic germs in [15].

Acknowledgments. The authors thank the USP-Cofecub project “UcMa163/17 - Topologie et géométrie des espaces singuliers et applications”.

The first author is supported by the ANR project LISA.

The second author is partially supported by Fundação de Amparo à Pesquisa do Estado de São Paulo - FAPESP, Brazil, grant 2017/09620-2 and Conselho Nacional de Desenvolvimento Científico e Tecnológico - CNPq, Brazil, grant 303046/2016-3.

2. PREREQUISITES ON THE TOPOLOGY OF COMPLEX ALGEBRAIC SETS

In this section, we work with a reduced complex algebraic set $X \subset \mathbb{C}^n$ of dimension d . We assume that X is equipped with a finite Whitney stratification \mathcal{V} whose

strata are connected. We denote by X_{reg} the regular part of X , i.e. the union of all the strata of dimension d .

2.1. Stratified Morse functions. The main reference for this subject is [20].

Definition 2.1. Let x be a point in X and let V_b be the stratum that contains it. A degenerate tangent plane of the stratification \mathcal{V} is an element T (of an appropriate Grassmannian) such that $T = \lim_{x_i \rightarrow x} T_{x_i} V_a$, where V_a is a stratum that contains V_b in its frontier and where the x_i 's belong to V_a .

Definition 2.2. A degenerate covector of \mathcal{V} at a point $x \in X$ is a covector which vanishes on a degenerate tangent plane of \mathcal{V} at x , i.e., an element η of $T_x^* \mathbb{C}^n$ such that there exists a degenerate tangent plane T of the stratification at x with $\eta(T) = 0$.

Let $f : X \rightarrow \mathbb{C}$ be an analytic function. We assume that f is the restriction to X of an analytic function $F : \mathbb{C}^n \rightarrow \mathbb{C}$, i.e. $f = F|_X$. A point x in X is a critical point of f if it is a critical point of $F|_{V(x)}$, where $V(x)$ is the stratum containing x .

Definition 2.3. Let x be a critical point of f . We say that f is general at x with respect to the stratification \mathcal{V} if $DF(x)$ is not a degenerate covector of \mathcal{V} at x .

We say that f is general with respect to \mathcal{V} if it is general at all critical points with respect to \mathcal{V} .

Definition 2.4. Let x be a critical point of f . We say that x is a stratified Morse critical point of f if f is general at x and the function $f|_{V(x)} : V(x) \rightarrow \mathbb{C}$ has a non-degenerate critical point at x when $\dim V(x) > 0$.

We say that f is a stratified Morse function if it admits only stratified Morse critical points.

2.2. The complex link and the normal Morse datum. The complex link is an important object in the study of the topology of complex analytic sets. It is analogous to the Milnor fibre and was studied first in [23]. It plays a crucial role in complex stratified Morse theory (see [20]) and appears in general bouquet theorems for the Milnor fibre of a function with isolated singularity (see [24, 35, 39]).

Let V be a stratum of the stratification \mathcal{V} of X and let x be a point in V . Let $g : (\mathbb{C}^n, x) \rightarrow (\mathbb{C}, 0)$ be an analytic complex function-germ such that the differential form $Dg(x)$ is not a degenerate covector of \mathcal{V} at x . Let $N_{x,V}^{\mathbb{C}}$ be a normal slice to V at x , i.e. $N_{x,V}^{\mathbb{C}}$ is a closed complex submanifold of \mathbb{C}^n which is transversal to V at x and $N_{x,V}^{\mathbb{C}} \cap V = \{x\}$.

Definition 2.5. The complex link \mathcal{L}_V^X of V is defined by

$$\mathcal{L}_V^X = X \cap N_{x,V}^{\mathbb{C}} \cap B_\epsilon(x) \cap \{g = \delta\},$$

where $0 < |\delta| \ll \epsilon \ll 1$. Here $B_\epsilon(x)$ is the closed ball of radius ϵ centered at x .

The normal Morse datum $\text{NMD}(V)$ of V is the pair of spaces

$$\text{NMD}(V) = (X \cap N_{x,V}^{\mathbb{C}} \cap B_\epsilon(x), X \cap N_{x,V}^{\mathbb{C}} \cap B_\epsilon(x) \cap \{g = \delta\}).$$

The fact that these two notions are well-defined, i.e. independent of all the choices made to define them, is explained in [20].

2.3. Constructible functions. We start with a presentation of Viro's method of integration with respect to the Euler characteristic with compact support [43]. We work in the semi-algebraic setting.

Definition 2.6. Let $Y \subset \mathbb{R}^n$ be a semi-algebraic set. A constructible function $\alpha : Y \rightarrow \mathbb{Z}$ is a \mathbb{Z} -valued function that can be written as a finite sum:

$$\alpha = \sum_{i \in I} m_i \mathbf{1}_{Y_i},$$

where Y_i is a semi-algebraic subset of Y and $\mathbf{1}_{Y_i}$ is the characteristic function on Y_i .

The sum and the product of two constructible functions on Y are again constructible. The set of constructible functions on Y is thus a commutative ring, denoted by $F(Y)$.

Definition 2.7. If $\alpha \in F(Y)$ and $W \subset Y$ is a semi-algebraic set then the Euler characteristic $\chi(W, \alpha)$ is defined by

$$\chi(W, \alpha) = \sum_{i \in I} m_i \chi_c(W \cap Y_i),$$

where $\alpha = \sum_{i \in I} m_i \mathbf{1}_{Y_i}$ and χ_c is the Euler characteristic of Borel-Moore homology.

The Euler characteristic $\chi(W, \alpha)$ is also called the Euler integral of α and denoted by $\int_W \alpha d\chi_c$. Here we follow the terminology and notations used in [4, 16, 32].

Definition 2.8. Let $f : Y \rightarrow Z$ be a continuous semi-algebraic map and let $\alpha : Y \rightarrow \mathbb{Z}$ be a constructible function. The pushforward $f_*\alpha$ of α along f is the function $f_*\alpha : Z \rightarrow \mathbb{Z}$ defined by:

$$f_*\alpha(z) = \chi(f^{-1}(z), \alpha).$$

Proposition 2.9. *The pushforward of a constructible function is a constructible function.*

Theorem 2.10 (Fubini's theorem). *Let $f : Y \rightarrow Z$ be a continuous semi-algebraic map and let α be a constructible function on Y . Then we have:*

$$\chi(Z, f_*\alpha) = \chi(Y, \alpha).$$

Let us go back to the complex situation. Here we write $\mathcal{V} = \{V_1, \dots, V_t\}$ for the Whitney stratification of X .

Definition 2.11. A constructible function with respect to the stratification \mathcal{V} of X is a function $\alpha : X \rightarrow \mathbb{Z}$ which is constant on each stratum V of the stratification.

This means that there exist integers $n_i, i \in \{1, \dots, t\}$, such that $\alpha = \sum_{i=1}^t n_i \cdot \mathbf{1}_{V_i}$. In most of the cases that we will consider, we can use the topological Euler characteristic χ instead of χ_c . First since each V_i is an even-dimensional submanifold, by Poincaré duality $\chi_c(V_i)$ is equal to $\chi(V_i)$ and so $\chi(X, \alpha) = \sum_{i=1}^t n_i \chi(V_i)$. Now let $B \subset \mathbb{C}^n$ be an euclidian closed ball that intersects X transversally (in the stratified sense). We will give four equalities for $\chi(X \cap B, \alpha)$. By additivity of χ_c , we have

$$\chi(X \cap B, \alpha) = \chi(X \cap \overset{\circ}{B}, \alpha) + \chi(X \cap \partial B, \alpha).$$

But $X \cap \partial B$ is Whitney stratified by odd dimensional strata and so $\chi(X \cap \partial B) = 0$ (see Lemma 5.0.3 in [31] or Proposition 1.6 in [28]). Therefore, we have

$$\chi(X \cap B, \alpha) = \chi(X \cap \mathring{B}, \alpha) = \sum_{i=1}^t n_i \chi_c(V_i \cap \mathring{B}),$$

and by Poincaré duality,

$$\chi(X \cap B, \alpha) = \sum_{i=1}^t n_i \chi(V_i \cap \mathring{B}).$$

But each $V_i \cap B$ is a manifold with boundary, so $\chi(V_i \cap \mathring{B}) = \chi(V_i \cap B)$ and

$$\chi(X \cap B, \alpha) = \sum_{i=1}^t n_i \chi(V_i \cap B).$$

Similarly, if $E = \mathbb{C}^n \setminus \mathring{B}$ then

$$\begin{aligned} \chi(X \cap E, \alpha) &= \sum_{i=1}^t n_i \chi_c(V_i \cap E) = \sum_{i=1}^t n_i \chi_c(V_i \cap \mathring{E}) \\ &= \sum_{i=1}^t n_i \chi(V_i \cap \mathring{E}) = \sum_{i=1}^t n_i \chi(V_i \cap E). \end{aligned}$$

If the radius of B is sufficiently big, then $X \cap \partial B$ is homeomorphic to the link at infinity of X , denoted by $\text{Lk}^\infty(X)$, and $X \cap B$ is a retract by deformation of X which implies that $\chi(X) = \chi(X \cap B)$. Since $X \cap B$ is compact, $\chi(X \cap B) = \chi_c(X \cap B)$ and so, by additivity, $\chi_c(X) = \chi(X) + \chi_c(X \cap \mathring{E})$. But $X \cap \mathring{E}$ is homeomorphic to the product of $\text{Lk}^\infty(X)$ and an open interval in \mathbb{R} . Since $\chi_c(\text{Lk}^\infty(X)) = 0$, by multiplicativity of χ_c we obtain that $\chi_c(X \cap \mathring{E}) = 0$ and finally that $\chi(X) = \chi_c(X)$.

Definition 2.12. Let $\alpha : X \rightarrow Z$ be a constructible function with respect to the stratification \mathcal{V} . Its normal Morse index $\eta(V, \alpha)$ along V is defined by

$$\eta(V, \alpha) = \chi(\text{NMD}(V), \alpha) = \chi(X \cap N_{x,V}^{\mathbb{C}} \cap B_\epsilon(x), \alpha) - \chi(\mathcal{L}_V^X, \alpha),$$

where x is a point in V .

If $Z \subset X$ is a closed union of strata, then $\eta(V, \mathbf{1}_Z) = 1 - \chi(\mathcal{L}_V \cap Z)$.

2.4. The local Euler obstruction and the Brasselet number. Here we assume that X is equidimensional. The Euler obstruction at $x \in X$, denoted by $\text{Eu}_X(x)$, was defined by MacPherson, using 1-forms and the Nash blow-up (see [27] for the original definition). An equivalent definition of the Euler obstruction was given by Brasselet and Schwartz in the context of vector fields [5]. Roughly speaking, $\text{Eu}_X(x)$ is the obstruction for extending a continuous stratified radial vector field around x in X to a non-zero section of the Nash bundle over the Nash blow-up of X .

The Euler obstruction is a constructible function and there are two distinguished bases for the free abelian group of constructible functions: the characteristic functions $\mathbf{1}_{\overline{V}}$ and the Euler obstruction $\text{Eu}_{\overline{V}}$ of the closure \overline{V} of all strata V . Moreover, the key role of the Euler obstruction comes from the following identities (see [32] p.34 or [31] p.292 and p.323-324):

$$\eta(V', \text{Eu}_{\overline{V}}) = 1 \text{ if } V' = V,$$

and:

$$\eta(V', \text{Eu}_{\overline{V}}) = 0 \text{ if } V' \neq V.$$

In [3], Brasselet, Lê and Seade study the Euler obstruction using hyperplane sections, following ideas of Dubson and Kato. Let us assume that 0 belongs to X .

Theorem 2.13 ([3]). *For each generic linear form l , there is ϵ_0 such that for any ϵ with $0 < \epsilon < \epsilon_0$, the Euler obstruction of $(X, 0)$ is equal to:*

$$\text{Eu}_X(0) = \chi(X \cap B_\epsilon(0) \cap l^{-1}(\delta), \text{Eu}_X),$$

where $0 < |\delta| \ll \epsilon \ll 1$.

Let $f : X \rightarrow \mathbb{C}$ be a holomorphic function. We assume that f has an isolated singularity (or an isolated critical point) at 0, i.e. that f has no critical point in a punctured neighborhood of 0 in X .

In [4] Brasselet, Massey, Parameswaran and Seade introduced an invariant which measures, in a way, how far the equality given in Theorem 2.13 is from being true if we replace the generic linear form l with some other function on X with at most an isolated stratified critical point at 0. This number is called the Euler obstruction of a function and denoted by $\text{Eu}_{f,X}(0)$. The following result is the Brasselet, Massey, Parameswaran and Seade formula [4] that compares, in the same point, the local Euler obstruction with the Euler obstruction of a function.

Theorem 2.14. *Let $f : X \rightarrow \mathbb{C}$ be a function with an isolated singularity at 0. For $0 < |\delta| \ll \epsilon \ll 1$ we have:*

$$\text{Eu}_X(0) - \text{Eu}_{f,X}(0) = \chi(X \cap B_\epsilon(0) \cap f^{-1}(\delta), \text{Eu}_X),$$

where $0 < |\delta| \ll \epsilon \ll 1$.

In [33], J. Seade, Tibăr and Verjovsky show that the Euler obstruction of f is closely related to the number of Morse points of a Morsefication of f , as it is stated in the next proposition.

Proposition 2.15 ([33]). *Let $f : X \rightarrow \mathbb{C}$ be the an analytic function with isolated singularity at the origin. Then:*

$$\text{Eu}_{f,X}(0) = (-1)^d n_{\text{reg}},$$

where n_{reg} is the number of Morse points on X_{reg} in a stratified Morsefication of f lying in a small neighborhood of 0.

Definition 2.16. A good stratification of X relative to f is a stratification \mathcal{V} of X which is adapted to X^f , (i.e. X^f is a union of strata), where $X^f = X \cap f^{-1}(0)$, such that $\{V_i \in \mathcal{V}; V_i \not\subset X^f\}$ is a Whitney stratification of $X \setminus X^f$ and such that for any pair of strata (V_a, V_b) such that $V_a \not\subset X^f$ and $V_b \subset X^f$, the (a_f) -Thom condition is satisfied.

Let us now recall the definition of the Brasselet number, defined in [16].

Definition 2.17. Let \mathcal{V} be a good stratification of X relative to f . We define $B_{f,X}(0)$ by:

$$B_{f,X}(0) = \chi(X \cap B_\epsilon(0) \cap f^{-1}(\delta), \text{Eu}_X),$$

where $0 < |\delta| \ll \epsilon \ll 1$.

Remark 2.18. Note that if f has a stratified isolated singularity at the origin then, by Theorem 2.14, we have that $B_{f,X}(0) = \text{Eu}_X(0) - \text{Eu}_{f,X}(0)$.

2.5. Global Euler obstruction. Here we assume that X is equidimensional and we write $\mathcal{V} = \{V_1, \dots, V_t\}$. In [34], Seade, Tibăr and Verjovsky introduced a global analogous of the Euler obstruction called the global Euler obstruction and denoted by $\text{Eu}(X)$. Let $\tilde{X} \xrightarrow{\nu} X$ denote the Nash modification of X , and let us consider a stratified real vector field v on a subset $V \subset X$: this means that the vector field is continuous and tangent to the strata. The restriction of v to V has a well-defined canonical lifting \tilde{v} to $\nu^{-1}(V)$ as a section of the real bundle underlying the Nash bundle $\tilde{T} \rightarrow \tilde{X}$.

Definition 2.19. We say that the stratified vector field v on X is radial-at-infinity if it is defined on the restriction to X of the complement of a sufficiently large ball B_M centered at the origin of \mathbb{C}^N , and it is transversal to S_R , pointing outwards, for any $R > M$. In particular, v is without zeros on $X \setminus B_M$.

The “sufficiently large” radius M is furnished by the following well-known result.

Lemma 2.20. *There exists $M \in \mathbb{R}$ such that, for any $R \geq M$, the sphere S_R centered at the origin of \mathbb{C}^N and of radius R is stratified transversal to X , i.e. transversal to all strata of the stratification \mathcal{V} .*

Using this last lemma and inspired by [5] and [11], Seade, Tibăr and Verjovsky defined the global Euler obstruction in [34] as follows:

Definition 2.21. Let \tilde{v} be the lifting to a section of the Nash bundle \tilde{T} of a radial-at-infinity stratified vector field v over $X \setminus B_R$. We call global Euler obstruction of X , and denote it by $\text{Eu}(X)$, the obstruction for extending \tilde{v} as a nowhere zero section of \tilde{T} within $\nu^{-1}(X \cap B_R)$.

To be precise, the obstruction to extend \tilde{v} as a nowhere zero section of \tilde{T} within $\nu^{-1}(X \cap B_R)$ is in fact a relative cohomology class

$$o(\tilde{v}) \in H^{2d}(\nu^{-1}(X \cap B_R), \nu^{-1}(X \cap S_R)) \simeq H_c^{2d}(\tilde{X}).$$

The global Euler obstruction of X is the evaluation of $o(\tilde{v})$ on the fundamental class of the pair $(\nu^{-1}(X \cap B_R), \nu^{-1}(X \cap S_R))$. Thus $\text{Eu}(X)$ is an integer and does not depend on the radius of the sphere defining the link at infinity of X . Since two radial-at-infinity vector fields are homotopic as stratified vector fields, it does not depend on the choice of v either.

Remark 2.22. The global Euler obstruction has the following properties (see [34] p. 396):

- (1) if X is non-singular, then $\text{Eu}(X) = \chi(X)$,
- (2) $\text{Eu}(X) = \chi(X, \text{Eu}_X)$.

3. REGULARITY CONDITIONS AT INFINITY

A natural question is if the concepts of the Euler obstruction and the Brasselet number of a function could be extended to the global setting, as Seade, Tibăr and Verjovsky did for the local Euler obstruction, and what kind of information we could obtain with these possible new global invariants.

But, before extending the local notions of the Euler obstruction and Brasselet number of a function, we recall in this section some definitions and results about the study of singularities at infinity and we adapted some results to the stratified

setting. The main references for this section are [9, 10, 41] and we refer to these papers for details.

We consider $X \subset \mathbb{C}^N$ a reduced algebraic set of dimension d . We use coordinates (x_1, \dots, x_N) for the space \mathbb{C}^N and coordinates $[x_0 : x_1 : \dots : x_N]$ for the projective space \mathbb{P}^N . We consider the algebraic closure \overline{X} of X in the complex projective space \mathbb{P}^N and we denote by

$$H^\infty = \{[x_0 : x_1 : \dots : x_N] \mid x_0 = 0\},$$

the hyperplane at infinity of the embedding $\mathbb{C}^N \subset \mathbb{P}^N$.

One may endow \overline{X} with a semi-algebraic Whitney stratification \mathcal{W} such that X_{reg} is a stratum and the part at infinity $\overline{X} \cap H^\infty$ is a union of strata.

Since \overline{X} is projective and since the stratification of \overline{X} is locally finite, it follows that \mathcal{W} has finitely many strata. We denote by X_{sing} the set of singular points of X , i.e. $X_{\text{sing}} = X \setminus X_{\text{reg}}$.

In order to recall the definition of the t -regularity, let us recall first the definition of the conormal spaces.

Definition 3.1. We denote by $C(X)$ the conormal modification of X , defined as:

$$C(X) = \text{closure}\{(x, H) \in X_{\text{reg}} \times \check{\mathbb{P}}^{N-1} \mid T_x X_{\text{reg}} \subset H\} \subset \overline{X} \times \check{\mathbb{P}}^{N-1}.$$

Let $\pi : C(X) \rightarrow \overline{X}$ denote the projection $\pi(x, H) = x$.

Definition 3.2. Let $g : X \rightarrow \mathbb{C}$ be an analytic function defined in some neighbourhood of X in \mathbb{C}^N . Let X_0 denote the subset of X_{reg} where g is a submersion. The relative conormal space of g is defined as follows:

$$C_g(X) = \text{closure}\{(x, H) \in X_0 \times \check{\mathbb{P}}^{N-1} \mid T_x g^{-1}((g(x))) \subset H\} \subset \overline{X} \times \check{\mathbb{P}}^{N-1},$$

together with the projection $\pi : C_g(X) \rightarrow \overline{X}$, $\pi(x, H) = x$.

Let $f : X \rightarrow \mathbb{C}$ be a function such that $f = F|_X$, where $F : \mathbb{C}^N \rightarrow \mathbb{C}$ is a polynomial function.

Let $\mathbb{X} = \text{graph} f$ be the closure of the graph of f in $\mathbb{P}^N \times \mathbb{C}$ and let $\mathbb{X}^\infty = \mathbb{X} \cap (H^\infty \times \mathbb{C})$. One has the isomorphism $\text{graph} f \simeq X$.

We consider the affine charts $U_j \times \mathbb{C}$ of $\mathbb{P}^N \times \mathbb{C}$, where

$$U_j = \{[x_0 : \dots : x_N] \mid x_j \neq 0\}, j = 0, 1, \dots, N.$$

Identifying the chart U_0 with the affine space \mathbb{C}^N , we have $\mathbb{X} \cap (U_0 \times \mathbb{C}) = \mathbb{X} \setminus \mathbb{X}^\infty = \text{graph} f$, and \mathbb{X}^∞ is covered by the charts $U_1 \times \mathbb{C}, \dots, U_N \times \mathbb{C}$.

If g denotes the projection to the variable x_0 in some affine chart $U_j \times \mathbb{C}$, then the relative conormal $C_g(\mathbb{X} \setminus \mathbb{X}^\infty \cap U_j \times \mathbb{C}) \subset \mathbb{X} \times \check{\mathbb{P}}^N$ is well defined.

With the projection $\pi(y, H) = y$, let us then consider the space $\pi^{-1}(\mathbb{X}^\infty)$, which is well defined for every chart $U_j \times \mathbb{C}$ as a subset of $C_g(\mathbb{X} \setminus \mathbb{X}^\infty \cap U_j \times \mathbb{C})$.

Definition 3.3. We call space of characteristic covectors at infinity the set $C^\infty = \pi^{-1}(\mathbb{X}^\infty)$. For some $p_0 \in \mathbb{X}^\infty$, we denote $C_{p_0}^\infty := \pi^{-1}(p_0)$.

By Lemma 2.8 in [41], these notions are well-defined, i.e. they do not depend on the chart U_j .

Let $\tau : \mathbb{P}^N \times \mathbb{C} \rightarrow \mathbb{C}$ denote the second projection. One defines the relative conormal space $C_\tau(\mathbb{P}^N \times \mathbb{C})$ as in Definition 3.2 where the function g is replaced by the mapping τ .

Definition 3.4. We say that f is t -regular at $p_0 \in \mathbb{X}^\infty$ if

$$C_\tau(\mathbb{P}^N \times \mathbb{C}) \cap C_{p_0}^\infty = \emptyset.$$

We say that $f^{-1}(t_0)$ is t -regular if f is t -regular at all points $p_0 \in \mathbb{X}^\infty \cap \tau^{-1}(t_0)$.

Let us now recall the definition of ρ -regularity. Let $K \subset \mathbb{C}^N$ be some compact (eventually empty) set and let $\rho : \mathbb{C}^N \setminus K \rightarrow \mathbb{R}_{\geq 0}$ be a proper analytic submersion.

Definition 3.5 (ρ -regularity at infinity). We say that f is ρ -regular at $p_0 \in \mathbb{X}^\infty$ if there is an open neighbourhood $U \subset \mathbb{P}^N \times \mathbb{C}$ of p_0 and an open neighbourhood $D \subset \mathbb{C}$ of $\tau(p_0)$ such that, for all $t \in D$, the fibre $f^{-1}(t) \cap X_{\text{reg}} \cap U$ intersects all the levels of the restriction $\rho|_{U \cap X_{\text{reg}}}$ and this intersection is transversal.

We say that the fibre $f^{-1}(t_0)$ is ρ -regular at infinity if f is ρ -regular at all points $p_0 \in \mathbb{X}^\infty \cap \tau^{-1}(t_0)$. We say that t_0 is an asymptotic ρ -non-regular value if $f^{-1}(t_0)$ is not ρ -regular at infinity.

The next proposition relates t -regularity to ρ_E -regularity, where ρ_E denotes the Euclidian norm.

Proposition 3.6. *If f is t -regular at $p_0 \in \mathbb{X}^\infty$, then f is ρ_E -regular at p_0 .*

Proof. This is just an adaptation to our setting of the proof of Proposition 2.11 in [41]. \square

Corollary 3.7. *The set of asymptotic non- ρ_E -regular values of f is finite.*

Proof. It is enough to prove that there are only finitely many values t_0 such that $f^{-1}(t_0)$ is not t -regular. The proof of this fact is as in Corollary 2.12 in [41]. We can equip \mathbb{X} with a Whitney stratification such that \mathbb{X}^∞ is a union of strata and such that any pair of strata (V, W) , with $V \subset \mathbb{X} \setminus \mathbb{X}^\infty$, $W \subset \mathbb{X}^\infty$ and $W \subset \bar{V} \setminus V$, satisfies the Thom (a_g) -regularity condition for some function g defining \mathbb{X}^∞ in \mathbb{X} . If $\tau^{-1}(t_0)$ is transversal to \mathbb{X}^∞ in the stratified sense, then $f^{-1}(t_0)$ is t -regular. But the mapping $\tau|_{\mathbb{X}^\infty} : \mathbb{X}^\infty \rightarrow \mathbb{C}$ has a finite number of critical values (in the stratified sense). \square

Proposition 3.8. *Let $l : X \rightarrow \mathbb{C}$ be a generic linear projection then for all $p_0 \in \mathbb{X}^\infty$, l is t -regular at p_0 .*

Proof. With U_j defined as before, let us work in the chart $U_j \times \mathbb{C}$ and with $C_g(\mathbb{X} \setminus \mathbb{X}^\infty \cap U_j \times \mathbb{C})$ with $\pi(x, H) = x$, as defined above.

Let us suppose that l is not t -regular at $p_0 = (q_0, t_0) \in \mathbb{X}^\infty$. It means that there exists a sequence $p_n = (q_n, l(q_n)) \rightarrow p_0$, with $q_n \in X_{\text{reg}}$ such that

$$T_{p_n} \text{graph}(l) \cap T_{p_n} g^{-1}(g(p_n)) \subset H_n,$$

and $H_n \rightarrow H$ and a sequence of hyperplanes $\{L_n\}$ such that $\mathbb{C}^N \times \{0\} \subset L_n$ and $L_n \rightarrow H$, where $(p_0, H) \in C_\tau(\mathbb{P}^N \times \mathbb{C}) \cap C_{p_0}^\infty$.

Since in fact each $L_n = \mathbb{C}^N \times \{0\}$, we conclude that $H = \mathbb{C}^N \times \{0\}$. Note also that,

$$\begin{aligned} & T_{p_n} \text{graph}(l) \cap T_{p_n} g^{-1}(g(p_n)) \\ &= \text{graph}(l : T_{q_n} X_{\text{reg}} \rightarrow \mathbb{C}) \cap T_{q_n} g^{-1}(g(q_n)) \times \mathbb{C}. \end{aligned}$$

This implies that $\lim_{n \rightarrow +\infty} l(u_n) = 0$ for any bounded sequence (u_n) of vectors such that $u_n \in T_{q_n} X_{\text{reg}} \cap T_{q_n} g^{-1}(g(q_n))$. As in the previous corollary, we can equip \bar{X}

with a Whitney stratification such that $\overline{X} \cap H^\infty$ is a union of strata and such that any pair of strata (V, W) , with $V \subset X$, $W \subset \overline{X} \cap H^\infty$ and $W \subset \overline{V} \setminus V$, satisfies the Thom (a_g) -regularity. Therefore we see that the axis of the pencil defined by l is not transversal to $\overline{X} \cap H^\infty$. By Lemma 3.1 in [34], this is not possible if l is generic. So we conclude that l generic is t -regular. \square

We assume now that X is equipped with a finite Whitney stratification $\mathcal{V} = \{V_i\}_{i=1}^t$ such that V_1, \dots, V_{t-1} are connected, $\overline{V}_1, \dots, \overline{V}_t$ are reduced and $V_t = X_{\text{reg}}$. For $i = \{1, \dots, t\}$, let $f_i : \overline{V}_i \rightarrow \mathbb{C}$ be the restriction to \overline{V}_i of the polynomial function F . Note that $f_t = f$.

Definition 3.9 (stratified t -regularity at infinity). We say that f is stratified t -regular at $p_0 \in \mathbb{X}^\infty$ if for $i = 1, \dots, t$, f_i is t -regular at p_0 .

We say that $f^{-1}(t_0)$ is stratified t -regular if f is stratified t -regular at all points $p_0 \in \mathbb{X}^\infty \cap \tau^{-1}(t_0)$.

Definition 3.10 (stratified ρ -regularity at infinity). We say that f is stratified ρ -regular at $p_0 \in \mathbb{X}^\infty$ if for $i = 1, \dots, t$, f_i is ρ -regular at p_0 .

We say that the fibre $f^{-1}(t_0)$ is stratified ρ -regular at infinity if f is stratified ρ -regular at all points $p_0 \in \mathbb{X}^\infty \cap \tau^{-1}(t_0)$. We say that t_0 is a stratified asymptotic ρ -non-regular value if $f^{-1}(t_0)$ is not stratified ρ -regular at infinity.

The following statements are easy consequences of the definitions of stratified t -regularity and stratified ρ -regularity.

Proposition 3.11. *Stratified t -regularity implies stratified ρ_E -regularity.*

Corollary 3.12. *The set of stratified asymptotic non- ρ_E -regular values of f is finite.*

Corollary 3.13. *Let $l : X \rightarrow \mathbb{C}$ be a generic linear projection, then for all $p_0 \in \mathbb{X}^\infty$, l is stratified t -regular (and therefore stratified ρ_E -regular) at p_0 . Moreover the set of stratified asymptotic non- ρ_E -regular values of f is empty.*

4. GLOBAL BRASSELET NUMBERS AND BRYLINSKY-DUBSON-KASHIWARA FORMULAS

Let $X \subset \mathbb{C}^n$ be a reduced algebraic set of dimension d , equipped with a finite Whitney stratification $\mathcal{V} = \{V_i\}_{i=1}^t$. We assume that V_1, \dots, V_{t-1} are connected, $\overline{V}_1, \dots, \overline{V}_t$ are reduced and that $V_t = X_{\text{reg}}$, where X_{reg} has dimension d . Let $f : X \rightarrow \mathbb{C}$ be a complex polynomial, restriction to X of a polynomial function $F : \mathbb{C}^n \rightarrow \mathbb{C}$, i.e., $f = F|_X$. We assume that f has a finite number of critical points q_1, \dots, q_s and we denote by $\{a_1, \dots, a_r\}$ the set of stratified asymptotic non- ρ_E -regular values of f .

For simplicity, we will write B_R for the ball $B_R(0)$ and S_R for ∂B_R .

Lemma 4.1. *Let $\alpha : X \rightarrow \mathbb{Z}$ be a constructible function with respect to \mathcal{V} . The function $c \mapsto \chi(f^{-1}(c), \alpha)$ is constant on $\mathbb{C} \setminus \left(\{f(q_1), \dots, f(q_s)\} \cup \{a_1, \dots, a_r\} \right)$.*

Proof. Let $c \in \mathbb{C} \setminus \left(\{f(q_1), \dots, f(q_s)\} \cup \{a_1, \dots, a_r\} \right)$ and let us choose $R_c > 0$ such that $f^{-1}(c) \cap \{\rho_E \geq R_c\}$ does not contain any critical point of $\rho_E|_{f^{-1}(c)}$ (here $f^{-1}(c)$ is equipped with the Whitney stratification $\sqcup V_i \cap f^{-1}(c)$). This implies

that $f^{-1}(c) \cap B_R$ is a retract by deformation of $f^{-1}(c)$ and that $\chi(f^{-1}(c)) = \chi(f^{-1}(c) \cap B_R)$ for any $R \geq R_c$. Since $f^{-1}(c)$ is stratified ρ_E -regular at infinity, there is $R \geq R_c$ and $\epsilon > 0$ such that the mapping

$$(\rho_E, f) : f^{-1}(D_\epsilon(c)) \cap \{\rho \geq R\} \rightarrow \mathbb{R} \times \mathbb{C},$$

where $D_\epsilon(c)$ is the closed disc of radius ϵ centered at c in \mathbb{C} , is a stratified submersion and so for $c' \in D_\epsilon(c)$, $f^{-1}(c') \cap B_R$ is also a retract by deformation of $f^{-1}(c')$. But since c is a regular value of f ,

$$\chi(f^{-1}(c') \cap B_R) = \chi(f^{-1}(c) \cap B_R),$$

for c' in a small neighborhood of c . Hence the result is proved for the function $c \mapsto \chi(f^{-1}(c))$, i.e., when $\alpha = \mathbf{1}_X$ (see the discussion after Definition 2.11).

Let V be a stratum of X . By additivity, we have

$$\chi_c(\bar{V} \cap f^{-1}(c)) = \chi_c(V \cap f^{-1}(c)) + \chi_c(\bar{V} \setminus V \cap f^{-1}(c)),$$

and by the arguments after Definition 2.11, we get that

$$\chi(\bar{V} \cap f^{-1}(c)) = \chi(V \cap f^{-1}(c)) + \chi(\bar{V} \setminus V \cap f^{-1}(c)).$$

But since \bar{V} and $\bar{V} \setminus V$ are algebraic subsets of X stratified by strata of \mathcal{V} , c is a regular value and $f^{-1}(c)$ is stratified ρ_E -regular at infinity for $f : \bar{V} \rightarrow \mathbb{C}$ and $f : \bar{V} \setminus V \rightarrow \mathbb{C}$. Therefore the functions

$$c \mapsto \chi(\bar{V} \cap f^{-1}(c)) \text{ and } c \mapsto \chi((\bar{V} \setminus V) \cap f^{-1}(c)),$$

are constant on $\mathbb{C} \setminus \left(\{f(q_1), \dots, f(q_s)\} \cup \{a_1, \dots, a_r\} \right)$ and so is the function $c \mapsto \chi(V \cap f^{-1}(c))$. Hence for $i \in \{1, \dots, t\}$, the function $c \mapsto \chi(V_i \cap f^{-1}(c))$ is constant on $\mathbb{C} \setminus \left(\{f(q_1), \dots, f(q_s)\} \cup \{a_1, \dots, a_r\} \right)$. This implies that $c \mapsto \chi(f^{-1}(c), \alpha)$ is also constant on $\mathbb{C} \setminus \left(\{f(q_1), \dots, f(q_s)\} \cup \{a_1, \dots, a_r\} \right)$ for any constructible function α . \square

Definition 4.2. When X is equidimensional, we define the global Brasselet number of f at c by

$$B_{f,c}^X = \chi(f^{-1}(c), \text{Eu}_X),$$

and the global Euler obstruction of f at c by

$$\text{Eu}_{f,c}^X = \text{Eu}(X) - B_{f,c}^X.$$

Let $a \in \mathbb{C}$ and let $R_a > 0$ be such that $f^{-1}(a) \cap B_{R_a}$ does not contain any critical point of $\rho_E|_{f^{-1}(a)}$. Then there exists $\delta_1 > 0$ such that

$$f : f^{-1}(D_{\delta_1}(a) \setminus \{a\}) \rightarrow D_{\delta_1}(a) \setminus \{a\},$$

is a locally trivial topological fibration (this is just a singular version of the Milnor-Lê fibration) and so $\chi(f^{-1}(c) \cap B_{R_a})$ is constant for $c \in D_{\delta_1}(a) \setminus \{a\}$.

Since $\chi(f^{-1}(c))$ is constant on $\mathbb{C} \setminus \left(\{f(q_1), \dots, f(q_s)\} \cup \{a_1, \dots, a_r\} \right)$, there exists $\delta_2 > 2$ such that $\chi(f^{-1}(c))$ is constant for $c \in D_{\delta_2}(a) \setminus \{a\}$. Since

$$\begin{aligned} \chi(f^{-1}(c)) &= \chi(f^{-1}(c) \cap B_{R_a}) + \chi(f^{-1}(c) \cap \{\rho_E \geq R_a\}) - \chi(f^{-1}(c) \cap S_{R_a}) \\ &= \chi(f^{-1}(c) \cap B_{R_a}) + \chi(f^{-1}(c) \cap \{\rho_E \geq R_a\}), \end{aligned}$$

we see that $\chi(f^{-1}(c) \cap \{\rho_E \geq R_a\})$ is constant for c in $D_\delta(a) \setminus \{a\}$, where $\delta = \min\{\delta_1, \delta_2\}$.

Let $R'_a > 0$ be such that $f^{-1}(a) \cap B_{R'_a}$ does not contain any critical point of $\rho_{E|f^{-1}(a)}$. Then there exists $\delta' > 0$ such that $\chi(f^{-1}(c) \cap \{\rho_E \geq R'_a\})$ is constant for c in $D'_{\delta'}(a) \setminus \{a\}$. We can suppose that $R'_a > R_a$. Since there are no critical points of $\rho_{E|f^{-1}(a)}$ on $\{R_a \leq \rho_E \leq R'_a\}$, a is a regular value of $f : \{R_a \leq \rho_E \leq R'_a\} \rightarrow \mathbb{C}$ (note that a critical point of f on $f^{-1}(a)$ is also a critical point of $\rho_{E|f^{-1}(a)}$). Hence there exists $\nu > 0$ such that

$$f : f^{-1}(D_\nu(a)) \cap \{R_a \leq \rho_E \leq R'_a\} \rightarrow D_\nu(a),$$

is a locally trivial topological fibration. Let $\nu' = \min\{\delta, \delta', \nu\}$. For c in $D_{\nu'}(a) \setminus \{a\}$, we have

$$\begin{aligned} \chi(f^{-1}(c) \cap \{\rho_E \geq R_a\}) &= \chi(f^{-1}(c) \cap \{\rho_E \geq R'_a\}) + \chi(f^{-1}(c) \cap \{R_a \leq \rho_E \leq R'_a\}) \\ &= \chi(f^{-1}(c) \cap \{\rho_E \geq R'_a\}) + \chi(f^{-1}(a) \cap \{R_a \leq \rho_E \leq R'_a\}). \end{aligned}$$

But $\rho : f^{-1}(a) \cap \{R_a \leq \rho_E \leq R'_a\} \rightarrow \mathbb{C}$ is a stratified submersion and so

$$\chi(f^{-1}(a) \cap \{R_a \leq \rho_E \leq R'_a\}) = \chi(f^{-1}(a) \cap \{\rho_E = R_a\}) = 0.$$

We have proved that

$$\chi(f^{-1}(c) \cap \{\rho_E \geq R_a\}) = \chi(f^{-1}(c) \cap \{\rho_E \geq R'_a\}),$$

if c is sufficiently close to a .

Definition 4.3. Let $\alpha : X \rightarrow \mathbb{Z}$ be a constructible function with respect to \mathcal{V} . For any $a \in \mathbb{C}$, we set

$$B_{f,a}^{X,\infty}(\alpha) = \lim_{c \rightarrow a} \chi(f^{-1}(c) \cap \{\rho_E \geq R_a\}, \alpha) \text{ and } \lambda_{f,a}^{X,\infty} = B_{f,a}^{X,\infty}(\mathbf{1}_X),$$

where $R_a > 0$ is such that $f^{-1}(a) \cap \{\rho_E \geq R_a\}$ does not contain any critical point of $\rho_{E|f^{-1}(a)}$.

Note that $B_{f,a}^{X,\infty}(\alpha)$ is well-defined since, by the previous considerations,

$$\lim_{c \rightarrow a} \chi(\overline{V}_i \cap f^{-1}(c) \cap \{\rho_E \geq R_a\}),$$

is well-defined and so is

$$\lim_{c \rightarrow a} \chi(V_i \cap f^{-1}(c) \cap \{\rho_E \geq R_a\}).$$

Lemma 4.4. Let $\alpha : X \rightarrow \mathbb{Z}$ be a constructible function with respect to \mathcal{V} . If $c \in \mathbb{C}$ is such that $f^{-1}(c)$ is stratified ρ_E -regular at infinity then $B_{f,c}^{X,\infty}(\alpha) = 0$.

Proof. It is enough to prove that $\lambda_{f,c}^{X,\infty} = 0$. Since $f^{-1}(c)$ is stratified ρ_E -regular at infinity, there is $R_c > 0$ and $\epsilon > 0$ such that the mapping

$$(\rho_E, f) : f^{-1}(D_\epsilon(c)) \cap \{\rho_E \geq R_c\} \rightarrow \mathbb{R} \times \mathbb{C},$$

is a stratified submersion. Let $c' \in f^{-1}(D_\epsilon(c))$. Then the mapping

$$\rho_E : \{\rho_E \geq R_c\} \cap f^{-1}(c') \rightarrow \mathbb{R},$$

is a proper stratified submersion and so

$$\chi(f^{-1}(c') \cap \{\rho_E \geq R_c\}) = \chi(f^{-1}(c') \cap \{\rho_E = R_c\}) = 0.$$

□

Definition 4.5. Let $\alpha : X \rightarrow \mathbb{Z}$ be a constructible function with respect to \mathcal{V} . We set

$$B_f^{X,\infty}(\alpha) = \sum_{c \in \mathbb{C}} B_{f,c}^{X,\infty}(\alpha) \text{ and } \lambda_f^{X,\infty} = B_f^{X,\infty}(\mathbf{1}_X).$$

Definition 4.6. When X is equidimensional, we define the Brasselet numbers at infinity of f by:

$$B_{f,c}^{X,\infty} = B_{f,c}^{X,\infty}(\text{Eu}_X),$$

for $c \in \mathbb{C}$, and the total Brasselet number at infinity of f by:

$$B_f^{X,\infty} = B_f^{X,\infty}(\text{Eu}_X).$$

We start comparing the global Brasselet numbers of f and the Euler obstructions of the fibres of f .

Proposition 4.7. *Let $a \in \mathbb{C}$, we have*

$$B_{f,a}^X = \text{Eu}(f^{-1}(a)) + \sum_{j \mid f(q_j)=a} \text{Eu}_X(q_j) - \text{Eu}_{f^{-1}(a)}(q_j).$$

Proof. By definition,

$$B_{f,a}^X = \sum_{i=1}^t \chi(V_i \cap f^{-1}(a)) \text{Eu}_X(V_i).$$

For each $i \in \{1, \dots, t\}$, let us denote by Γ_i the set consisting of the q_j 's such that $q_j \in V_i \cap f^{-1}(a)$. The partition

$$f^{-1}(a) = (\sqcup_{i \mid \Gamma_i = \emptyset} V_i \cap f^{-1}(a)) \cup (\sqcup_{i \mid \Gamma_i \neq \emptyset} V_i \cap f^{-1}(a) \setminus \Gamma_i) \cup (\sqcup_{i \mid \Gamma_i \neq \emptyset} \Gamma_i),$$

gives a Whitney stratification of $f^{-1}(a)$, and

$$\begin{aligned} \text{Eu}(f^{-1}(a)) &= \sum_{i \mid \Gamma_i = \emptyset} \chi(V_i \cap f^{-1}(a)) \text{Eu}_{f^{-1}(a)}(V_i \cap f^{-1}(a)) \\ &+ \sum_{i \mid \Gamma_i \neq \emptyset} \chi(V_i \cap f^{-1}(a) \setminus \Gamma_i) \text{Eu}_{f^{-1}(a)}(V_i \cap f^{-1}(a) \setminus \Gamma_i) \\ &+ \sum_{i \mid \Gamma_i \neq \emptyset} \sum_{q \in \Gamma_i} \text{Eu}_{f^{-1}(a)}(q). \end{aligned}$$

If Γ_i is empty then the intersection $V_i \cap f^{-1}(a)$ is transverse (necessarily $\dim V_i > 0$) and by [12], Proposition IV. 4.1.1

$$\text{Eu}_X(V_i) = \text{Eu}_{f^{-1}(a)}(V_i \cap f^{-1}(a)).$$

If Γ_i is not empty and $\dim V_i > 0$, then

$$\chi(V_i \cap f^{-1}(a)) = \chi(V_i \cap f^{-1}(a) \setminus \Gamma_i) + \#\Gamma_i,$$

and $\text{Eu}_X(V_i) = \text{Eu}_{X \cap f^{-1}(a)}(V_i \cap f^{-1}(a) \setminus \Gamma_i)$ because outside Γ_i , $f^{-1}(a)$ intersects V_i transversally. If Γ_i is not empty and $\dim V_i = 0$, then

$$\chi(V_i \cap f^{-1}(a)) = 1 \text{ and } \chi(V_i \cap f^{-1}(a) \setminus \Gamma_i) = 0.$$

Therefore we get

$$B_{f,a}^X = \sum_{i \mid \Gamma_i = \emptyset} \chi(V_i \cap f^{-1}(a)) \text{Eu}_{f^{-1}(a)}(V_i \cap f^{-1}(a))$$

$$\begin{aligned}
& + \sum_{i \mid \Gamma_i \neq \emptyset} \chi(V_i \cap f^{-1}(a) \setminus \Gamma_i) \text{Eu}_{f^{-1}(a)}(V_i \cap f^{-1}(a) \setminus \Gamma_i) \\
& \quad + \sum_{i \mid \Gamma_i \neq \emptyset} \sum_{q \in \Gamma_i} \text{Eu}_X(q) \\
& = \text{Eu}(f^{-1}(a)) + \sum_{i \mid \Gamma_i \neq \emptyset} \text{Eu}_X(q) - \text{Eu}_{f^{-1}(a)}(q).
\end{aligned}$$

□

Note that for a regular value c of f , $B_{f,c}^X = \text{Eu}(f^{-1}(c))$. Furthermore if $X = \mathbb{C}^n$ then $\text{Eu}_X(q_j) = 1$ and $\text{Eu}_{f^{-1}(a)} = 1 + (-1)^{n-2} \mu'(f, q_j)$, where $\mu'(f, q_j)$ is the first Milnor-Teissier number of f at q_j , so

$$B_{f,a}^X = \chi(f^{-1}(a)) = \text{Eu}(f^{-1}(a)) + (-1)^{n-1} \sum_{j \mid f(q_j)=a} \mu'(f, q_j),$$

and we recover Equality (3.3) in [42].

A direct corollary of the previous proposition is a global relative version of the local index formula of Brylinski, Dubson and Kashiwara.

Corollary 4.8. *Let $\alpha : X \rightarrow \mathbb{Z}$ be a constructible function with respect to \mathcal{V} . For any $a \in \mathbb{C}$, we have*

$$\chi(f^{-1}(a), \alpha) = \sum_{i=1}^t B_{f,a}^{\overline{V}_i} \eta(V_i, \alpha).$$

Proof. We keep the notations of the previous proposition and apply Equality (0.2) of [42] to get

$$\begin{aligned}
\chi(f^{-1}(a)) & = \sum_{i \mid \Gamma_i = \emptyset} \text{Eu}(\overline{V}_i \cap f^{-1}(a)) \left(1 - \chi(\mathcal{L}_{V_i \cap f^{-1}(a)}^{f^{-1}(a)})\right) \\
& \quad + \sum_{i \mid \Gamma_i \neq \emptyset} \text{Eu}(\overline{V}_i \cap f^{-1}(a) \setminus \Gamma_i) \left(1 - \chi(\mathcal{L}_{V_i \cap f^{-1}(a) \setminus \Gamma_i}^{f^{-1}(a)})\right) \\
& \quad + \sum_{i \mid \Gamma_i \neq \emptyset} \sum_{q \in \Gamma_i} 1 - \chi(\mathcal{L}_{\{q\}}^{f^{-1}(a)}) \\
& = \sum_{i \mid \Gamma_i \neq V_i} \text{Eu}(\overline{V}_i \cap f^{-1}(a)) \left(1 - \chi(\mathcal{L}_{V_i \cap f^{-1}(a) \setminus \Gamma_i}^{f^{-1}(a)})\right) \\
& \quad + \sum_{i \mid \Gamma_i \neq \emptyset} \sum_{q \in \Gamma_i} 1 - \chi(\mathcal{L}_{\{q\}}^{f^{-1}(a)}),
\end{aligned}$$

because $\Gamma_i = V_i$ if and only if V_i is just a 0-dimensional stratum and in this case, $V_i \cap f^{-1}(a) \setminus \Gamma_i = \emptyset$. By the previous proposition, we obtain the equality

$$\begin{aligned}
\chi(f^{-1}(a)) & = \sum_{i \mid \Gamma_i \neq V_i} \\
& \quad \left[B_{f,a}^{\overline{V}_i} - \sum_{j \mid f(q_j)=a} -\text{Eu}_{\overline{V}_i}(q_j) + \text{Eu}_{\overline{V}_i \cap f^{-1}(a)}(q_j) \right] \left(1 - \chi(\mathcal{L}_{V_i \cap f^{-1}(a) \setminus \Gamma_i}^{f^{-1}(a)})\right)
\end{aligned}$$

$$+ \sum_{i \mid \Gamma_i \neq \emptyset} \sum_{q \in \Gamma_i} 1 - \chi(\mathcal{L}_{\{q\}}^{f^{-1}(a)}),$$

that we rewrite

$$\begin{aligned} \chi(f^{-1}(a)) &= \sum_{i \mid \Gamma_i \neq V_i} \left[B_{f,a}^{\overline{V}_i} - \sum_{j \mid f(q_j)=a} -\text{Eu}_{\overline{V}_i}(q_j) + \text{Eu}_{\overline{V}_i \cap f^{-1}(a)}(q_j) \right] \left(1 - \chi(\mathcal{L}_{V_i \cap f^{-1}(a) \setminus \Gamma_i}^{f^{-1}(a)}) \right) \\ &\quad + \sum_{j \mid f(q_j)=a} 1 - \chi(\mathcal{L}_{\{q_j\}}^{f^{-1}(a)}). \end{aligned}$$

If $\Gamma_i \neq V_i$ then $f^{-1}(a)$ intersects $V_i \setminus \Gamma_i$ transversally and so

$$\chi(\mathcal{L}_{V_i \cap f^{-1}(a) \setminus \Gamma_i}^{f^{-1}(a)}) = \chi(\mathcal{L}_{V_i}^X).$$

Hence we have

$$\begin{aligned} \chi(f^{-1}(a)) &= \sum_{i \mid \Gamma_i \neq V_i} B_{f,a}^{\overline{V}_i} \left(1 - \chi(\mathcal{L}_{V_i}^X) \right) + \sum_{j \mid f(q_j)=a} \left[1 - \chi(\mathcal{L}_{\{q_j\}}^{f^{-1}(a)}) \right] \\ &\quad + \sum_{i \mid \Gamma_i \neq V_i} -\text{Eu}_{\overline{V}_i}(q_j) \left(1 - \chi(\mathcal{L}_{V_i}^X) \right) + \text{Eu}_{\overline{V}_i \cap f^{-1}(a)}(q_j) \left(1 - \chi(\mathcal{L}_{V_i \cap f^{-1}(a) \setminus \Gamma_i}^{f^{-1}(a)}) \right). \end{aligned}$$

Let us evaluate the second part of this sum and fix q a critical point of f such that $f(q) = a$. Two cases are possible.

If q belongs to a stratum V_k with $\Gamma_k \neq V_k$ then we add the stratum $V_0 = \{q\}$ to the Whitney stratification of X . By the Brylinski, Dubson and Kashiwara index formula ([6] or [31], p294), we know that

$$1 = \sum_{i=0}^t \text{Eu}_{\overline{V}_i}(q) \left(1 - \chi(\mathcal{L}_{V_i}^X) \right),$$

and so

$$1 = \sum_{i \mid \Gamma_i \neq V_i} \text{Eu}_{\overline{V}_i}(q) \left(1 - \chi(\mathcal{L}_{V_i}^X) \right) + \text{Eu}_{V_0}(q) \left(1 - \chi(\mathcal{L}_{V_0}^X) \right),$$

because $q \notin \overline{V}_i$ if $\Gamma_i = V_i$ ($i \geq 1$). But $\text{Eu}_{V_0}(q) = 1$ and $\chi(\mathcal{L}_{V_0}^X) = 1$ because a generic linear form is a stratified submersion at q (see [17], p90 for details). The same index formula applied to $f^{-1}(a)$ gives

$$1 = \sum_{i=1}^t \text{Eu}_{\overline{V}_i \cap f^{-1}(a) \setminus \Gamma_i}(q) \left(1 - \chi(\mathcal{L}_{V_i \cap f^{-1}(a) \setminus \Gamma_i}^{f^{-1}(a)}) \right) + 1 - \chi(\mathcal{L}_{\{q\}}^{f^{-1}(a)}).$$

Therefore we get that

$$\chi(\mathcal{L}_{\{q\}}^{f^{-1}(a)}) = \sum_{i \mid \Gamma_i \neq V_i} \text{Eu}_{\overline{V}_i \cap f^{-1}(a)}(q) \left(1 - \chi(\mathcal{L}_{V_i \cap f^{-1}(a) \setminus \Gamma_i}^{f^{-1}(a)}) \right),$$

and so the contribution of q in the second summand of the above sum is zero.

If q belongs to a stratum V_k with $\Gamma_k = V_k$ then, actually $V_k = \{q\}$. By the index formula and the same arguments, we find that

$$1 = \sum_{i \mid \Gamma_i \neq V_i} \text{Eu}_{\overline{V}_i}(q) \left(1 - \chi(\mathcal{L}_{V_i}^X) \right) + 1 - \chi(\mathcal{L}_{\{q\}}^X),$$

and

$$1 = \sum_{i \mid \Gamma_i \neq V_i} \text{Eu}_{\overline{V_i} \cap f^{-1}(a)}(q) \left(1 - \chi(\mathcal{L}_{V_i \cap f^{-1}(a) \setminus \Gamma_i}^{f^{-1}(a)}) \right) + 1 - \chi(\mathcal{L}_{\{q\}}^{f^{-1}(a)}).$$

Hence the contribution of q in the above second summand is $1 - \chi(\mathcal{L}_{\{q\}}^X)$, which we can write $B_{f,a}^{\overline{V_k}} \left(1 - \chi(\mathcal{L}_{V_k}^X) \right)$. Finally we have proved that

$$\chi(f^{-1}(a)) = \sum_{i=1}^t B_{f,a}^{\overline{V_i}} \left(1 - \chi(\mathcal{L}_{V_i}^X) \right),$$

and the theorem follows because both sides of the equality are linear in α . \square

5. GLOBAL BRASSELET NUMBERS AND CRITICAL POINTS

In this section, we prove several formulas that relate the number of critical points of a Morsefication of a polynomial function f on an algebraic set X , to the global Brasselet numbers and the Brasselet numbers at infinity of f . We note that when $X = \mathbb{C}^n$, similar formulas have already appeared in the literature ([8, 22, 38, 40, 41, 36, 30, 1]).

The setting is the same as in the previous section: $X \subset \mathbb{C}^n$ is a reduced algebraic set of dimension d , equipped with a finite Whitney stratification $\mathcal{V} = \{V_i\}_{i=1}^t$ such that V_1, \dots, V_{t-1} are connected, $\overline{V_1}, \dots, \overline{V_t}$ are reduced and $V_t = X_{\text{reg}}$; $f : X \rightarrow \mathbb{C}$ is a complex polynomial, restriction to X of a polynomial function $F : \mathbb{C}^n \rightarrow \mathbb{C}$. We assume that f has a finite number of critical points q_1, \dots, q_s and we denote by $\{a_1, \dots, a_r\}$ the set of stratified asymptotic non- ρ_E -regular values of f .

Definition 5.1. We say that $\tilde{f} : X \rightarrow \mathbb{C}$ is a Morsefication of f if \tilde{f} is a small deformation of f which is a local (stratified) Morsefication at all isolated critical points of f .

Let \tilde{f} be a Morsefication of f . As in the local, we can take \tilde{f} of the form $f + tl$ where t is a sufficiently small complex number and l is the restriction to X of a generic linear form (see Theorem 2.2 in [24]). Note that \tilde{f} has two kinds of critical points: those appearing in a small neighborhood of a critical point of f and those appearing at infinity, i.e., outside a ball of sufficiently big radius. We will only consider the first ones.

Let $n_i, i = 1, \dots, t$, be the number of critical points of \tilde{f} appearing in a small neighborhood of a critical point of f on the stratum V_i . Note that

$$n_i \geq \mu^T(f|_{V_i}) = \sum_{j \mid q_j \in V_i} \mu(f|_{V_i}, q_j),$$

where $\mu(f|_{V_i}, q_j)$ is the Milnor number of $f|_{V_i}$ at q_j , since we do not assume that f is general with respect to \mathcal{V} .

The following theorem relates the number of stratified critical points of \tilde{f} appearing on the stratum V_i to the topology of X and a generic fibre of f .

Theorem 5.2. *Let $c \in \mathbb{C}$ be a regular value of f , which is not a stratified asymptotic non- ρ_E -regular value. We have*

$$\chi(X) - \chi(f^{-1}(c)) = \sum_{i=1}^t (-1)^{d_i} n_i \left(1 - \chi(\mathcal{L}_{V_i}^X) \right) - \lambda_f^{X, \infty}.$$

Moreover if f is general with respect to \mathcal{V} , then we have

$$\chi(X) - \chi(f^{-1}(c)) = \sum_{i=1}^t (-1)^{d_i} \mu^T(f|_{V_i}) \left(1 - \chi(\mathcal{L}_{V_i}^X)\right) - \lambda_f^{X,\infty}.$$

Proof. For $x \in X$, let $\varphi(x) = \chi(f^{-1}(\tilde{x}) \cap B_\epsilon(x)) = \chi_c(f^{-1}(\tilde{x}) \cap B_\epsilon(x))$, where \tilde{x} is a regular value of f close to $f(x)$ and $0 \leq |\tilde{x} - f(x)| \ll \epsilon \ll 1$. Note that $\varphi(x) = 1$ if x is not a critical point of f and so φ is a constructible function. By Fubini theorem, we have

$$\chi(X, \varphi) = \chi(\mathbb{C}, f_*\varphi),$$

that we can rewrite

$$\int_X \varphi(x) d\chi_c(x) = \int_{\mathbb{C}} \left[\int_{f^{-1}(y)} \varphi(x) d\chi_c(x) \right] dy.$$

Let us compute the integral $\int_{f^{-1}(y)} \varphi(x) d\chi_c(x)$ for y in \mathbb{C} . Let \tilde{y} be a regular value of f , which is not a stratified asymptotic non- ρ_E -regular value. Let $R_y \gg 1$ be such that $f^{-1}(y) \cap B_{R_y}$ is a retract by deformation of $f^{-1}(y)$ and let us denote by z_1, \dots, z_l the critical points of f in $f^{-1}(y)$. On the one hand, we have

$$\begin{aligned} \chi(f^{-1}(\tilde{y}) \cap B_{R_y}) &= \chi\left(f^{-1}(\tilde{y}) \cap B_{R_y} \setminus (\cup_{i=1}^l \mathring{B}_\epsilon(z_i))\right) + \sum_{i=1}^l \chi(f^{-1}(\tilde{y}) \cap B_\epsilon(z_i)) \\ &= \chi_c\left(f^{-1}(y) \cap B_{R_y} \setminus (\cup_{i=1}^l B_\epsilon(z_i))\right) + \sum_{i=1}^l \varphi(z_i) \\ &= \chi_c\left(f^{-1}(y) \cap B_{R_y} \setminus \{z_1, \dots, z_l\}\right) + \sum_{i=1}^l \varphi(z_i) = \int_{f^{-1}(y) \cap B_{R_y}} \varphi(x) d\chi_c(x). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \int_{f^{-1}(y) \cap B_{R_y}} \varphi(x) d\chi_c(x) &= \int_{f^{-1}(y)} \varphi(x) d\chi_c(x) - \int_{f^{-1}(y) \cap \{\rho_E > R_y\}} \varphi(x) d\chi_c(x) \\ &= \int_{f^{-1}(y)} \varphi(x) d\chi_c(x) - \chi_c(f^{-1}(y) \cap \{\rho_E > R_y\}). \end{aligned}$$

But the function $f^{-1}(y) \cap \{\rho_E > R_y\} \rightarrow]R_y, +\infty[$, $x \mapsto \rho_E(x)$ is a proper stratified submersion, so

$$\chi_c(f^{-1}(y) \cap \{\rho_E > R_y\}) = \chi_c(f^{-1}(y) \cap S_R) \times \chi_c(]R_y, +\infty[),$$

where $R \in]R_y, +\infty[$. Since $\chi_c(f^{-1}(y) \cap S_R) = 0$, we find that

$$\begin{aligned} \int_{f^{-1}(y)} \varphi(x) d\chi_c(x) &= \chi(f^{-1}(\tilde{y}) \cap B_{R_y}) \\ &= \chi_c(f^{-1}(\tilde{y}) \cap B_{R_{\tilde{y}}}) - \chi_c(f^{-1}(\tilde{y}) \cap \{R_y < \rho_E \leq R_{\tilde{y}}\}) \\ &= \chi(f^{-1}(\tilde{y}) \cap B_{R_{\tilde{y}}}) - \chi(f^{-1}(\tilde{y}) \cap \{R_y \leq \rho_E \leq R_{\tilde{y}}\}) \\ &= \chi(f^{-1}(\tilde{y})) - \lambda_{f,y}^{X,\infty}, \end{aligned}$$

because $\chi(f^{-1}(\tilde{y}) \cap \{R_y \leq \rho_E \leq R_{\tilde{y}}\}) = \chi(f^{-1}(\tilde{y}) \cap \{R_y \leq \rho_E\})$. Therefore, we get

$$\int_X \varphi(x) d\chi_c(x) = \chi(X) - \sum_{j=1}^s 1 - \varphi(q_j) = \int_{\mathbb{C}} \chi(f^{-1}(\tilde{y})) dy - \int_{\mathbb{C}} \lambda_{f,y}^{X,\infty} dy$$

$$= \chi(f^{-1}(c)) - \sum_{j=1}^r \lambda_{f, a_j}^{X, \infty} = \chi(f^{-1}(c)) - \lambda_f^{X, \infty},$$

and so,

$$\chi(X) - \chi(f^{-1}(c)) = \sum_{j=1}^s 1 - \chi(f^{-1}(\tilde{q}_j) \cap B_\epsilon(q_j)) - \lambda_f^{X, \infty}.$$

By Theorem 3.2 in [29] applied to the sheaf \mathbb{Z}_X^\bullet , we know that

$$1 - \chi(f^{-1}(\tilde{q}_j) \cap B_\epsilon(q_j)) = \sum_{i=1}^t (-1)^{d_i} n_{ij} (1 - \chi(\mathcal{L}_{V_i}^X)), \quad (*)$$

where n_{ij} is the number of critical points of a Morsefication of f that lie on V_i in a small neighborhood of q_j . Summing over all the critical points of f , we obtain the result. \square

Corollary 5.3. *Let $\alpha : X \rightarrow \mathbb{Z}$ be a constructible function with respect to \mathcal{V} and let $c \in \mathbb{C}$ be a regular value of f , which is not a stratified asymptotic non- ρ_E -regular value. We have*

$$\chi(X, \alpha) - \chi(f^{-1}(c), \alpha) = \sum_{i=1}^t (-1)^{d_i} n_i \eta(V_i, \alpha) - B_f^{X, \infty}(\alpha).$$

Moreover if f is general with respect to \mathcal{V} , then we have

$$\chi(X, \alpha) - \chi(f^{-1}(c), \alpha) = \sum_{i=1}^t (-1)^{d_i} \mu^T(f|_{V_i}) \eta(V_i, \alpha) - B_f^{X, \infty}(\alpha).$$

Proof. By the previous theorem, the result is true for $\alpha = \mathbf{1}_{\overline{V_i}}$. Since both sides of the equality are linear in α , we see that the result is valid for any constructible function α . \square

If X is equidimensional then by [33], Proposition 2.3, the term $(-1)^{d_i} n_{ij}$ that appears in Equality (*) is equal to $\text{Eu}_{f, \overline{V_i}}(q_j)$. Hence the above corollary can be refined.

Corollary 5.4. *Assume that X is equidimensional. Let $\alpha : X \rightarrow \mathbb{Z}$ be a constructible function with respect to \mathcal{V} and let $c \in \mathbb{C}$ be a regular value of f , which is not a stratified asymptotic non- ρ_E -regular value. We have*

$$\begin{aligned} \chi(X, \alpha) - \chi(f^{-1}(c), \alpha) &= \sum_{i=1}^t (-1)^{d_i} n_i \eta(V_i, \alpha) - B_f^{X, \infty}(\alpha) \\ &= \sum_{i=1}^t \left(\sum_{j=1}^q \text{Eu}_{f, \overline{V_i}}(q_j) \right) \eta(V_i, \alpha) - B_f^{X, \infty}(\alpha). \end{aligned}$$

Moreover if f is general with respect to \mathcal{V} , then we have

$$\chi(X, \alpha) - \chi(f^{-1}(c), \alpha) = \sum_{i=1}^t (-1)^{d_i} \mu^T(f|_{V_i}) \eta(V_i, \alpha) - B_f^{X, \infty}(\alpha).$$

An interesting application occurs when $\alpha = \text{Eu}_X$.

Corollary 5.5. *Assume that X is equidimensional. Let $c \in \mathbb{C}$ be a regular value of f , which is not a stratified asymptotic non- ρ_E -regular value. We have*

$$\mathrm{Eu}_{f,c}^X = \mathrm{Eu}(X) - \mathrm{B}_{f,c}^X = (-1)^d n_t - \mathrm{B}_f^{X,\infty} = \sum_{j=1}^s \mathrm{Eu}_{f,X}(q_j) - \mathrm{B}_f^{X,\infty}.$$

Moreover if f is general with respect to \mathcal{V} , then we have

$$\mathrm{Eu}_{f,c}^X = \mathrm{Eu}(X) - \mathrm{B}_{f,c}^X = \sum_{j \mid q_j \in X_{\mathrm{reg}}} (-1)^d \mu(f|_{X_{\mathrm{reg}}}, q_j) - \mathrm{B}_f^{X,\infty}.$$

Proof. By definition, $\chi(X \cap f^{-1}(c), \mathrm{Eu}_X) = \mathrm{B}_{f,c}^X$ and $\mathrm{B}_f^{X,\infty}(\mathrm{Eu}_X) = \mathrm{B}_f^{X,\infty}$. By Remark 2.22, $\chi(X, \mathrm{Eu}_X) = \mathrm{Eu}(X)$. But if $V_i \neq V_t$, then $\eta(V_i, \mathrm{Eu}_X) = 0$ and $\eta(V_t, \mathrm{Eu}_X) = 1$. \square

Another corollary is a Brylinski-Dubson-Kashiwara type formula for the global Brasselet number at infinity.

Corollary 5.6. *Assume that X is equidimensional and let $\alpha : X \rightarrow \mathbb{Z}$ be a constructible function with respect to \mathcal{V} . We have*

$$\mathrm{B}_f^{X,\infty}(\alpha) = \sum_{i=1}^t \mathrm{B}_f^{\overline{V}_i,\infty} \eta(V_i, \alpha).$$

Proof. Applying Corollaries 5.4 and 5.5 to each set \overline{V}_i , we obtain that

$$\begin{aligned} \chi(X, \alpha) - \chi(f^{-1}(c), \alpha) + \mathrm{B}_f^{X,\infty}(\alpha) &= \\ \sum_{i=1}^t \left[\mathrm{Eu}(\overline{V}_i) - \mathrm{B}_{f,c}^{\overline{V}_i} - \mathrm{B}_f^{\overline{V}_i,\infty} \right] \eta(V_i, \alpha). \end{aligned}$$

But we know that $\chi(X, \alpha) = \sum_{i=1}^t \mathrm{Eu}(\overline{V}_i) \eta(V_i, \alpha)$ (see [15], Corollary 5.4) and that $\chi(f^{-1}(c), \alpha) = \sum_{i=1}^t \mathrm{B}_{f,c}^{\overline{V}_i} \eta(V_i, \alpha)$. \square

Let us study what happens if we replace the generic regular value c with any value a . First we do not assume that X is equidimensional. For $i = 1, \dots, t$, let n_i^a be the number of critical points of \tilde{f} on V_i appearing in a small neighborhood of a critical point of f , but that do not lie in a small neighborhood of a critical point q of f such that $f(q) = a$. Similarly, we set

$$\mu_a^T(f|_{V_i}) = \sum_{j \mid f(q_j) \neq a} \mu(f|_{V_j}, q_j).$$

Proposition 5.7. *Let $\alpha : X \rightarrow \mathbb{Z}$ be a constructible function with respect to \mathcal{V} and let $a \in \mathbb{C}$. We have*

$$\chi(X, \alpha) - \chi(f^{-1}(a), \alpha) = \sum_{i=1}^t (-1)^{d_i} n_i^a \eta(V_i, \alpha) - \mathrm{B}_f^{X,\infty}(\alpha) + \mathrm{B}_{f,a}^{X,\infty}(\alpha).$$

Moreover if f is general with respect to \mathcal{V} then we have

$$\chi(X, \alpha) - \chi(f^{-1}(a), \alpha) = \sum_{i=1}^t (-1)^{d_i} \mu_a^T(f|_{V_i}) \eta(V_i, \alpha) - \mathrm{B}_f^{X,\infty}(\alpha) + \mathrm{B}_{f,a}^{X,\infty}(\alpha).$$

Proof. Let c be a generic value (i.e., regular and not stratified asymptotic non- ρ_E -regular value) of f close to a . Let $R_a \gg 1$ be such that $f^{-1}(a) \cap B_{R_a}$ is a deformation retract of $f^{-1}(a)$. We have

$$\begin{aligned}
\chi(f^{-1}(c) \cap B_{R_a}) &= \chi(f^{-1}(c) \cap B_{R_a} \setminus (\cup_j |_{f(q_j)=a} B_\epsilon(q_j))) \\
&\quad + \sum_{j | f(q_j)=a} \chi(f^{-1}(c) \cap B_\epsilon(q_j)) \\
&= \chi_c(f^{-1}(a) \cap B_{R_a} \setminus (\cup_j |_{f(q_j)=a} B_\epsilon(q_j))) + \sum_{j | f(q_j)=a} \chi(f^{-1}(c) \cap B_\epsilon(q_j)) \\
&= \chi_c(f^{-1}(a) \cap B_{R_a} \setminus (\cup_j |_{f(q_j)=a} B_\epsilon(q_j))) + \sum_{j | f(q_j)=a} \chi(f^{-1}(c) \cap B_\epsilon(q_j)) \\
&= \chi_c(f^{-1}(a) \cap B_{R_a}) - \sum_{j | f(q_j)=a} 1 - \chi(f^{-1}(c) \cap B_\epsilon(q_j)) \\
&= \chi(f^{-1}(a) \cap B_{R_a}) - \sum_{j | f(q_j)=a} 1 - \chi(f^{-1}(c) \cap B_\epsilon(q_j)) \\
&= \chi(f^{-1}(a)) - \sum_{j | f(q_j)=a} 1 - \chi(f^{-1}(c) \cap B_\epsilon(q_j)).
\end{aligned}$$

But

$$\begin{aligned}
\chi(f^{-1}(c)) &= \chi(f^{-1}(c) \cap B_{R_a}) + \chi(f^{-1}(c) \cap \{R_a \leq \rho_E \leq R_c\}) \\
&= \chi(f^{-1}(a)) - \sum_{j | f(q_j)=a} 1 - \chi(f^{-1}(c) \cap B_\epsilon(q_j)) + \lambda_{f,a}^{X,\infty}.
\end{aligned}$$

Combining this with the equality proved in Theorem 5.2, we get

$$\chi(X) - \chi(f^{-1}(a)) = \sum_{j | f(q_j) \neq a} 1 - \chi(f^{-1}(c) \cap B_\epsilon(q_j)) - \lambda_f^{X,\infty} + \lambda_{f,a}^{X,\infty}.$$

Using Massey's results ([29], Theorem 3.2), we obtain the result for $\alpha = \mathbf{1}_X$. The general case easily follows because of the linearity in α of both sides of the equality. \square

If X is equidimensional, we can refine the above proposition.

Corollary 5.8. *Let $\alpha : X \rightarrow \mathbb{Z}$ be a constructible function with respect to \mathcal{V} and let $a \in \mathbb{C}$. We have*

$$\begin{aligned}
\chi(X, \alpha) - \chi(f^{-1}(a), \alpha) &= \sum_{i=1}^t (-1)^{d_i} n_i^a \eta(V_i, \alpha) - B_f^{X,\infty}(\alpha) + B_{f,a}^{X,\infty}(\alpha) \\
&= \sum_{i=1}^t \left(\sum_{j | f(q_j) \neq a} \text{Eu}_{f, \bar{V}_i}(q_j) \right) \eta(V_i, \alpha) - B_f^{X,\infty}(\alpha) + B_{f,a}^{X,\infty}(\alpha).
\end{aligned}$$

Moreover if f is general with respect to \mathcal{V} then we have

$$\chi(X, \alpha) - \chi(f^{-1}(a), \alpha) = \sum_{i=1}^t (-1)^{d_i} \mu_a^T(f|_{V_i}) \eta(V_i, \alpha) - B_f^{X,\infty}(\alpha) + B_{f,a}^{X,\infty}(\alpha).$$

As above, we can specify these equalities to the case $\alpha = \text{Eu}_X$.

Corollary 5.9. *Assume that X is equidimensional. Let $a \in \mathbb{C}$. We have*

$$\begin{aligned} \mathrm{Eu}_{f,a}^X &= \mathrm{Eu}(X) - \mathrm{B}_{f,a}^X = (-1)^d n_t^a - \mathrm{B}_f^{X,\infty} + \mathrm{B}_{f,a}^{X,\infty} \\ &= \sum_{j \mid f(q_j) \neq a} \mathrm{Eu}_{f,X}(q_j) - \mathrm{B}_f^{X,\infty} + \mathrm{B}_{f,a}^{X,\infty}. \end{aligned}$$

Moreover if f is general with respect to \mathcal{V} , then we have

$$\mathrm{Eu}_{f,a}^X = \mathrm{Eu}(X) - \mathrm{B}_{f,a}^X = \sum_{\substack{j \mid q_j \in X_{\mathrm{reg}} \\ f(q_j) \neq a}} (-1)^d \mu(f|_{X_{\mathrm{reg}}}, q_j) - \mathrm{B}_f^{X,\infty} + \mathrm{B}_{f,a}^{X,\infty}.$$

We can also give a version of the Brylinski-Dubson-Kashiwara formula for the Brasselet numbers at infinity $\mathrm{B}_{f,a}^{X,\infty}$.

Corollary 5.10. *Assume that X is equidimensional. Let $\alpha : X \rightarrow \mathbb{Z}$ be a constructible function with respect to \mathcal{V} and let $a \in \mathbb{C}$. We have*

$$\mathrm{B}_{f,a}^{X,\infty}(\alpha) = \sum_{i=1}^t \mathrm{B}_{f,a}^{\overline{V}_i,\infty} \eta(V_i, \alpha).$$

Proof. Apply the previous two corollaries and proceed as in the proof of Corollary 5.6. \square

An easy corollary is a relation between $\mathrm{Eu}_{f,a}^X$ (resp. $\mathrm{B}_{f,a}^X$) and $\mathrm{Eu}_{f,c}^X$ (resp. $\mathrm{B}_{f,c}^X$) where c is generic.

Corollary 5.11. *Assume that X is equidimensional. Let $c \in \mathbb{C}$ be a regular value of f , which is not a stratified asymptotic non- ρ_E -regular value, and let $a \in \mathbb{C}$. We have*

$$\begin{aligned} \mathrm{Eu}_{f,c}^X - \mathrm{Eu}_{f,a}^X &= \mathrm{B}_{f,a}^X - \mathrm{B}_{f,c}^X = (-1)^d (n_t - n_t^a) - \mathrm{B}_{f,a}^{X,\infty} \\ &= \sum_{j \mid f(q_j) = a} \mathrm{Eu}_{f,X}(q_j) - \mathrm{B}_{f,a}^{X,\infty}. \end{aligned}$$

Moreover if f is general with respect to \mathcal{V} , then we have

$$\mathrm{Eu}_{f,c}^X - \mathrm{Eu}_{f,a}^X = \mathrm{B}_{f,a}^X - \mathrm{B}_{f,c}^X = \sum_{\substack{j \mid q_j \in X_{\mathrm{reg}} \\ f(q_j) = a}} (-1)^d \mu(f|_{X_{\mathrm{reg}}}, q_j) - \mathrm{B}_{f,a}^{X,\infty}.$$

When f has no stratified asymptotic non- ρ_E -regular values then in all the above equalities, the terms $\lambda_f^{X,\infty}$, $\lambda_{f,a}^{X,\infty}$, $\mathrm{B}_f^{X,\infty}(\alpha)$, $\mathrm{B}_{f,a}^{X,\infty}(\alpha)$, $\mathrm{B}_f^{X,\infty}$ and $\mathrm{B}_{f,a}^{X,\infty}$ vanish. From now on, we assume that X is equidimensional. If $f = l$ is the restriction to X of a generic linear function $L : \mathbb{C}^N \rightarrow \mathbb{C}$, then l has no stratified asymptotic non- ρ_E -regular values and moreover l is a stratified Morse function (see [34], Lemma 3.1).

Keeping the notations introduced in [34], we denote by $\alpha_X^{(d)}$ the number of (Morse) critical points of l on X_{reg} and by $\alpha_{X,a}^{(d)}$ those not occurring on $l^{-1}(a)$. In this case, if c is a regular value of l then $\mathrm{Eu}_{l,c}^X = (-1)^d \alpha_X^{(d)}$ and if a is a critical value of l , then $\mathrm{Eu}_{l,a}^X = (-1)^d \alpha_{X,a}^{(d)}$. By the relation between $\mathrm{B}_{l,a}^X$ and $\mathrm{Eu}(l^{-1}(a))$, we obtain

$$\mathrm{Eu}(X) - \mathrm{Eu}(l^{-1}(a)) = (-1)^d \alpha_{X,a}^{(d)} + \sum_{j \mid l(q_j) = a} \mathrm{Eu}_X(q_j) - \mathrm{Eu}_{l^{-1}(a)}(q_j),$$

where the q_j 's are the critical points of l . For a regular value c of l , this gives

$$\text{Eu}(X) - \text{Eu}(l^{-1}(c)) = (-1)^d \alpha_X^{(d)},$$

and we remark that we have recovered Equality (2), page 401 in [34]. Based on this equality, Seade, Tibăr and Verjovsky could express the global Euler obstruction as an alternating sum of global polar invariants. In the sequel, we will establish a relative version of this result for the global Brasselet number and the Brasselet numbers at infinity.

So we consider a polynomial function $f : X \rightarrow \mathbb{C}$, restriction to X of a polynomial function $F : \mathbb{C}^N \rightarrow \mathbb{C}$. We assume that f has a finite number of critical points $\{q_1, \dots, q_s\}$. For $a \in \mathbb{C}$, we put $X_a = f^{-1}(a)$. The algebraic set X_a is equidimensional and if q_1, \dots, q_u , $u \leq s$, are the critical points of f on $f^{-1}(a)$, then

$$\mathcal{V}_a = \left(\bigsqcup_{i=1}^t V_i \cap f^{-1}(a) \setminus \{q_1, \dots, q_u\} \right) \cup \left(\bigsqcup_{j=1}^u \{q_j\} \right),$$

is a Whitney stratification of X_a .

Let $L : \mathbb{C}^N \rightarrow \mathbb{C}$ be a linear function and $l : X \rightarrow \mathbb{C}$ be its restriction to X . We denote by $\Gamma_{f,l}^X$ the relative polar variety of f and l . It is defined as follows:

$$\Gamma_{f,l}^X = \overline{\{x \in X_{\text{reg}} \mid \text{rank}[df(x), dl(x)] < 2\}}.$$

It is well-known that for L generic, $\Gamma_{f,l}^X$ is a reduced algebraic curve. Moreover if L is generic, we can assume the following fact:

$l|_{X_a} : X_a \rightarrow \mathbb{C}$ is ρ -regular at infinity and Morse stratified.

Let $I_X(\Gamma_{f,l}^X, X_a)$ be the global intersection multiplicity of $\Gamma_{f,l}^X$ and X_a , namely

$$I_X(\Gamma_{f,l}^X, X_a) = \sum_{p \in \Gamma_{f,l}^X \cap f^{-1}(a)} I_p(\Gamma_{f,l}^X, X_a),$$

where $I_p(\Gamma_{f,l}^X, X_a)$ is the local intersection multiplicity of $\Gamma_{f,l}^X$ and X_a at p . If $\dim(X) = 1$ then $\Gamma_{f,l}^X = X$ and in this case $I_p(\Gamma_{f,l}^X, X_a)$ is the degree of $l : (X, p) \rightarrow (\mathbb{C}, a)$, that is the cardinality of $l^{-1}(c) \cap X \cap B_\epsilon(p)$ for $0 < |c - a| \ll \epsilon \ll 1$.

Proposition 5.12. *We have*

$$B_{f,a}^X - B_{f,a}^{X \cap H} = (-1)^{d-1} I_X(\Gamma_{f,l}^X, X_a) + \sum_{j=1}^r \text{Eu}_{f,X}(q_j),$$

where H is a generic hyperplane given by $H = L^{-1}(g)$ for a regular value g of $l|_{X_a}$ and $l|_X$.

Proof. Let us treat first the case $\dim(X) > 1$. Applying Equality (2) of [34] that we have mentioned above to X_a and $X_a \cap L^{-1}(g)$, we get

$$\text{Eu}(X_a) - \text{Eu}(X_a \cap H) = (-1)^{d-1} \alpha_{X_a}^{(d-1)},$$

where $\alpha_{X_a}^{(d-1)}$ is the number of critical points of $l|_{X_a}$ on $X_{\text{reg}} \cap (f^{-1}(a) \setminus \{q_1, \dots, q_u\})$. Since g is a regular value of $l|_{X_a}$ then

$$\text{Eu}(X_a \cap H) = \sum_{i=1}^t \chi(V_i \cap f^{-1}(a) \cap H) \text{Eu}_{X_a \cap H}(V_i \cap H \cap f^{-1}(a)).$$

The hyperplane H intersects X transversally. Furthermore, because the intersections $V_i \cap f^{-1}(a) \cap H$ are transverse, we know that

$$\mathrm{Eu}_{X_a \cap H}(V_i \cap H \cap f^{-1}(a)) = \mathrm{Eu}_{X \cap H}(V_i \cap H),$$

which implies that $\mathrm{Eu}(X_a \cap H) = \mathrm{B}_{f,a}^{X \cap H}$. Applying Proposition 4.7, we obtain

$$\mathrm{B}_{f,a}^X - \mathrm{B}_{f,a}^{X \cap H} = (-1)^{d-1} \alpha_{X_a}^{(d-1)} + \sum_{j=1}^r \mathrm{Eu}_X(q_j) - \mathrm{Eu}_{X_a}(q_j).$$

But $I_X(\Gamma_{f,l}^X, X_a)$ is equal to

$$\alpha_{X_a}^{(d-1)} + \sum_{j=1}^r I_{q_j}(\Gamma_{f,l}^X, X_a).$$

By Corollary 5.2 in [16], we have

$$(-1)^{d-1} I_{q_j}(\Gamma_{f,l}^X, X_a) = \mathrm{B}_{f,X}(q_j) - \mathrm{B}_{f,X \cap H_j}(q_j),$$

where $H_j = L^{-1}(L(q_j))$. Corollary 6.6 in [16] implies that

$$(-1)^{d-1} I_{q_j}(\Gamma_{f,l}^X, X_a) = \mathrm{B}_{f,X}(q_j) - \mathrm{Eu}_{X_a}(q_j) = \mathrm{Eu}_X(q_j) - \mathrm{Eu}_{f,X}(q_j) - \mathrm{Eu}_{X_a}(q_j),$$

and so

$$\mathrm{Eu}_X(q_j) - \mathrm{Eu}_{X_a}(q_j) = (-1)^{d-1} I_{q_j}(\Gamma_{f,l}^X, X_a) + \mathrm{Eu}_{f,X}(q_j).$$

If $\dim(X) = 1$ then $\mathrm{B}_{f,a}^{X \cap H} = 0$ and

$$\mathrm{B}_{f,a}^X = \sum_{p \in f^{-1}(a)} \mathrm{Eu}_X(p) = \#f^{-1}(a) \setminus \{q_1, \dots, q_r\} + \sum_{j=1}^r \mathrm{Eu}_X(q_j).$$

Applying Theorem 3.1 in [4], it is easy to see that

$$\mathrm{Eu}_X(q_j) = \mathrm{Eu}_{f,X}(q_j) + I_{q_j}(\Gamma_{f,l}^X, X_a).$$

Since $I_p(\Gamma_{f,l}^X, X_a) = 1$ if p is a regular point of f , we obtain the result. \square

By a standard connectivity argument, $I_X(\Gamma_{f,l}^X, X_a)$ does not depend on the choice of the generic linear function L . Following Tibăr's notation [40], we denote it by $\gamma_{X,a}^{(d-1)}$. Similarly for $i = 2, \dots, d$, we define

$$\gamma_{X,a}^{(d-i)} = I_{X \cap H^{i-1}}(\Gamma_{f,l}^{X \cap H^{i-1}}, X_a \cap H^{i-1}),$$

where H^{i-1} is a generic linear space of codimension $i-1$. The following statement is a relative version of the Seade-Tibăr-Verjovsky polar formula for the global Euler obstruction.

Corollary 5.13. *We have*

$$\mathrm{B}_{f,a}^X = \sum_{i=1}^d (-1)^{d-i} \gamma_{X,a}^{(d-i)} + \sum_{j=1}^r \mathrm{Eu}_{f,X}(q_j).$$

Proof. We apply the previous result to $X \cap H^{i-1}$. Note that for $i \geq 2$, we can choose H^{i-1} generic enough so that H^{i-1} intersects X_a transversally, which implies that $f^{-1}(a)$ intersects $H^{i-1} \cap X$ transversally. \square

If $a = c$ is a generic value of f , then the above corollary becomes

$$B_{f,c}^X = \sum_{i=1}^d (-1)^{d-i} \gamma_{X,c}^{(d-i)}.$$

If we apply this to $f = l$, the restriction to X of a generic linear function $L : \mathbb{C}^n \rightarrow \mathbb{C}$, then for $i = 1, \dots, d$, $\gamma_{X,c}^{(d-i)}$ is exactly equal to the number $\alpha_X^{(d-i)}$ defined in [34], which is the number of critical points of a generic linear function on $X_{\text{reg}} \cap H^{i-1}$. Combining this fact with the equality

$$B_{f,c}^X = \text{Eu}(X) - \text{Eu}_{f,c}^X = \text{Eu}(X) - (-1)^d \alpha_X^{(d)},$$

we obtain

$$\text{Eu}(X) = \sum_{i=0}^d (-1)^{d-i} \alpha_X^{(d-i)} = \sum_{i=0}^d (-1)^i \alpha_X^{(i)},$$

that is, the main result of [34].

Another corollary is a characterization of the Brasselet numbers at infinity in terms of critical points of generic linear forms.

Corollary 5.14. *Let $a \in \mathbb{C}$ be a stratified asymptotic non- ρ_E -regular value of f and let $c \in \mathbb{C}$ be a generic regular value of f . We have*

$$B_{f,a}^{X,\infty} = \sum_{i=1}^d (-1)^{d-i} \left(\gamma_{X,c}^{(d-i)} - \gamma_{X,a}^{(d-i)} \right).$$

Proof. Use the previous corollary and the equality

$$B_{f,a}^{X,\infty} = B_{f,c}^X - B_{f,a}^X + \sum_{j=1}^r \text{Eu}_{f,X}(q_j).$$

□

If $\alpha : X \rightarrow \mathbb{Z}$ is a constructible function relative to \mathcal{V} , then the previous equality, combined with the Brylinski-Dubson-Kashiwara formula for $B_{f,a}^{X,\infty}$ proved in Section 3, gives

$$B_{f,a}^{X,\infty}(\alpha) = \sum_{j=1}^t \left(\sum_{i=1}^{d_j} (-1)^{d_j-i} \left(\gamma_{V_j,c}^{(d_j-i)} - \gamma_{V_j,a}^{(d_j-i)} \right) \right) \eta(V_j, \alpha).$$

In particular for $\alpha = \mathbf{1}_X$, we get

$$\lambda_{f,a}^{X,\infty} = \sum_{j=1}^t \left(\sum_{i=1}^{d_j} (-1)^{d_j-i} \left(\gamma_{V_j,c}^{(d_j-i)} - \gamma_{V_j,a}^{(d_j-i)} \right) \right) (1 - \chi(\mathcal{L}_{V_j}^X)).$$

We end this section with an application. We assume that $\dim(X) \geq 2$ and that f is general with respect to \mathcal{V} . We also suppose that there exists $l : X \rightarrow \mathbb{C}$, restriction to X of a linear form $L : \mathbb{C}^N \rightarrow \mathbb{C}$, such that l has no stratified asymptotic non- ρ_E -regular values and is general with respect to \mathcal{V} and such that the mapping $(f, l)|_{X_{\text{reg}}} : X_{\text{reg}} \rightarrow \mathbb{C}^2$ is a submersion. In this situation,

Corollary 5.5 gives $\text{Eu}_{f,c}^X = -\text{B}_f^{X,\infty}$ because f has no critical points on X_{reg} . Similarly $\text{Eu}_{f,c}^{X \cap H} = -\text{B}_f^{X \cap H, \infty}$, where $H = L^{-1}(g)$ for a regular value g of $l : X \rightarrow \mathbb{C}$. But, by Proposition 5.12 applied to f and l , we find

$$\text{B}_{f,c}^X - \text{B}_{f,c}^{X \cap H} = 0 = \text{Eu}(X) - \text{Eu}_{f,c}^X - \text{Eu}(X \cap H) + \text{Eu}_{f,c}^{X \cap H}.$$

But $\text{Eu}(X) = \text{Eu}(X \cap H)$ because l has no critical points on X_{reg} . Finally, we obtain that $\text{B}_f^{X,\infty} = \text{B}_f^{X \cap H, \infty}$. When we apply this equality to \mathbb{C}^2 , we recover a well-known result. Indeed, if $X = \mathbb{C}^2$ then $\lambda_f^\infty = 0$ because $f|_H$ is proper and so $\lambda_f^{H,\infty} = 0$. But $\lambda_f^\infty = \sum_{i=1}^r \lambda_{f,a_i}^\infty$. In this case,

$$\lambda_{f,a_i}^\infty = \chi(f^{-1}(c) \cap \{R_a \leq \rho_E \leq R_{\bar{a}_i}\}),$$

and $f^{-1}(c) \cap \{R_a \leq \rho_E \leq R_{\bar{a}_i}\}$ is a smooth compact orientable surface with at least two boundary components. Therefore $\lambda_{f,a_i}^\infty \leq 0$ and so $\lambda_{f,a_i}^\infty = 0$ for each $i \in \{1, \dots, r\}$. Applying Theorem 5.2 and Proposition 5.7, we find that for all $c \in \mathbb{C}$, $\chi(f^{-1}(c)) = 1$. Hence, by Suzuki [38] or Hâ-Lê's results [22], f has no bifurcation value at infinity and so $f : \mathbb{C}^2 \rightarrow \mathbb{C}$ is a fibration.

6. GLOBAL EULER OBSTRUCTION AND THE GAUSS-BONNET MEASURE

In this section, we relate the global Euler obstruction of an equidimensional algebraic set $X \subset \mathbb{C}^N$ to the Gauss-Bonnet curvature of its regular part and the Gauss-Bonnet curvature of the regular part of its link at infinity. Actually the result we will prove is the global counterpart of the formula that the first author established for analytic germs in [15] and that gave a positive answer to a question of Fu [19].

Before recalling this formula, let us give a brief presentation of the Lipschitz-Killing curvatures. In [18], Fu developed integral geometry for compact subanalytic sets. Using the technology of the normal cycle, he associated with every compact subanalytic set $X \subset \mathbb{R}^N$ a sequence of curvature measures

$$\Lambda_0(X, -), \dots, \Lambda_N(X, -),$$

called the Lipschitz-Killing measures. He proved several integral geometry formulas, among them a Gauss-Bonnet formula and a kinematic formula. Later another description of the measures using stratified Morse theory was given by Broecker and Kuppe [7] (see also [2]). The reader can refer to [14], Section 2, for a rather complete presentation of these two approaches and for the definition of the Lipschitz-Killing measures.

Let us give some comments on these Lipschitz-Killing curvatures. If $\dim(X) = d$ then

$$\Lambda_{d+1}(X, U') = \dots = \Lambda_N(X, U') = 0,$$

for any Borel set U' of X and $\Lambda_d(X, U') = \mathcal{H}_d(U')$, where \mathcal{H}_d is the d -dimensional Hausdorff measure in \mathbb{R}^N . Furthermore if X is smooth then for any Borel set U' of X and for $k \in \{0, \dots, d\}$, $\Lambda_k(X, U')$ is related to the classical Lipschitz-Killing-Weil curvature K_{d-k} through the following equality:

$$\Lambda_k(X, U') = \frac{1}{s_{N-k-1}} \int_{U'} K_{d-k}(x) dx,$$

where s_{N-k-1} is the volume of the $N-k-1$ -dimensional unit sphere. The measure $\Lambda_0(X, -)$ is also called the Gauss-Bonnet measure. This terminology is justified by the following Gauss-Bonnet formula (see [18] or [7]): $\Lambda_0(X, X) = \chi(X)$.

In [15], Corollary 6.10, the first author showed that if $(X, 0) \subset (\mathbb{C}^N, 0)$ is the germ of an equidimensional analytic set then

$$\text{Eu}_X(0) = \lim_{\epsilon \rightarrow 0} \Lambda_0(X \cap B_\epsilon, X_{\text{reg}} \cap S_\epsilon).$$

Roughly speaking, this means that the Euler obstruction is in the limit equal to the Gauss-Bonnet curvature of $X \cap B_\epsilon$ within $X_{\text{reg}} \cap S_\epsilon$.

The method we will follow is exactly the same as the one used in [15], Section 6, and that is why we will omit some details and often refer to [15]. However, we will see that there are differences between the two cases.

First we will work with closed semi-algebraic sets. So let $X \subset \mathbb{R}^N$ be a closed semi-algebraic set. We assume that X is equipped with a finite Whitney semi-algebraic stratification $X = \bigcup_{a \in A} V_a$. For $R \gg 1$ sufficiently big, B_R intersects X transversally and so $X \cap B_R$ admits the following Whitney stratification:

$$X \cap B_R = \bigcup_{a \in A} V_a \cap \overset{\circ}{B}_R \cup \bigcup_{a \in A} V_a \cap S_R,$$

(note that $V_a \cap S_R = \emptyset$ if V_a is bounded). By the Gauss-Bonnet theorem mentioned above, we have

$$\chi(X \cap B_R) = \Lambda_0(X \cap B_R, X \cap B_R),$$

which implies that

$$\chi(X \cap B_R) = \Lambda_0(X \cap B_R, X \cap \overset{\circ}{B}_R) + \Lambda_0(X \cap B_R, X \cap S_R).$$

As in [15], Section 6, we can deduce that

$$\chi(X \cap B_R) = \Lambda_0(X, X \cap B_R) + \Lambda_0(X \cap B_R, X \cap S_R).$$

Since $\lim_{R \rightarrow +\infty} \chi(X \cap B_R) = \chi(X)$ and since by Corollary 5.7 in [13],

$$\lim_{R \rightarrow +\infty} \Lambda_0(X, X \cap B_R),$$

exists and is finite, we find that

$$\lim_{R \rightarrow +\infty} \Lambda_0(X \cap B_R, X \cap S_R),$$

exists and is finite.

We can describe this last limit topologically by means of critical points on $X \cap S_R$ of generic linear functions. Namely, as in [15], Section 6, we can prove that for almost all $v \in S^{N-1}$, $v|_X^*$ has a finite number of critical points and there exists $R_v > 0$ such that for $R \geq R_v$, the function $v|_{X \cap S_R}^*$ is a stratified Morse function. Here $v^*(x) = \langle v, x \rangle$. A critical point p of $v|_{X \cap S_R}^*$ is an inwards-pointing critical point of $v|_{X \cap B_R}^*$ if

$$\nabla(v|_V^*)(p) = \lambda(p) \nabla(\rho_{E|V})(p),$$

with $\lambda(p) < 0$ and where V is the stratum that contains p . Let us denote by $\mathcal{I}_{v,R}$ the set of inwards-pointing critical points of $v|_{X \cap B_R}^*$.

Proposition 6.1. *We have*

$$\lim_{R \rightarrow +\infty} \Lambda_0(X \cap B_R, X \cap S_R)$$

$$= \frac{1}{s_{N-1}} \int_{S_{N-1}} \lim_{R \rightarrow +\infty} \sum_{p \in \mathcal{I}_{v,R}} (-1)^{\sigma(p)} \cdot \text{ind}_{\text{nor}}(v^*, X \cap S_R, p) dv,$$

where $\sigma(p)$ is the Morse index of $v^*_{|V \cap S_R}$, V being the stratum that contains p , and $\text{ind}_{\text{nor}}(v^*, X \cap S_R, p)$ is the normal Morse index of $v^*_{|X \cap S_R}$ at p .

Proof. See [15], Proposition 6.6. \square

Let us apply this equality in the case where X is a complex algebraic set. Let $X \subset \mathbb{C}^N$ be an algebraic set. We keep the notation of the previous sections. We consider a vector v in S^{2N-1} generic as above and we choose $R \geq R_v$. Let $p \in V \cap S_R$ be an inwards-pointing critical point of $v^*_{|X \cap B_R}$. It is explained in [15] that in this case $\text{ind}_{\text{nor}}(v^*, X \cap S_R, p) = \eta(V, \mathbf{1}_X)$. If we denote by $\{q_1^v, \dots, q_s^v\}$ the critical points of $v^*_{|X}$ then, by stratified Morse theory, we can write

$$\chi(X \cap B_R) = \sum_{j=1}^s \text{ind}(v^*, X, q_j^v) + \sum_{i=1}^t \eta(V_i, \mathbf{1}_X) \sum_{p \in \mathcal{I}_{v,R}^{V_i}} (-1)^{\sigma(p)},$$

for $R \gg 1$ sufficiently big and where $\mathcal{I}_{v,R}^{V_i}$ is the set of inwards-pointing critical points of $v^*_{|X \cap B_R}$ on the stratum V_i .

When we apply this relation to $X = \bar{V}$ where V is a stratum of depth 0, this gives that $\lim_{R \rightarrow +\infty} \sum_{p \in \mathcal{I}_{v,R}^V} (-1)^{\sigma(p)}$ exists and is finite. Note that if $\dim(V) = 0$ then $\mathcal{I}_{v,R}^V$ is empty and $\lim_{R \rightarrow +\infty} \sum_{p \in \mathcal{I}_{v,R}^V} (-1)^{\sigma(p)} = 0$. Applied to $X = \bar{W}$, for W a stratum of depth 1, it gives that $\lim_{R \rightarrow +\infty} \sum_{p \in \mathcal{I}_{v,R}^W} (-1)^{\sigma(p)}$ exists and is finite. By induction on the depth of the stratum, we see that for $i \in \{1, \dots, t\}$, $\lim_{R \rightarrow +\infty} \sum_{p \in \mathcal{I}_{v,R}^{V_i}} (-1)^{\sigma(p)}$ exists and is finite. Proposition 6.1 becomes

Proposition 6.2. *We have*

$$\lim_{R \rightarrow +\infty} \Lambda_0(X \cap B_R, X \cap S_R) = \sum_{i=1}^t \eta(V_i, \mathbf{1}_X) \frac{1}{s_{2N-1}} \int_{S_{2N-1}} \lim_{R \rightarrow +\infty} \sum_{p \in \mathcal{I}_{v,R}^{V_i}} (-1)^{\sigma(p)} dv.$$

Corollary 6.3. *For $i \in \{1, \dots, t\}$, $\lim_{R \rightarrow +\infty} \Lambda_0(\bar{V}_i \cap B_R, V_i \cap S_R)$ exists and is finite. Furthermore, we have*

$$\lim_{R \rightarrow +\infty} \Lambda_0(X \cap B_R, X \cap S_R) = \sum_{i=1}^t \eta(V, \mathbf{1}_X) \cdot \lim_{R \rightarrow +\infty} \Lambda_0(\bar{V}_i \cap B_R, V_i \cap S_R).$$

Proof. Same proof as Corollary 6.8 in [15]. \square

Theorem 6.4. *For any stratum V_i , we have*

$$\lim_{R \rightarrow +\infty} \Lambda_0(\bar{V}_i \cap B_R, V_i \cap S_R) = \sum_{e=1}^{d_{V_i}} \lim_{R \rightarrow +\infty} \frac{1}{b_{2e} R^{2e}} \frac{1}{s_{2N-2e-1}} \int_{V_i \cap B_R} K_{2d_{V_i}-2e}(x) dx,$$

where b_{2e} is the volume of the $2e$ -dimensional unit ball and d_{V_i} is the dimension of V_i .

Proof. Let us treat first the case of a stratum V of depth 0 i.e., $\bar{V} = V$. This is trivial if $\dim(V) = 0$. If $\dim(V) > 0$, then we have

$$\chi(V \cap B_R) = \Lambda_0(V, V \cap B_R) + \Lambda_0(V \cap B_R, V \cap S_R),$$

and so

$$\lim_{R \rightarrow +\infty} \Lambda_0(V \cap B_R, V \cap S_R) = \chi(V) - \lim_{R \rightarrow \infty} \Lambda_0(V, V \cap B_R).$$

The result is then just an application of Theorem 4.3 in [14].

If V is a stratum of depth greater or equal to 1, then we also have

$$\chi(\bar{V} \cap B_R) = \Lambda_0(\bar{V}, \bar{V} \cap B_R) + \Lambda_0(\bar{V} \cap B_R, \bar{V} \cap S_R),$$

and

$$\lim_{R \rightarrow +\infty} \Lambda_0(\bar{V} \cap B_R, \bar{V} \cap S_R) = \chi(V) - \lim_{R \rightarrow \infty} \Lambda_0(\bar{V}, \bar{V} \cap B_R).$$

By Theorem 3.5 in [14], we obtain that

$$\lim_{R \rightarrow +\infty} \Lambda_0(\bar{V} \cap B_R, \bar{V} \cap S_R) = \sum_{k=1}^{d_V} \lim_{R \rightarrow +\infty} \frac{\Lambda_k(\bar{V}, \bar{V} \cap B_R)}{b_k R^k}.$$

Using the description of the Lipschitz-Killing measures for complex analytic sets done in [15], Section 4, we find that

$$\begin{aligned} \lim_{R \rightarrow +\infty} \Lambda_0(\bar{V} \cap B_R, \bar{V} \cap S_R) &= \sum_{e=1}^{d_V} \lim_{R \rightarrow +\infty} \frac{1}{b_{2e} R^{2e}} \frac{1}{s_{2N-2e-1}} \int_{V \cap B_R} K_{2d_V-2e}(x) dx \\ &+ \sum_{W \subset \bar{V} \setminus V} \eta(W, \mathbf{1}_V) \sum_{e=1}^{d_W} \lim_{R \rightarrow +\infty} \frac{1}{b_{2e} R^{2e}} \frac{1}{s_{2N-2e-1}} \int_{W \cap B_R} K_{2d_W-2e}(x) dx. \end{aligned}$$

Comparing this equality with the previous corollary and applying the induction hypothesis gives the result. \square

Corollary 6.5. *If X is equidimensional then*

$$\text{Eu}(X) = \lim_{R \rightarrow +\infty} \Lambda_0(X \cap B_R, X_{\text{reg}} \cap B_R),$$

and

$$\lim_{R \rightarrow +\infty} \Lambda_0(X \cap B_R, X_{\text{reg}} \cap S_R) = \sum_{i=0}^{d-1} (-1)^i \alpha_X^{(i)}.$$

Proof. By Corollary 5.3 in [15], we know that

$$\text{Eu}(X) = \sum_{e=0}^d \lim_{R \rightarrow +\infty} \frac{1}{b_{2e} R^{2e}} \frac{1}{s_{2N-2e-1}} \int_{X_{\text{reg}} \cap B_R} K_{2(d-e)}(x) dx,$$

and so

$$\begin{aligned} \text{Eu}(X) &= \lim_{R \rightarrow +\infty} \frac{1}{s_{2N-1}} \int_{X_{\text{reg}} \cap B_R} K_{2d}(x) dx + \lim_{R \rightarrow +\infty} \Lambda_0(X \cap B_R, X_{\text{reg}} \cap S_R) \\ &= \lim_{R \rightarrow +\infty} \Lambda_0(X, X_{\text{reg}} \cap B_R) + \lim_{R \rightarrow +\infty} \Lambda_0(X \cap B_R, X_{\text{reg}} \cap S_R) \\ &= \lim_{R \rightarrow +\infty} \Lambda_0(X, X_{\text{reg}} \cap \overset{\circ}{B}_R) + \lim_{R \rightarrow +\infty} \Lambda_0(X \cap B_R, X_{\text{reg}} \cap S_R) \\ &= \lim_{R \rightarrow +\infty} \Lambda_0(X \cap B_R, X_{\text{reg}} \cap \overset{\circ}{B}_R) + \lim_{R \rightarrow +\infty} \Lambda_0(X \cap B_R, X_{\text{reg}} \cap S_R) \\ &= \lim_{R \rightarrow +\infty} \Lambda_0(X \cap B_R, X_{\text{reg}} \cap B_R). \end{aligned}$$

To prove the second equality, we use the fact that

$$\frac{1}{s_{2N-1}} \int_{X_{\text{reg}} \cap B_R} K_{2d}(x) dx = \int_{X_{\text{reg}} \cap B_R} \text{ch}_d(X_{\text{reg}}),$$

where $\text{ch}_d(X_{\text{reg}})$ is the d -th Chern form on X_{reg} and the exchange formula proved by Shifrin in [37], page 103. Passing to the limit as $R \rightarrow +\infty$, this gives that

$$\lim_{R \rightarrow +\infty} \frac{1}{S_{2N-1}} \int_{X_{\text{reg}} \cap B_R} K_{2d}(x) dx = (-1)^d \alpha_X^{(d)}.$$

We just have to apply Theorem 3.4 in [34] to conclude. \square

REFERENCES

- [1] ARTAL, E., LUENGO, I. and MELLE, A.: On the topology of a generic fibre of a polynomial function, *Comm. Algebra* **28** (2000), no. 4, 1767–1787.
- [2] BERNIG, A., BRÖCKER, L.: Courbures intrinsèques dans les catégories analytico-géométriques, *Ann. Inst. Fourier (Grenoble)* **53**(6) (2003), 1897–1924.
- [3] BRASSELET, J.P., LÊ, D. T. and SEADE, J.: Euler obstruction and indices of vector fields, *Topology*, **6** (2000) 1193–1208.
- [4] BRASSELET, J.P., MASSEY, D. , PARAMESWARAN, A. and SEADE, J.: Euler obstruction and defects of functions on singular varieties, *Journal London Math. Soc* (2) **70** (2004), no. 1, 59–76.
- [5] BRASSELET, J.P. and SCHWARTZ, M.H.: Sur les classes de Chern d’un ensemble analytique complexe, *Astérisque* **82-83** (1981) 93–147.
- [6] BRYLINSKI, J., DUBSON, A. and KASHIWARA, M.: Formule de l’indice pour modules holonomes et obstruction d’Euler locale, *C. R. Acad. Sci. Paris Sér. I Math.* **293** (1981), 573–576.
- [7] BRÖCKER, L. and KUPPE, M.: Integral geometry of tame sets, *Geometriae Dedicata* **82** (2000), 285–323.
- [8] BROUGHTON, S. A.: Milnor number and the topology of polynomial hypersurfaces, *Invent. Math.* **92** (1988) 217–241.
- [9] DIAS, L. R. G.: On regularity conditions at infinity, *J. Singul.* **10** (2014), 54–66.
- [10] DIAS, L. R. G.; RUAS, M. A. S. and TIBĂR, M.: Regularity at infinity of real mappings and a Morse-Sard theorem, *J. Topol.* **5** (2012), no. 2, 323–340.
- [11] DUBSON, A.: Classes caractéristiques des variétés singulières, *C. R. Acad. Sci. Paris Sér. A-B* **287** (1978), no. 4, 237–240.
- [12] DUBSON, A.: Calcul des invariants numériques des singularités et applications, *Sonderforschungsbereich 40 Theoretische Mathematik* Universitaet Bonn (1981).
- [13] DUTERTRE, N.: A Gauss-Bonnet formula for closed semi-algebraic sets, *Advances in Geometry* **8**, no. 1 (2008), 33–51.
- [14] DUTERTRE, N.: Euler characteristic and Lipschitz-Killing curvatures of closed semi-algebraic sets, *Geom. Dedicata* **158**, no. 1 (2012), 167–189.
- [15] DUTERTRE, N.: Euler obstruction and Lipschitz-Killing curvatures, *Israel J. Math.* **213** (2016), 109–137.
- [16] DUTERTRE, N. and GRULHA, JR. N. G.: Lê-Greuel type formula for the Euler obstruction and applications, *Adv. Math.* **251** (2014), 127–146.
- [17] DUTERTRE, N. and GRULHA JR., N. G.: Some notes on the Euler obstruction of a function, *J. Singul.* **10** (2014), 82–91.
- [18] FU, J.H.G.: Curvature measures of subanalytic sets, *Amer. J. Math.* **116**, no. 4 (1994), 819–880.
- [19] FU, J.H.G.: Curvature measures and Chern classes of singular varieties, *J. Differential Geom.* **39**, no. 2 (1994), 251–280.
- [20] GORESKY, M. and MAC-PHERSON, R.: Stratified Morse theory, *Springer-Verlag*, Berlin, 1988.
- [21] GRULHA, JR. N. G.: The Euler obstruction and Bruce-Roberts’ Milnor number. *Q. J. Math.* **60** (2009), no. 3, 291–302.
- [22] HÀ, H.V. and LÊ, D.T.: Sur la topologie des polynômes complexes, *Acta Math. Vietnam.* **9** (1984), no. 1, 21–32.
- [23] LÊ D.T.: Vanishing cycles on complex analytic sets, *Proc. Sympos., Res. Inst. Math. Sci., Kyoto, Univ. Kyoto, 1975* Sûrikaiseikikenkyûsho Kôkyûroku, no. 266 (1976), 299–318.

- [24] LÊ D.T.: Complex analytic functions with isolated singularities, *J. Algebraic Geom.* **1** (1992), no. 1, 83–99.
- [25] LÊ D. T. and TEISSIER, B.: Variétés polaires locales et classes de Chern des variétés singulières, *Ann. of Math.* **114**, (1981), 457–491.
- [26] LOESER, F.: Formules intégrales pour certains invariants locaux des espaces analytiques complexes, *Comment. Math. Helv.* **59** (1984), no. 2, 204–225.
- [27] MAC-PHERSON, R. D.: Chern classes for singular algebraic varieties, *Ann. of Math.* **100** (1974), 423–432.
- [28] McCRORY, C. and PARUSINSKI, A.: Algebraically constructible functions, *Ann. Sci. École Norm. Sup. (4)* **30** (1997), no. 4, 527–552.
- [29] MASSEY, D. B.: Hypercohomology of Milnor fibres, *Topology* **35** (1996), no. 4, 969–1003.
- [30] PARUSIŃSKI, A.: On the bifurcation set of complex polynomial with isolated singularities at infinity, *Compositio Math.* **97** (1995), no. 3, 369–384.
- [31] SCHÜRMAN, J.: Topology of singular spaces and constructible sheaves, *Instytut Matematyczny Polskiej Akademii Nauk. Monografie Matematyczne (New Series)*, **63** Birkhauser Verlag, Basel, 2003.
- [32] SCHÜRMAN, J. and TIBĀR, M.: Index formula for MacPherson cycles of affine algebraic varieties, *Tohoku Math. J. (2)* **62** (2010), no. 1, 29–44.
- [33] SEADE, J., TIBĀR, M. and VERJOVSKY A.: Milnor Numbers and Euler obstruction, *Bull. Braz. Math. Soc. (N.S)* **36** (2005), no. 2, 275–283.
- [34] SEADE, J., TIBĀR, M. and VERJOVSKY, A. : Global Euler obstruction and polar invariants, *Math. Ann.* **333** (2005), no. 2, 393–403.
- [35] SIERSMA, D.: A bouquet theorem for the Milnor fibre, *J. Algebraic Geom.* **4** (1995), no. 1, 51–66.
- [36] SIERSMA, D. and TIBĀR, M.: Singularities at infinity and their vanishing cycles, *Duke Math. J.* **80** (1995), no. 3, 771–783.
- [37] SHIFRIN, T.: Formules cinématiques dans le cas complexe, *Introduction à la théorie des singularités, II*, 97–108, Travaux en Cours, **37**, Hermann, Paris, 1988.
- [38] SUZUKI, M.: Propriétés topologiques des polynômes de deux variables complexes, et automorphismes algébriques de l’espace \mathbb{C}^2 , *J. Math. Soc. Japan* **26** (1974), 241–257.
- [39] TIBĀR, M.: Bouquet decomposition of the Milnor fibre, *Topology* **35** (1996), no. 1, 227–241.
- [40] TIBĀR, M.: Asymptotic equisingularity and topology of complex hypersurfaces, *Internat. Math. Res. Notices* (1998), no. 18, 979–990.
- [41] TIBĀR, M.: Regularity at infinity of real and complex polynomial functions, *Singularity theory (Liverpool, 1996)*, London Math. Soc. Lecture Note Ser., **263**, 249–264, Cambridge Univ. Press, Cambridge, 1999.
- [42] TIBĀR, M.: Duality of Euler data for affine varieties, Singularities in geometry and topology 2004, 251–257, *Adv. Stud. Pure Math.*, **46**, Math. Soc. Japan, Tokyo, 2007.
- [43] VIRO, O.: Some integral calculus based on Euler characteristic, *Topology and geometry, Rohlin Seminar*, 127–138, Lecture Notes in Math., **1346**, Springer, Berlin, 1988.

NICOLAS DUTERTRE

LAREMA UMR 6093, CNRS, Univ. Angers, Univ. Bretagne-Loire, 49000

Angers, France.

E-MAIL: nicolas.dutertre@univ-angers.fr

NIVALDO G. GRULHA JR.

Universidade de São Paulo, Instituto de Ciências Matemáticas e de Computação

USP Av. Trabalhador São-carlense, 400 - Centro, Caixa Postal: 668

CEP: 13560-970 - São Carlos - SP - Brazil.

E-MAIL: njunior@icmc.usp.br

SÉRIE MATEMÁTICA

NOTAS DO ICMC

- 437/18** ARAÚJO, R.O.; MA, T.F.; MARINHO, S.S.; PRATES FILHO, J.S. - Uniform stability of a non-autonomous semilinear Bresse system with memory.
- 436/17** ANTONOV, V.; FERNANDES, W.; OMANOVSKI, V.G.; SHCHEGLOVA, N.L. - First integrals of the may-leonard asymmetric system.
- 435/17** ROMANOVSKI, V.G.; FERNANDES, W.; TANG, Y.; TIAN, Y. - Linearizability and critical period bifurcations of a generalized Riccati system.
- 434/17** BONOTTO, E.M.; DEMUNER, D.P.; JIMENEZ, M.Z. - Convergence for non-autonomous semidynamical systems with impulses.
- 433/17** BEZERRA, F.D.M.; CARVALHO, A.N.; CHOLEWA, J.W.; NASCIMENTO, M.J.D. - Parabolic approximation of damped wave equations via fractional powers: fast growing nonlinearities and continuity of the dynamics.
- 432/17** BORTOLAN, M.C.; CARVALHO, A.N. - Strongly damped wave equation and its Yosida approximations.
- 431/17** CARVALHO, A.N.; CHOLEWA, J.W. - NLS-like equations in bounded domains: parabolic approximation procedure.
- 430/17** CARVALHO, A.N.; CHOLEWA, J.W.; NASCIMENTO, M.J.D. - On the continuation of solutions of nonautonomous semilinear parabolic problems.
- 429/16** OLIVEIRA, R.D.S.; REZENDE, A.C.; SCHLOMIUK, D.; VULPE, N. - Geometric and algebraic classification of quadratic differential systems with invariant hyperbolas.
- 428/16** IZUMIYA, S.; NABARRO, A.C.; SACRAMENTO, A.J. - Horospherical and hyperbolic dual surfaces of spacelike curves in de Sitter space.