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Abstract. For the May-Leonard asymmetric system, which is a quadratic Lotka-Voltera system depending on six parameters, we first look for systems admitting invariant algebraic surfaces of degree two. Then for some families of such systems we construct first integrals of Darboux type identifying subfamilies of the May-Leonard asymmetric system with one first integral and with two independent first integrals.

1. Introduction

An important class of mathematical models describing different phenomena in biology, ecology and chemistry are the so-called Lotka-Volterra systems which are written in the form

$$\dot{x}_i = x_i(\sum_{j=1}^{n} a_{ij}x_j + b_i) \qquad (i = 1, \dots, n). \tag{1.1}$$

They were introduced independently by Lotka and Volterra in the 1920s to model the interaction among species, see [?, ?], and have been, and continue to be, intensively studied. For the class of systems (??) most studies are devoted to the case $n = 3$. One of simplest models of such type describing a competition of three species was introduced by May and Leonard in [?]. It is a model depending on two parameters and written as the differential system

$$\begin{aligned} \dot{x} &= x(1 - x - \alpha y - \beta z),\\ \dot{y} &= y(1 - \beta x - y - \alpha z),\\ \dot{z} &= z(1 - \alpha x - \beta y - z). \end{aligned} \tag{1.2}$$

where $x, y, z \geq 0$, $0 < \alpha < 1 < \beta$, and

$$\alpha + \beta > 2. \tag{1.3}$$

It was showed in [?] that system (??) has four singular points in $\mathbb{R}^3_+ = \{(x, y, z) \in \mathbb{R}^3, x, y, z \geq 0\}$. Three of them are on the boundary of $\mathbb{R}^3_+$,

$$E_1 = (1, 0, 0),\ E_2 = (0, 1, 0),\ E_3 = (0, 0, 1),$$

and the forth one is in the interior point

$$C = ((1 + \alpha + \beta)^{-1}, (1 + \alpha + \beta)^{-1}, (1 + \alpha + \beta)^{-1}).$$

There is a separatix cycle F formed by orbits connecting E_1, E_2 and E_3 on the boundary of R^3_+, and every orbit in R^3_+, except of the equilibrium point C has F as the ω-limit. It was showed [?] that in the degenerate case $\alpha+\beta=2$ the cycle F becomes a triangle on the invariant plane $x+y+z=1$, all orbits inside the triangle are closed and every orbit in the interior of $\mathbb{R}^3_+$ has one of these closed orbits as ω-limit. Latter on, the dynamics of (??) was studied in more details in [?, ?, ?] and some other works.

A generalization of model (??) is the model described by the differential system

$$\begin{aligned}\dot x &= x(1-x-\alpha_1 y-\beta_1 z) = X(x,y,z),\\ \dot y &= y(1-\beta_2 x-y-\alpha_2 z) = Y(x,y,z),\\ \dot z &= z(1-\alpha_3 x-\beta_3 y-z) = Z(x,y,z),\end{aligned} \tag{1.4}$$

where $x,y,z \geq 0$ and $\alpha_i,\ \beta_i\ (1\leq i\leq 3)$ are real parameters, which is called the asymmetric May-Leonard model. The dynamics of (??) was studied in [?, ?, ?, ?]. In particular, Chi, Hsu and Wu [?] studied (??) under the assumption

$$0<\alpha_i<1<\beta_i\ (1\leq i\leq 3) \tag{1.5}$$

and showed that under this assumption the system has a unique interior equilibrium P, which is locally asymptotically stable if $A_1A_2A_3 > B_1B_2B_3$, (where $A_i = 1-\alpha_i$, $B_i = \beta_i - 1$, $(1\leq i\leq 3)$) and if $A_1A_2A_3 < B_1B_2B_3$, then P is a hyperbolic point with a one-dimensional stable manifold. They also have shown that if $A_1A_2A_3 \neq B_1B_2B_3$, then the system does not have periodic solutions, and if

$$A_1A_2A_3 = B_1B_2B_3, \tag{1.6}$$

then there is a family of periodic solutions. It was showed in [?] that even if assumption (??) is dropped system (??) still can have a family of periodic solutions. Moreover, it was showed there, that the periodic solutions of the system do not arise as a result of Hopf bifurcations, but their existence is due to the so-called Lyapunov theorem on holomorphic integral.

First integrals of May-Leonard system (??) where studied by Leach and Miritzis [?] (see also [?]), who obtained the following first integrals:

(i) $H_1 = \frac{xyz}{(x+y+z)^3}$ if $\alpha+\beta=2$ and $\alpha\neq 1$

(ii) $H_2 = \frac{y(x-z)}{x(y-z)}$ if $\alpha=\beta\neq 1$,

(iii) $H_3 = x/z$ and $H_4 = y/z$ (two independent first integrals) if $\alpha=\beta=1$.

It was shown in [?] that system (??) is completely integrable, that is, it admits two independent first integrals, if either $\alpha+\beta=2$, or $\beta=\alpha$.

In this paper we study integrability of asymmetric May-Leonard model (??). Using algorithms of the elimination theory we first find systems of the form (??) admitting invariant planes and invariant surfaces defined by quadratic polynomials. Then we look for first integrals of the Darboux type constructed using these invariant surfaces and find subfamilies of (??) admitting one or two independent first integrals. As we show the set of systems with first integrals is much larger for system (**??**) than for classical May-Leonard system (**??**).

The proposed approach can be used to study integrability of many other mathematical models described by polynomial systems of differential equations.

2. Preliminaries

In this section we remind some general results on the elimination theory and the Darboux theory of integrability that we shall use in our study.

Considerer system of differential equations

$$\begin{aligned}\dot{x} =& P(x, y, z),\\ \dot{y} =& Q(x, y, z),\\ \dot{z} =& R(x, y, z),\end{aligned} \tag{2.1}$$

where P, Q and R are polynomials of degree at most m, and let $\mathfrak{X}$ be the corresponding vector field,

$$\mathfrak{X} = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z}.$$

A C^1 function $H : U \to \mathbb{R}$ non-constant in any open subset of $U \subset \mathbb{R}^2$ is a first integral of the differential system (**??**) if and only if $\mathfrak{X}H \equiv 0$ in U. Let $H_1 : U_1 \to \mathbb{R}$ and $H_2 : U_2 \to \mathbb{R}$ be two first integrals of system (**??**). It is said that H_1 and H_2 are independent in $U_1 \cap U_2$ if their gradients are independent in all the points of $U_1 \cap U_2$ except perhaps in a zero Lebesgue measure set. System (**??**) is completely integrable on $U_1 \cap U_2$ if it has two independent first integrals on $U_1 \cap U_2$.

A *Darboux polynomial* of system (**??**) is a polynomial $f(x, y, z)$ such that

$$\mathfrak{X}f := \frac{\partial f}{\partial x}P + \frac{\partial f}{\partial y}Q + \frac{\partial f}{\partial z}R = Kf, \tag{2.2}$$

where $K(x, y, z)$ is a polynomial of degree at most $m - 1$. The polynomial $K(x, y, z)$ is called the *cofactor of* f. It is easy to see that if f is a Darboux polynomial of (**??**) then the equation $f = 0$ defines an algebraic surface which is invariant under the flow of system (**??**). For this reason f often is referred as invariant algebraic surface of (**??**).

A simple computation shows that if there are Darboux polynomials $f_1, f_2, ..., f_k$ with the cofactors $K_1, K_2, ..., K_k$ satisfying

$$\sum_{i=1}^{k} \lambda_i K_i = 0, \tag{2.3}$$

then

$$H = f_1^{\lambda_1} \cdots f_k^{\lambda_k}, \tag{2.4}$$

is a first integral of (**??**). An integral of the form (**??**) is called *a Darboux integral* of system (**??**).

To find Darboux polynomials (algebraic invariant surfaces) of system (**??**) we will use the following result from computational commutative algebra. Let I be an ideal in a polynomial ring $k[x_1, \dots, x_n]$, where k is a field, and ℓ be a fixed number from the set $\{0, 1, \dots, n-1\}$. The *ℓ-th elimination ideal* of I is the ideal

$$I_\ell = I \cap k[x_{\ell+1}, \dots, x_n].$$

According to the Elimination Theorem (see, for example, [**?**, **?**]) in order to compute (for any $0 \leqslant \ell \leqslant n-1$) the ℓ-th elimination ideal I_ℓ of an ideal I of $k[x_1, \dots, x_n]$ one can choose the lexicographic term order on the ring $k[x_1, \dots, x_n]$ with $x_1 > x_2 > \cdots > x_n$ and compute a Gröbner basis G for the ideal I with respect to this order. Then, by the Elimination theorem, the set

$$G_\ell := G \cap k[x_{\ell+1}, \dots, x_n]$$

is a Gröbner basis for the ℓ-th elimination ideal I_ℓ. Geometrically, the elimination means projecting the variety $\mathbf{V}(I)$ of the ideal I to the affine space $k^{n-\ell}$ corresponding to the variables $x_{\ell+1}, \dots, x_n$.

3. Darboux polynomials of system (??)

Invariant planes of system (**??**) and Darboux integrals constructed from such planes were found in [**?**]. In this section using the Elimination Theorem we look for Darboux polynomials of system (**??**) of degree two. A general form of a polynomial of degree two is

$$\begin{aligned} f(x,y,z) = {} & h_{000} + h_{100}x + h_{010}y + h_{001}z + h_{200}x^2 + h_{110}xy \\ & + h_{101}xz + h_{020}y^2 + h_{011}yz + h_{002}z^2. \end{aligned} \tag{3.1}$$

A cofactor of any Darboux polynomials of system (**??**) is a polynomial of degree one which we write in the form

$$K(x,y,z) = c_0 + c_1x + c_2y + c_3z. \tag{3.2}$$

Polynomial (**??**) will be a Darboux polynomial of system (**??**) with cofactor (**??**) if

$$\mathfrak{X}f = Kf, \tag{3.3}$$

where now

$$\mathfrak{X}f = \frac{\partial f}{\partial x}X + \frac{\partial f}{\partial y}Y + \frac{\partial f}{\partial z}Z,$$

with X, Y and Z defined in (**??**).

Comparing the coefficients of similar terms in (**??**), we obtain the polynomial system

$$g_1 = g_2 = \dots = g_{19} = g_{20} = 0,$$

where

$$
\begin{aligned}
g_1 &= -c_0h_{000},\\
g_2 &= -c_3h_{000} + h_{001} - c_0h_{001},\\
g_3 &= -h_{001} - c_3h_{001} + 2h_{002} - c_0h_{002},\\
g_4 &= -2h_{002} - c_3h_{002},\\
g_5 &= -c_2h_{000} + h_{010} - c_0h_{010},\\
g_6 &= -\beta_3h_{001} - c_2h_{001} - \alpha_2h_{010} - c_3h_{010} + 2h_{011} - c_0h_{011},\\
g_7 &= -2\beta_3h_{002} - c_2h_{002} - h_{011} - \alpha_2h_{011} - c_3h_{011},\\
g_8 &= -h_{010} - c_2h_{010} + 2h_{020} - c_0h_{020},\\
g_9 &= -2h_{020} - c_2h_{020},\\
g_{10} &= -h_{011} - \beta_3h_{011} - c_2h_{011} - 2\alpha_2h_{020} - c_3h_{020},\\
g_{11} &= -c_1h_{000} + h_{100} - c_0h_{100},\\
g_{12} &= -\alpha_3h_{001} - c_1h_{001} - \beta_1h_{100} - c_3h_{100} + 2h_{101} - c_0h_{101},\\
g_{13} &= -2\alpha_3h_{002} - c_1h_{002} - h_{101} - \beta_1h_{101} - c_3h_{101},\\
g_{14} &= -\beta_2h_{010} - c_1h_{010} - \alpha_1h_{100} - c_2h_{100} + 2h_{110} - c_0h_{110},\\
g_{15} &= -2\beta_2h_{020} - c_1h_{020} - h_{110} - \alpha_1h_{110} - c_2h_{110},\\
g_{16} &= -\alpha_3h_{011} - \beta_2h_{011} - c_1h_{011} - \alpha_1h_{101} - \beta_3h_{101}\\
&\quad - c_2h_{101} - \alpha_2h_{110} - \beta_1h_{110} - c_3h_{110},\\
g_{17} &= -h_{100} - c_1h_{100} + 2h_{200} - c_0h_{200},\\
g_{18} &= -2h_{200} - c_1h_{200},\\
g_{19} &= -h_{110} - \beta_2h_{110} - c_1h_{110} - 2\alpha_1h_{200} - c_2h_{200},\\
g_{20} &= -h_{101} - \alpha_3h_{101} - c_1h_{101} - 2\beta_1h_{200} - c_3h_{200}.
\end{aligned}
\tag{3.4}
$$

We denote by $I = \langle g_1, g_2, ..., g_{19}, g_{20}\rangle$ the ideal generated by polynomials in (**??**). Since computations based on the Elimination Theorem are very laborious, to simplify them we consider separately the cases $h_{000} = 1$ and $h_{000} = 0$, that is, we look separately for invariant curves $f = 0$ not passing and passing through the origin, so *from now on in this section we assume that* $h_{000} = 1$.

To find Darboux polynomials of system (**??**) of degree two, we have to determine for which values of parameters α_i, β_i $(i = 1, 2, 3)$ system (**??**) has a solution with at least one of coefficient $h_{200}, h_{002}, h_{011}, h_{020}, h_{101}, h_{110}$ different from zero. To satisfy this condition we have six options that can be written in polynomial form as

$$
\begin{aligned}
&1 - wh_{200} = 0,\ \ 1 - wh_{110} = 0,\ \ 1 - wh_{101} = 0,\\
&1 - wh_{020} = 0,\ \ 1 - wh_{011} = 0,\ \ 1 - wh_{002} = 0,
\end{aligned}
\tag{3.5}
$$

with w being a new variable. For instance, to find systems of form (**??**) which have surfaces with $h_{200} \neq 0$, we can compute (for example, with the routine `eliminate` of the computer algebra system SINGULAR [?]) the 13-th elimination ideal of the ideal

$I^{(1)} = \langle I, 1 - wh_{200} \rangle$, in the ring

$$\mathbb{Q}[w, c_0, c_1, c_2, c_3, h_{001}, h_{002}, h_{010}, h_{011}, h_{020}, h_{100}, h_{101}, h_{110}, \alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3].$$

Denote this elimination ideal by $I_{13}^{(1)}$ and its variety by V_1 (that is, $V_1 = \mathbf{V}(I_{13}^{(1)})$). Proceeding analogously, we can find the other five eliminations ideals $I_{13}^{(2)}, \ldots, I_{13}^{(6)}$ corresponding to the other cases of (??). Denote the corresponding varieties $V_2 = \mathbf{V}(I_{13}^{(2)}), \ldots, \mathbf{V}(I_{13}^{(6)})$. It is clear that the union $V = V_1 \cup ... \cup V_6$ of these six varieties contains the set of all systems (??) having invariant surfaces of the form (??) not passing through the origin. To compute the irreducible decomposition of the variety V it is sufficient to compute the ideal $J = I_{13}^{(1)} \cap \cdots \cap I_{13}^{(6)}$, which defines the variety $V = V_1 \cup ... \cup V_6$ and then to find the irreducible decomposition of V. The intersection of ideals can be computed with the routine `intersect` of Singular, and the irreducible decomposition of V can be found with the routine `minAssGTZ`, which is based on the algorithm of [?]. Theoretically, such computations should give all systems in the family (??) having invariant surfaces of degree two. However all the routines `eliminate`, `intersect` and `minAssGTZ` rely on computations of many Groebner bases, and such computations can be rarely completed when computing over the field $\mathbb{Q}$ of rational numbers for polynomials in many variables. To be able to complete our computations we computed in the field of finite characteristic 32003 and then lifted the resulting ideals to the ring of polynomials with rational coefficients using the rational reconstruction algorithm of [?] (a Mathematica code for the algorithm can be found in [?]).

The primary decomposition of the radical of the ideal

$$J = \bigcap_{i=1}^{6} I_{13}^{(i)} \tag{3.6}$$

computed using the routine `minAssGTZ` in the field of characteristic 32003 consists of 88 ideals, that is, we have 88 irreducible components of the variety $\mathbf{V}(J)$ given in Appendix. It means there are 88 conditions on the parameters α_i, β_i of system (??) for existence of an invariant surface of degree two not passing through the origin.

However some of these conditions give systems with the same dynamics in the phase space, since system (??) has a symmetry with respect to simple linear transformations. Namely, it is easily seen that the transformations

$$x \to z,\ y \to x,\ z \to y, \tag{3.7}$$

$$x \to y,\ y \to z,\ z \to x, \tag{3.8}$$

$$x \to y,\ y \to x,\ z \to z, \tag{3.9}$$

$$x \to z,\ y \to y,\ z \to z, \tag{3.10}$$

$$x \to x,\ y \to z,\ z \to y, \tag{3.11}$$

which correspond to relabelling of the coordinate axes, do not change the shape of the system. For instance, under transformation (??) system (??) is changed into the system

$$\begin{aligned} \dot{x} =& x(1 - x - \alpha_2 y - \beta_2 z), \\ \dot{y} =& y(1 - \beta_3 x - y - \alpha_3 z), \\ \dot{z} =& z(1 - \alpha_1 x - \beta_1 y - z), \end{aligned} \tag{3.12}$$

which can be obtained from system (**??**) by the change of parameters

$$\alpha_1 \to \alpha_3,\ \beta_1 \to \beta_3,\ \alpha_2 \to \alpha_1,\ \beta_2 \to \beta_1,\ \alpha_3 \to \alpha_2,\ \beta_3 \to \beta_2. \tag{3.13}$$

Thus, if we have a condition on the parameters of (**??**) under which the system has an algebraic invariant surface, another condition will be obtained by the transformation of the parameters according to rule (**??**). For example, as we will see below, system (**??**) has the invariant surface

$$f = 2 - 4x + 2x^2 - 2y + yz$$

if condition (4) of Theorem **??** is fulfilled, that is, if

$$\beta_3 = \beta_1 = \alpha_3 + 1 = \beta_2 - 3 = \alpha_2 + 1 = \alpha_1 - 1/2 = 0.$$

Applying to (**??**) transformation (**??**) we obtain that system (**??**) has the invariant surface

$$f = 2 - 4z + 2z^2 - 2x + xy$$

if the condition

$$\beta_2 = \beta_3 = \alpha_2 + 1 = \beta_1 - 3 = \alpha_1 + 1 = \alpha_3 - 1/2 = 0$$

holds, that is, condition (4) is changed according to (**??**). Similarly, after substitutions (**??**)–(**??**) the conditions for existence of invariant surfaces are changed according to the rules

$$\alpha_1 \to \alpha_2,\ \beta_1 \to \beta_2,\ \alpha_2 \to \alpha_3,\ \beta_2 \to \beta_3,\ \alpha_3 \to \alpha_1,\ \beta_3 \to \beta_1, \tag{3.14}$$

$$\alpha_1 \to \beta_2,\ \beta_1 \to \alpha_2,\ \alpha_2 \to \beta_1,\ \beta_2 \to \alpha_1,\ \alpha_3 \to \beta_3,\ \beta_3 \to \alpha_3, \tag{3.15}$$

$$\alpha_1 \to \beta_3,\ \beta_1 \to \alpha_3,\ \alpha_2 \to \beta_2,\ \beta_2 \to \alpha_2,\ \alpha_3 \to \beta_1,\ \beta_3 \to \alpha_1, \tag{3.16}$$

$$\alpha_1 \to \beta_1,\ \beta_1 \to \alpha_1,\ \alpha_2 \to \beta_3,\ \beta_2 \to \alpha_3,\ \alpha_3 \to \beta_2,\ \beta_3 \to \alpha_2, \tag{3.17}$$

respectively.

We say, that two conditions for existence of invariant surfaces are *conjugate* if one can be obtained from another by means of one of transformations (**??**)–(**??**). For instance, condition (4) (which is the same as condition (7) from Appendix) and conditions (10), (19), (25), (33), (47) from Appendix can be obtained from each other by one of transformations (**??**)–(**??**), so all these conditions are conjugate.

Note that some of the obtained 88 conditions give Darboux polynomials of degree two which are not irreducible, but they are products of two polynomials of degree one. Namely, if

(i) $\alpha_1 = \beta_1 = 0$ (condition (1) of Appendix), then system (**??**) has the Darboux polynomial $(-1+x)^2$ (and the conjugate conditions are (22) and (36) from Appendix);

(ii) $\alpha_2 = \beta_1 = \beta_2 + \alpha_1 - 2 = 0$ (condition (5) of Appendix), then system (**??**) has the Darboux polynomial $(-1 + x + z)^2$ (and the conjugate conditions are (44) and (78) from Appendix);
(iii) $\beta_1 + \alpha_3 - 2 = \beta_2 + \alpha_1 - 2 = \beta_3 + \alpha_2 - 2 = 0$ (condition (88) of Appendix), then system (**??**) has the Darboux polynomial $(-1 + x + y + z)^2$.

From the analysis of the obtained 88 conditions we obtain the following result.

Theorem 3.1. *System* (**??**) *has an irreducible invariant surface of degree two not passing through the origin if one of the following conditions or conjugated to it holds:*

(1) $\alpha_2 = \beta_1 = \beta_2 - 1/2 = \alpha_1 - 3 = 0,$
(2) $\alpha_2 = \beta_1 = \beta_2 - 3 = \alpha_1 - 3 = 0,$
(3) $\beta_3 = \beta_1 = \alpha_3 + \beta_2 - 1 = \alpha_2 + 1 = \alpha_1 - \alpha_3 - 1 = 0,$
(4) $\beta_3 = \beta_1 = \alpha_3 + 1 = \beta_2 - 3 = \alpha_2 + 1 = \alpha_1 - 1/2 = 0,$
(5) $\beta_3 = \beta_1 = \alpha_3 - 3 = \beta_2 - 3 = \alpha_2 - 3/2 = \alpha_1 + 1 = 0,$
(6) $\beta_1 = \beta_3 - 3 = \alpha_3 - 3 = \beta_2 - 1 = \alpha_2 - 1/2 = \alpha_1 - 1 = 0,$
(7) $\beta_1 = \beta_3 - 3 = \alpha_3 - 1/2 = \beta_2 - 1/2 = \alpha_2 + 1 = \alpha_1 - 3 = 0,$
(8) $\beta_1 = \beta_3 - 1/2 = \alpha_3 - 3 = \beta_2 - 3 = \alpha_2 - 3/2 = \alpha_1 - 1/2 = 0,$
(9) $\beta_1 = \beta_3 - 3 = \alpha_3 + 3 = \beta_2 - 3 = \alpha_2 + 1 = \alpha_1 - 3 = 0,$
(10) $\beta_1 = \beta_3 - 1/2 = \alpha_3 - 2 = \beta_2 - 3 = \alpha_2 - 3/2 = \alpha_1 - 1/2 = 0,$
(11) $\beta_1 = \alpha_3 = \beta_2 - \beta_3 - 1 = \alpha_2 + \beta_3 - 2 = \alpha_1 + \beta_3 - 1 = 0,$
(12) $\beta_1 = \beta_3 - 3 = \alpha_3 + \beta_2 - 4 = \alpha_2 + 1 = \alpha_1 - \alpha_3 + 2 = 0,$
(13) $\beta_3 - 1/2 = \alpha_3 - 1/2 = \alpha_2 - 3 = \beta_1 - 3 = \alpha_1 + \beta_2 - 2 = 0,$
(14) $\beta_3 - 1/2 = \alpha_3 - 3 = \beta_2 - 3 = \alpha_2 - 3 = \beta_1 - 3 = \alpha_1 - 1/2 = 0,$
(15) $\beta_3 - 1/2 = \beta_2 - 3 = \alpha_2 - 3 = \alpha_3 + \beta_1 - 2 = \alpha_1 - 1/2 = 0,$
(16) $\beta_3 - 3 = \alpha_3 - 3 = \alpha_2 - 3 = \beta_1 - 3 = \alpha_1 + \beta_2 - 2 = 0,$
(17) $\beta_3 - 3 = \alpha_3 + \beta_2 - 4 = \alpha_2 - 3 = \alpha_3 + \beta_1 - 2 = \alpha_1 - \alpha_3 + 2 = 0.$

Proof. For each case of the theorem we give below the irreducible Darboux polynomial f of degree two which defines quadratic invariant surface $f = 0$ not passing through the origin and the corresponding cofactor:

(1) $f = 1 - x - 2y + y^2; \quad K = -x - 2y;$
(2) $f = 1 - 2x + x^2 - 2y - 2xy + y^2; \quad K = -2(x + y);$
(3) $f = 2 - 2x - 2y + yz; \quad K = -x - y;$
(4) $f = 2 - 4x + 2x^2 - 2y + yz; \quad K = -2x - y;$
(5) $f = 2 - 4x + 2x^2 + 2xy - 2z + xz; \quad K = -2x - z;$
(6) $f = 2 - 4x + 2x^2 - 4y + 4xy + 2y^2 - 2z + xz; \quad K = -2x - 2y - z;$
(7) $f = 1 - x - 2y + y^2 + yz; \quad K = -x - 2y;$
(8) $f = 2 - 4x + 2x^2 - 2y - 2z + xz; \quad K = -2x - y - z;$
(9) $f = 1 - 2x + x^2 - 2y - 2xy + y^2 + yz; \quad K = -2(x + y);$
(10) $f = 1 - 2x + x^2 - y - z + xz; \quad K = -2x - y - z;$
(11) $f = 1 - x - y - z + xz; \quad K = -x - y - z;$
(12) $f = 1 - 2x + x^2 - 2y + 2xy + y^2 + yz; \quad K = -2(x + y);$
(13) $f = 1 - x - y - 2z + z^2; \quad K = -x - y - 2z;$

(14) $f = 1 - 2x + x^2 - y - 2z - 2xz + z^2$; $K = -2x - y - 2z$;
(15) $f = 1 - 2x + x^2 - y - 2z + 2xz + z^2$; $K = -2x - y - 2z$;
(16) $f = 1 - 2x + x^2 - 2y + 2xy + y^2 - 2z - 2xz - 2yz + z^2$; $K = -2(x + y + z)$;
(17) $f = 1 - 2x + x^2 - 2y + 2xy + y^2 - 2z + 2xz - 2yz + z^2$; $K = -2(x + y + z)$.

□

4. First integrals of system (??)

In this section we look for Darboux first integrals of system (??), which can be constructed using the invariant surfaces obtained in the previous section.

Theorem 4.1. *a) If one of conditions 1-3, 11, 12, 17 of Theorem* **??** *holds, then the corresponding system* (**??**) *admits at least one Darboux first integral.*
b) If one of conditions 4-10, 13-16 of Theorem **??** *holds, then the corresponding system* (**??**) *is completely integrable on* $\mathbb{R}^2$ *(it admits two independent Darboux first integrals).*

Proof. First note that system (??) always has the following three invariant surfaces of degree one, with the respective cofactors:

$$\begin{aligned} f_1 &= x;\ \ K_1 = 1 - x - \alpha_1 y - \beta_1 z;\\ f_2 &= y;\ \ K_2 = 1 - \beta_2 x - y - \alpha_2 z;\\ f_3 &= z;\ \ K_3 = 1 - \alpha_3 x - \beta_3 y - z. \end{aligned} \tag{4.1}$$

Case a). To prove statement a) of the theorem we present the Darboux first integrals for each case mentioned in the statement.

If condition 1) of Theorem ?? is satisfied the system has the form

$$\dot{x} = x(1 - x - 3y), \quad \dot{y} = y(1 - x/2 - y)y, \quad \dot{z} = z(1 - \alpha_3 x - \beta_3 y - z). \tag{4.2}$$

Besides the invariant surface f_1, f_2 and f_3 given above and the invariant surface $f = 1 - x - 2y + y^2$, system (??) has the following surfaces f_4, f_5 (with cofactors K_4, K_5, respectively),

$$f_4 = x + 4y;\ \ K_4 = 1 - x - y;$$
$$f_5 = x + 2y - 2y^2;\ \ K_5 = 1 - x - 2y.$$

From the corresponding equation (??) we find that $\lambda_1 = \lambda_3/2, \lambda_2 = \lambda_4, \lambda_5 = -\lambda_3 - 2\lambda_4, \lambda_6 = 0$. Thus, for arbitrary λ_3, λ_4 not both equal to zero system (??) has a Darboux first integral

$$\widetilde{H} = x^{\lambda_4} y^{\lambda_3} (x + 4y)^{\lambda_4} \left(x - 2y^2 + 2y\right)^{-\lambda_3 - 2\lambda_4} \left(-x + y^2 - 2y + 1\right)^{\frac{\lambda_3}{2}}.$$

In particular, taking $\lambda_4 = 1$ and $= \lambda_3 = 0$ we have the Darboux first integral

$$H = \frac{x(x + 4y)}{(x + 2y - 2y^2)^2}.$$

Since first two equations of (??) are independent of z we cannot construct another independent first integral $H_2(x,y)$ of (??) using only the surfaces given above, since if such integral would exist then the two-dimensional system

$$\dot{x} = x(1-x-3y), \quad \dot{y} = y(1-x/2-y)y,$$

would have two independent first integrals, which is impossible.

Similarly as above we find families of Darboux integrals for other cases. We list below representatives of the families for the corresponding cases:

2) $H = \dfrac{xy}{(-x+x^2-y-2xy+y^2)^2}$;

3) $H = \dfrac{(x+y-yz)^2}{x^2+2xy+y^2-2yz}$;

11) $H = \dfrac{xz(1-x-y-z+xz)}{(-x-y-z+2xz)^2}$;

12) $H = \dfrac{yz}{(-x+x^2-y+2xy+y^2+yz)^2}$;

17) $H = \dfrac{yz}{(-x+x^2-y+2xy+y^2-z+2xz-2yz+z^2)^2}$.

Case b). For each system of this case we present two independent Darboux first integrals.

Condition 4). Besides the invariant surface f_1, f_2, f_3 above and f of the previous theorem, we have the invariant surfaces $f_4 = 4x+y-2z$ with the cofactor $K_4 = 1-x-y-z$. Using these polynomials we can find the following two Darboux first integrals:

$$H_1 = \frac{z(2-4x+2x^2-2y+yz)}{(4x+y-2z)},$$
$$H_2 = \frac{yz}{x^2}.$$

To check if these first integrals are independent we compute their gradients and obtain, that are

$$G_1 = \{\frac{4(-2+2x+y)(1+x-z)z}{(4x+y-2z)^2}, \ -\frac{2(1+x-z)^2 z}{(4x+y-2z)^2}, \frac{2(4x-8x^2+4x^3+y-6xy+x^2y-y^2+4xyz+y^2z-yz^2)}{(4x+y-2z)^2}\},$$
$$G_2 = \{-\frac{2yz}{x^3}, \ \frac{z}{x^2}, \ \frac{y}{x^2}\},$$

respectively. Computations show that the linear combination aG_1+bG_2, where $a, b \in \mathbb{R}$, is equal to 0 if and only if $a = b = 0$.

Therefore the Darboux first integrals H_1 and H_2 are independents.

Condition 5). Besides the invariant surface f_1, f_2, f_3 given in (??) and f of the previous theorem, we have the following invariant surfaces passing through the origin with respective cofactors:

$$f_4 = -4xy + 2xz + z^2; \;\; K_4 = -2(-1 + 2x + z);$$
$$f_5 = 2y + z; \;\; K_5 = 1 - 3x - y - z;$$
$$f_6 = 2x + 2y + z; \;\; K_6 = 1 - x - y - z;$$
$$f_7 = -2x + 2x^2 + 2xy - z + xz; \;\; K_7 = 1 - 2x - z.$$

Using these polynomials we can find the following two Darboux first integrals:

$$H_1 = \frac{xy^2}{z^2(2x + 2y + z)},$$
$$H_2 = \frac{xy^2}{(2y + z)(-2x + 2x^2 + 2xy - z + xz)^2}.$$

The gradients of them are

$$G_1 = \{\frac{y^2(2y + z)}{z^2(2x + 2y + z)^2}, \; \frac{2xy(2x + y + z)}{z^2(2x + 2y + z)^2}, \; -\frac{xy^2(4x + 4y + 3z)}{z^3(2x + 2y + z)^2}\},$$
$$G_2 = \{ -\frac{y^2(-2x + 6x^2 + 2xy + z + xz)}{(2y + z)(-2x + 2x^2 + 2xy - z + xz)^3)},$$
$$\frac{2xy(-2xy + 2x^2y - 2xy^2 - 2xz + 2x^2z - yz + xyz - z^2 + xz^2)}{(2y + z)^2(-2x + 2x^2 + 2xy - z + xz)^3},$$
$$-\frac{xy^2(-2x + 2x^2 - 4y + 6xy - 3z + 3xz)}{(2y + z)^2(-2x + 2x^2 + 2xy - z + xz)^3)}\},$$

respectively. Analogously to the previous case to show that these two Darboux first integrals are independent we verify that a linear combination $aG_1 + bG_2$, where $a, b \in \mathbb{R}$, is equal to 0 if and only if $a = b = 0$.

Therefore the Darboux first integrals H_1 and H_2 are independents.

Using the similar computations we get the following pairs of independents Darboux first integrals for the remaining cases:

6. $H_1 = \dfrac{z}{x(2 - 4x + 2x^2 - 4y + 4xy + 2y^2 - 2z + xz)},$

   $H_2 = \dfrac{z(2x + 4y + z)}{y^2(2 - 4x + 2x^2 - 4y + 4xy + 2y^2 - 2z + xz)};$

7. $H_1 = -\dfrac{x^2}{yz(-1 + x + 2y - y^2 - yz)},$

   $H_2 = -\dfrac{(x - 2z)^2}{z(y + z)(-1 + x + 2y - y^2 - yz)};$

8. $H_1 = \dfrac{xz(2 - 4x + 2x^2 - 2y - 2z + xz)}{(y + z)^2},$

   $H_2 = \dfrac{(2x + z)(y + z)^2}{xy^2};$

9. $H_1 = \frac{yz(1-2x+x^2-2y-2xy+y^2+yz)}{x^2}$,
$H_2 = \frac{y^2z(4x-z)}{x^2(x^2-2xy+y^2+yz)}$;

10. $H_1 = -\frac{x(y-z+xz+z^2)}{z(-2x+2x^2-y-z+2xz)}$,
$H_2 = \frac{y^2(x+z)}{z(2x-2x^2+y+z-2xz)(y-z+xz+z^2)}$;

13. $H_1 = \frac{(-x-y-2z+2z^2)^2}{z^2(1-x-y-2z+z^2)}$,
$H_2 = \frac{xz^{2\alpha_1-2}(x+y+4z)^{2-2\alpha_1}}{y}$;

14. $H_1 = \frac{(-2x+2x^2-y-2z-4xz+2z^2)^2}{(x-z)^2(1-2x+x^2-y-2z-2xz+z^2)}$,
$H_2 = \frac{x(y+4z)^2(-2x+2x^2-y-2z-4xz+2z^2)^2}{z(x-z)^4(1-2x+x^2-y-2z-2xz+z^2)^2}$;

15. $H_1 = \frac{y(4x+y+4z)}{(2x-2x^2+y+2z-4xz-2z^2)^2}$,
$H_2 = \frac{x(4x+y+4z)^{1-\alpha_3}(-2x+2x^2-y-2z+4xz+2z^2)^{1+\alpha_3}}{z(x+z)^2(1-2x+x^2-y-2z+2xz+z^2)}$;

16. $H_1 = \frac{(x+y)z}{(-x+x^2-y+2xy+y^2-z-2xz-2yz+z^2)^2}$,
$$H_2 = \frac{1}{y^4}x^4z^{2-2\alpha_1}(1-2x+x^2-2y+2xy+y^2-2z-2xz-2yz+z^2)^{1-\alpha_1}$$
$$(-x+x^2-y+2xy+y^2-z-2xz-2yz+z^2)^{2\alpha_1+2}.$$

□

To summarize, we have found some Darboux first integral of May-Leonard asymetric system (??) which are constructed using Darboux polynomials of degree one and two. We do not know if we found all independent first integrals of system (??) which can be constructed from Darboux polynomials of degree one and two. To verify if the list is complete, we have to find Darboux polynomials of (??), which define invariant algebraic surfaces passing through the origin, that is, polynomials (??) with $h_{000} = 0$. A naïve expectation is that this case should be simpler, than the case $h_{000} = 1$, which we have successfully investigated in this paper. However it turns out that the case $h_{000} = 0$ is computationally much more difficult and we were not able to complete computations for this case using our computational facilities. We believe that a reason for this difficulty is that since the origin is a singular point there are many invariant surfaces passing through the origin and it implies a complicate structure of the elimination ideals which we have to compute using our approach.

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Appendix

Here we list the irreducible components of the variety of ideal (**??**), which give conditions for existence in system (**??**) invariant surfaces of degree two not passing through the origin of the system:

(1) $\alpha_1 = \beta_1 = 0$
(2) $-(1/2) + \beta_2 = \alpha_2 = \beta_1 = -3 + \alpha_1 = 0$
(3) $-3 + \beta_2 = \alpha_2 = \beta_1 = -(1/2) + \alpha_1 = 0$
(4) $-3 + \beta_2 = \alpha_2 = \beta_1 = -3 + \alpha_1 = 0$
(5) $\alpha_2 = \beta_1 = \beta_2 + \alpha_1 - 2 = 0$
(6) $\beta_3 = -1 + \alpha_3 + \beta_2 = 1 + \alpha_2 = \beta_1 = -1 + \alpha_1 - \alpha_3 = 0$
(7) $\beta_3 = 1 + \alpha_3 = -3 + \beta_2 = 1 + \alpha_2 = \beta_1 = -(1/2) + \alpha_1 = 0$
(8) $\beta_3 = -3 + \alpha_3 = -3 + \beta_2 = -(3/2) + \alpha_2 = \beta_1 = 1 + \alpha_1 = 0$
(9) $-3 + \beta_3 = -(3/2) + \alpha_3 = \beta_2 = 1 + \alpha_2 = \beta_1 = -3 + \alpha_1 = 0$
(10) $-3 + \beta_3 = 1 + \alpha_3 = \beta_2 = -(1/2) + \alpha_2 = \beta_1 = 1 + \alpha_1 = 0$
(11) $-3 + \beta_3 = -3 + \alpha_3 = -1 + \beta_2 = -(1/2) + \alpha_2 = \beta_1 = -1 + \alpha_1 = 0$
(12) $-3 + \beta_3 = -(1/2) + \alpha_3 = -(1/2) + \beta_2 = 1 + \alpha_2 = \beta_1 = -3 + \alpha_1 = 0$
(13) $1 + \alpha_3 = \beta_2 = -2 + \alpha_2 + \beta_3 = \beta_1 = -1 + \alpha_1 + \beta_3 = 0$
(14) $-(1/2) + \beta_3 = -3 + \alpha_3 = -3 + \beta_2 = -(3/2) + \alpha_2 = \beta_1 = -(1/2) + \alpha_1 = 0$
(15) $-3 + \beta_3 = 3 + \alpha_3 = -3 + \beta_2 = 1 + \alpha_2 = \beta_1 = -3 + \alpha_1 = 0$
(16) $-(1/2) + \beta_3 = -2 + \alpha_3 = -3 + \beta_2 = -(3/2) + \alpha_2 = \beta_1 = -(1/2) + \alpha_1 = 0$
(17) $\alpha_3 = -1 + \beta_2 - \beta_3 = -2 + \alpha_2 + \beta_3 = \beta_1 = -1 + \alpha_1 + \beta_3 = 0$
(18) $-3 + \beta_3 = -4 + \alpha_3 + \beta_2 = 1 + \alpha_2 = \beta_1 = 2 + \alpha_1 - \alpha_3 = 0$
(19) $1 + \beta_3 = -3 + \alpha_3 = 1 + \beta_2 = \alpha_2 = -(1/2) + \beta_1 = \alpha_1 = 0$
(20) $1 + \beta_3 = -1 + \alpha_3 + \beta_2 = \alpha_2 = -2 + \alpha_3 + \beta_1 = \alpha_1 = 0$
(21) $-(3/2) + \beta_3 = -3 + \alpha_3 = -3 + \beta_2 = \alpha_2 = 1 + \beta_1 = \alpha_1 = 0$
(22) $\alpha_2 = \beta_2 = 0$
(23) $-2 + \beta_3 = -(1/2) + \alpha_3 = -(1/2) + \beta_2 = \alpha_2 = -(3/2) + \beta_1 = -3 + \alpha_1 = 0$
(24) $-3 + \beta_3 = -(1/2) + \alpha_3 = -(1/2) + \beta_2 = \alpha_2 = -(3/2) + \beta_1 = -3 + \alpha_1 = 0$
(25) $1 + \beta_3 = \alpha_3 = -(1/2) + \beta_2 = \alpha_2 = 1 + \beta_1 = -3 + \alpha_1 = 0$
(26) $3 + \beta_3 = -3 + \alpha_3 = -3 + \beta_2 = \alpha_2 = 1 + \beta_1 = -3 + \alpha_1 = 0$
(27) $-(1/2) + \beta_3 = -3 + \alpha_3 = -3 + \beta_2 = \alpha_2 = 1 + \beta_1 = -(1/2) + \alpha_1 = 0$
(28) $\alpha_3 = -1 + \beta_2 - \beta_3 = \alpha_2 = 1 + \beta_1 = -1 + \alpha_1 + \beta_3 = 0$
(29) $-3 + \beta_3 = \alpha_3 = 1 + \beta_2 = \alpha_2 = -(3/2) + \beta_1 = -3 + \alpha_1 = 0$
(30) $-3 + \alpha_3 = 2 + \beta_2 - \beta_3 = \alpha_2 = 1 + \beta_1 = -4 + \alpha_1 + \beta_3 = 0$
(31) $-3 + \beta_3 = -3 + \alpha_3 = -1 + \beta_2 = \alpha_2 = -(1/2) + \beta_1 = -1 + \alpha_1 = 0$

(32) $\beta_3 = -1 + \alpha_3 + \beta_2 = \alpha_2 = -2 + \alpha_3 + \beta_1 = -1 + \alpha_1 - \alpha_3 = 0$
(33) $\beta_3 = -(1/2) + \alpha_3 = \beta_2 = 1 + \alpha_2 = -3 + \beta_1 = 1 + \alpha_1 = 0$
(34) $\beta_3 = \beta_2 = 1 + \alpha_2 - \alpha_3 = -2 + \alpha_3 + \beta_1 = 1 + \alpha_1 = 0$
(35) $\beta_3 = 1 + \alpha_3 = \beta_2 = -3 + \alpha_2 = -3 + \beta_1 = -(3/2) + \alpha_1 = 0$
(36) $\alpha_3 = \beta_3 = 0$
(37) $\beta_3 = -(1/2) + \alpha_3 = -3 + \beta_1 = \alpha_1 = 0$
(38) $\beta_3 = -(1/2) + \alpha_3 = -(1/2) + \beta_2 = -2 + \alpha_2 = -3 + \beta_1 = -(3/2) + \alpha_1 = 0$
(39) $\beta_3 = -(1/2) + \alpha_3 = -(1/2) + \beta_2 = -3 + \alpha_2 = -3 + \beta_1 = -(3/2) + \alpha_1 = 0$
(40) $\beta_3 = -3 + \alpha_3 = -(1/2) + \beta_1 = \alpha_1 = 0$
(41) $\beta_3 = -3 + \alpha_3 = -3 + \beta_1 = \alpha_1 = 0$
(42) $\beta_3 = -3 + \alpha_3 = -3 + \beta_2 = -(1/2) + \alpha_2 = -(1/2) + \beta_1 = 1 + \alpha_1 = 0$
(43) $\beta_3 = -3 + \alpha_3 = -3 + \beta_2 = 3 + \alpha_2 = -3 + \beta_1 = 1 + \alpha_1 = 0$
(44) $\alpha_1 = \beta_1 + \alpha_3 - 2 = \beta_3 = 0$
(45) $\beta_3 = -1 + \alpha_3 = -3 + \beta_2 = -3 + \alpha_2 = -1 + \beta_1 = -(1/2) + \alpha_1 = 0$
(46) $\beta_3 = -3 + \beta_2 = -2 + \alpha_2 - \alpha_3 = -2 + \alpha_3 + \beta_1 = 1 + \alpha_1 = 0$
(47) $-(1/2) + \beta_3 = \alpha_3 = 1 + \beta_2 = -3 + \alpha_2 = 1 + \beta_1 = \alpha_1 = 0$
(48) $-(1/2) + \beta_3 = -1 + \alpha_3 = -3 + \beta_2 = -3 + \alpha_2 = -1 + \beta_1 = \alpha_1 = 0$
(49) $-(1/2) + \beta_3 = \alpha_3 = \beta_2 = -3 + \alpha_2 = 0$
(50) $-(1/2) + \beta_3 = 1 + \alpha_3 = \beta_2 = -3 + \alpha_2 = -3 + \beta_1 = -(1/2) + \alpha_1 = 0$
(51) $-(1/2) + \beta_3 = \alpha_3 = -(3/2) + \beta_2 = -3 + \alpha_2 = -2 + \beta_1 = -(1/2) + \alpha_1 = 0$
(52) $-(1/2) + \beta_3 = \alpha_3 = -(3/2) + \beta_2 = -3 + \alpha_2 = -3 + \beta_1 = -(1/2) + \alpha_1 = 0$
(53) $-(1/2) + \beta_3 = -(1/2) + \alpha_3 = -3 + \alpha_2 = -3 + \beta_1 = -2 + \alpha_1 + \beta_2 = 0$
(54) $-(1/2) + \beta_3 = -3 + \alpha_3 = -3 + \beta_2 = -3 + \alpha_2 = -3 + \beta_1 = -(1/2) + \alpha_1 = 0$
(55) $-(1/2) + \beta_3 = -3 + \beta_2 = -3 + \alpha_2 = -2 + \alpha_3 + \beta_1 = -(1/2) + \alpha_1 = 0$
(56) $-3 + \beta_3 = \alpha_3 = \beta_2 = -(1/2) + \alpha_2 = 0$
(57) $-3 + \beta_3 = \alpha_3 = \beta_2 = -3 + \alpha_2 = 0$
(58) $-3 + \beta_3 = -(3/2) + \alpha_3 = \beta_2 = -(1/2) + \alpha_2 = -(1/2) + \beta_1 = -2 + \alpha_1 = 0$
(59) $-3 + \beta_3 = -(3/2) + \alpha_3 = \beta_2 = -(1/2) + \alpha_2 = -(1/2) + \beta_1 = -3 + \alpha_1 = 0$
(60) $-3 + \beta_3 = 1 + \alpha_3 = \beta_2 = -3 + \alpha_2 = -3 + \beta_1 = 3 + \alpha_1 = 0$
(61) $-3 + \beta_3 = \alpha_3 = 1 + \beta_2 = -(1/2) + \alpha_2 = -(1/2) + \beta_1 = -3 + \alpha_1 = 0$
(62) $-3 + \beta_3 = \alpha_3 = 1 + \beta_2 = -3 + \alpha_2 = 3 + \beta_1 = -3 + \alpha_1 = 0$
(63) $-3 + \beta_3 = -(1/2) + \alpha_3 = -(1/2) + \beta_2 = -3 + \alpha_2 = -3 + \beta_1 = -3 + \alpha_1 = 0$
(64) $-3 + \beta_3 = -3 + \alpha_3 = -(1/2) + \alpha_2 = -(1/2) + \beta_1 = -2 + \alpha_1 + \beta_2 = 0$
(65) $-3 + \beta_3 = -3 + \alpha_3 = -3 + \alpha_2 = -3 + \beta_1 = -2 + \alpha_1 + \beta_2 = 0$
(66) $-3 + \beta_3 = -3 + \alpha_3 = -3 + \beta_2 = -(1/2) + \alpha_2 = -(1/2) + \beta_1 = -3 + \alpha_1 = 0$
(67) $-3 + \beta_3 = -4 + \alpha_3 + \beta_2 = -3 + \alpha_2 = -2 + \alpha_3 + \beta_1 = 2 + \alpha_1 - \alpha_3 = 0$
(68) $-3 + \beta_3 = -(1/2) + \beta_2 = -(1/2) + \alpha_2 = -2 + \alpha_3 + \beta_1 = -3 + \alpha_1 = 0$
(69) $-3 + \beta_3 = -3 + \beta_2 = -3 + \alpha_2 = -2 + \alpha_3 + \beta_1 = -3 + \alpha_1 = 0$
(70) $-3 + \alpha_3 + \beta_3 = \beta_2 = -2 + \alpha_2 + \beta_3 = 1 + \beta_1 - \beta_3 = \alpha_1 = 0$
(71) $\alpha_3 = 1 + \beta_2 = -2 + \alpha_2 + \beta_3 = 1 + \beta_1 - \beta_3 = \alpha_1 = 0$
(72) $1 + \beta_3 = \alpha_3 = -(3/2) + \beta_2 = -3 + \alpha_2 = -3 + \beta_1 = \alpha_1 = 0$
(73) $1 + \beta_3 = -(1/2) + \alpha_3 = -(1/2) + \beta_2 = -3 + \alpha_2 = -3 + \beta_1 = \alpha_1 = 0$

(74) $-(3/2) + \beta_3 = -3 + \alpha_3 = -2 + \beta_2 = -(1/2) + \alpha_2 = -(1/2) + \beta_1 = \alpha_1 = 0$
(75) $-(3/2) + \beta_3 = -3 + \alpha_3 = -3 + \beta_2 = -(1/2) + \alpha_2 = -(1/2) + \beta_1 = \alpha_1 = 0$
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(77) $1 + \beta_3 = -4 + \alpha_3 + \beta_2 = -3 + \alpha_2 = -2 + \alpha_3 + \beta_1 = \alpha_1 = 0$
(78) $\alpha_3 = \beta_2 = \beta_3 + \alpha_1 - 2 = 0$
(79) $1 + \alpha_3 = \beta_2 = -2 + \alpha_2 + \beta_3 = -3 + \beta_1 = -4 + \alpha_1 + \beta_3 = 0$
(80) $-1 + \beta_3 = -(1/2) + \alpha_3 = \beta_2 = -1 + \alpha_2 = -3 + \beta_1 = -3 + \alpha_1 = 0$
(81) $-1 + \beta_3 = \alpha_3 = -(1/2) + \beta_2 = -1 + \alpha_2 = -3 + \beta_1 = -3 + \alpha_1 = 0$
(82) $-(1/2) + \alpha_3 = -(1/2) + \beta_2 = -2 + \alpha_2 + \beta_3 = -3 + \beta_1 = -3 + \alpha_1 = 0$
(83) $-3 + \alpha_3 = -3 + \beta_2 = -2 + \alpha_2 + \beta_3 = -(1/2) + \beta_1 = -(1/2) + \alpha_1 = 0$
(84) $-3 + \alpha_3 = -3 + \beta_2 = -2 + \alpha_2 + \beta_3 = -3 + \beta_1 = -3 + \alpha_1 = 0$
(85) $\alpha_3 + \beta_3 = -3 + \beta_2 = -2 + \alpha_2 + \beta_3 = -2 + \beta_1 - \beta_3 = -3 + \alpha_1 = 0$
(86) $\alpha_3 = 1 + \beta_2 = -2 + \alpha_2 + \beta_3 = -2 + \beta_1 - \beta_3 = -3 + \alpha_1 = 0$
(87) $-3 + \alpha_3 = 2 + \beta_2 - \beta_3 = -2 + \alpha_2 + \beta_3 = -3 + \beta_1 = -4 + \alpha_1 + \beta_3 = 0$
(88) $\beta_1 + \alpha_3 - 2 = \beta_2 + \alpha_1 - 2 = \beta_3 + \alpha_1 - 2 = 0.$

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