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CONVERGENCE FOR NON-AUTONOMOUS SEMIDYNAMICAL SYSTEMS WITH IMPULSES

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Abstract

The present paper deals with impulsive non-autonomous systems with convergence. We show that the structure of the center of Levinson is preserved under homomorphism in impulsive convergent systems. Also, we present some criteria of convergence using Lyapunov functions.

1 Introduction

The theory of impulsive systems is an attractive area of investigation since many applications and complex problems can be modelled by such systems. The reader may consult [2], [16], [17] and [18], for instance.

The theory of dissipative impulsive autonomous systems has been started its study recently. In [7], the authors define various types of dissipativity and they present a study of the structure of the center of Levinson of a compact dissipative system. The reader may consult [9] to obtain properties of dissipative continuous dynamical systems.

On the other hand, there are systems that are identified by a surjective continuous mapping which takes orbits in orbits. These types of systems are called non-autonomous systems in the sense of [9].

The aim of this work is to consider non-autonomous systems subject to impulse conditions with convergence. These systems give us a one-to-one correspondence between the centers of Levinson of two homomorphic systems. Moreover, the centers of Levinson of homomorphic systems are homeomorphic, see Lemma 2.11 in [9]. In the next lines, we describe the organization of the paper and the main results.

In Section 2, we present the basis of the theory of impulsive semidynamical systems as basic definitions and notations.

In Section 3, we present additional useful definitions. We describe a brief resume of dissipative impulsive systems. In special, we exhibit the properties of the center of Levinson of a compact dissipative system.

Section 4 concerns with the main results. We divide this section in three subsections. In Subsection 4.1, we define the concept of a homomorphism between two impulsive systems. We show that several topological properties are preserved under homomorphism. A continuous section of a homomorphism is also considered in this subsection.

In Subsection 4.2, we define the concept of impulsive non-autonomous semidynamical systems with convergence. We present sufficient conditions to obtain convergence.

In the last subsection, we use functions of Lyapunov to get conditions of convergence for impulsive non-autonomous systems.

2 Preliminaries

Let X be a metric space, $\mathbb{R}_+$ be the set of non-negative real numbers and $\mathbb{N}$ be the set of natural numbers $\{0, 1, 2, 3, \ldots\}$. The triple $(X, \pi, \mathbb{R}_+)$ is called a *semidynamical system*, if the mapping $\pi : X \times \mathbb{R}_+ \to X$ is continuous with $\pi(x, 0) = x$ and $\pi(\pi(x, t), s) = \pi(x, t+s)$, for all $x \in X$ and $t, s \in \mathbb{R}_+$. We denote such system simply by (X, π). For every $x \in X$, we consider the continuous function $\pi_x : \mathbb{R}_+ \to X$ given by $\pi_x(t) = \pi(x, t)$ and we call it the *motion* of x.

Let (X, π) be a semidynamical system. Given $x \in X$, the *positive orbit* of x is given by $\pi^+(x) = \{\pi(x, t) : t \in \mathbb{R}_+\}$. Given $A \subset X$ and $\Delta \subset \mathbb{R}_+$, we define $\pi^+(A) = \bigcup\limits_{x \in A} \pi^+(x)$ and $\pi(A, \Delta) = \bigcup\limits_{x \in A,\, t \in \Delta} \pi(x, t)$. For $t \geq 0$ and $x \in X$, we define $F(x, t) = \{y \in X : \pi(y, t) = x\}$ and, for $\Delta \subset \mathbb{R}_+$ and $D \subset X$, we define $F(D, \Delta) = \cup\{F(x, t) : x \in D \text{ and } t \in \Delta\}$. Then a point $x \in X$ is called an *initial point* if $F(x, t) = \emptyset$ for all $t > 0$.

In the sequel, we define semidynamical systems with impulse action. An *impulsive semidynamical system* $(X, \pi; M, I)$ consists of a semidynamical system (X, π), a nonempty closed subset M of X such that for every $x \in M$ there exists $\epsilon_x > 0$ such that

$$F(x, (0, \epsilon_x)) \cap M = \emptyset \quad \text{and} \quad \pi(x, (0, \epsilon_x)) \cap M = \emptyset,$$

and a continuous function $I : M \to X$ whose action we explain below in the description of the impulsive trajectory of an impulsive semidynamical system. The set M is called the *impulsive set* and the function I is called *impulse function*. We also define

$$M^+(x) = \left(\bigcup_{t>0} \pi(x, t) \right) \cap M.$$

Given an impulsive semidynamical system $(X, \pi; M, I)$ and $x \in X$ such that $M^+(x) \neq \emptyset$, it is always possible to find a smallest number s such that the trajectory $\pi_x(t)$ does not intercept the set M for $0 < t < s$. This result is stated next and a proof of it can be found in [4].

Lemma 2.1. *Let $(X, \pi; M, I)$ be an impulsive semidynamical system. Then for every $x \in X$, there is a positive number s, $0 < s \leq +\infty$, such that $\pi(x, t) \notin M$ whenever $0 < t < s$ and $\pi(x, s) \in M$ if $M^+(x) \neq \emptyset$.*

By means of Lemma 2.1, it is possible to define the function $\phi : X \to (0, +\infty]$ by

$$\phi(x) = \begin{cases} s, & \text{if } \pi(x,s) \in M \text{ and } \pi(x,t) \notin M \text{ for } 0 < t < s, \\ +\infty, & \text{if } M^+(x) = \emptyset, \end{cases}$$

which represents the least positive time for which the trajectory of x meets M when $M^+(x) \neq \emptyset$. Thus for each $x \in X$, we call $\pi(x, \phi(x))$ the *impulsive point* of x.

The *impulsive trajectory* of x in $(X, \pi; M, I)$ is an $X-$valued function $\widetilde{\pi}_x$ defined on the subset $[0, s)$ of $\mathbb{R}_+$ (s may be $+\infty$). The description of such trajectory follows inductively as described in the following lines.

If $M^+(x) = \emptyset$, then $\phi(x) = +\infty$ and $\widetilde{\pi}_x(t) = \pi(x,t)$ for all $t \in \mathbb{R}_+$. However, if $M^+(x) \neq \emptyset$, then it follows from Lemma 2.1 that there is a smallest positive number s_0 such that $\pi(x, s_0) = x_1 \in M$ and $\pi(x,t) \notin M$ for $0 < t < s_0$. Thus we define $\widetilde{\pi}_x$ on $[0, s_0]$ by

$$\widetilde{\pi}_x(t) = \begin{cases} \pi(x,t), & 0 \leq t < s_0, \\ x_1^+, & t = s_0, \end{cases}$$

where $x_1^+ = I(x_1)$ and $\phi(x) = s_0$. Let us denote x by x_0^+.

Since $s_0 < +\infty$, the process now continues from x_1^+ onwards. If $M^+(x_1^+) = \emptyset$, then we define $\widetilde{\pi}_x(t) = \pi(x_1^+, t - s_0)$ for $s_0 \leq t < +\infty$ and in this case we have $\phi(x_1^+) = +\infty$. When $M^+(x_1^+) \neq \emptyset$, it follows again from Lemma 2.1 that there is a smallest positive number s_1 such that $\pi(x_1^+, s_1) = x_2 \in M$ and $\pi(x_1^+, t - s_0) \notin M$ for $s_0 < t < s_0 + s_1$. Then we define $\widetilde{\pi}_x$ on $[s_0, s_0 + s_1]$ by

$$\widetilde{\pi}_x(t) = \begin{cases} \pi(x_1^+, t - s_0), & s_0 \leq t < s_0 + s_1, \\ x_2^+, & t = s_0 + s_1, \end{cases}$$

where $x_2^+ = I(x_2)$ and $\phi(x_1^+) = s_1$, and so on. Notice that $\widetilde{\pi}_x$ is defined on each interval $[t_n, t_{n+1}]$, where $t_0 = 0$ and $t_{n+1} = \sum_{i=0}^{n} s_i$, $n = 0, 1, 2, \ldots$. Hence, $\widetilde{\pi}_x$ is defined on $[0, t_n]$ for each $n = 1, 2, \ldots$.

The process above ends after a finite number of steps, whenever $M^+(x_n^+) = \emptyset$ for some natural n. However, it continues infinitely if $M^+(x_n^+) \neq \emptyset$ for all $n = 0, 1, 2, \ldots$, and in this case the function $\widetilde{\pi}_x$ is defined on the interval $[0, T(x))$, where $T(x) = \sum_{i=0}^{\infty} s_i$.

Let $(X, \pi; M, I)$ be an impulsive semidynamical system. Given $x \in X$, the *impulsive positive orbit* of x is defined by the set $\widetilde{\pi}^+(x) = \{\widetilde{\pi}(x,t) : t \in [0, T(x))\}$.

Analogously to the non-impulsive case, an impulsive semidynamical system satisfies the following standard properties: $\widetilde{\pi}(x, 0) = x$ for all $x \in X$ and $\widetilde{\pi}(\widetilde{\pi}(x,t), s) = \widetilde{\pi}(x, t+s)$, for all $x \in X$ and for all $t, s \in [0, T(x))$ such that $t + s \in [0, T(x))$. See [5] for a proof of it.

For details about the structure of these types of impulsive semidynamical systems, the reader may consult [4, 5, 6, 7] and [10, 11, 12, 13, 14, 15].

Now, let us discuss the continuity of the function ϕ defined previously which indicates the moments of impulse action of a trajectory in an impulsive system. The theory below is borrowed from [10].

Let (X, π) be a semidynamical system. Any closed set $S \subset X$ containing x ($x \in X$) is called a *section* or a λ*-section* through x, with $\lambda > 0$, if there exists a closed set $L \subset X$ such that

i) $F(L, \lambda) = S$;

ii) $F(L, [0, 2\lambda])$ is a neighborhood of x;

iii) $F(L, \mu) \cap F(L, \nu) = \emptyset$, for $0 \leq \mu < \nu \leq 2\lambda$.

The set $F(L, [0, 2\lambda])$ is called a *tube* or a λ*-tube* and the set L is called a *bar.* Let $(X, \pi; M, I)$ be an impulsive semidynamical system. We now present the conditions TC and STC for a tube.

Any tube $F(L, [0, 2\lambda])$ given by a section S through $x \in X$ such that $S \subset M \cap F(L, [0, 2\lambda])$ is called *TC-tube* on x. We say that a point $x \in M$ fulfills the *Tube Condition* and we write TC, if there exists a TC-tube $F(L, [0, 2\lambda])$ through x. In particular, if $S = M \cap F(L, [0, 2\lambda])$ we have a *STC-tube* on x and we say that a point $x \in M$ fulfills the *Strong Tube Condition* (we write STC), if there exists a STC-tube $F(L, [0, 2\lambda])$ through x.

The following theorem concerns the continuity of ϕ which is accomplished outside M for M satisfying the condition TC.

Theorem 2.1. [10, Theorem 3.8] *Consider an impulsive system $(X, \pi; M, I)$. Assume that no initial point in (X, π) belongs to the impulsive set M and that each element of M satisfies the condition TC. Then ϕ is continuous at x if and only if $x \notin M$.*

3 Additional definitions

Let us consider a metric space X with metric ρ_X. By $B_X(x, \delta)$ we mean the *open ball in* X with center at $x \in X$ and radius $\delta > 0$. Given $A \subset X$, let $B_X(A, \delta) = \{x \in X : \rho_X(x, A) < \delta\}$ where $\rho_X(x, A) = \inf\{\rho_X(x, y) : y \in A\}$. Let $Comp(X)$ and $B(X)$ be the collection of all compact subsets and bounded subsets from X, respectively.

In what follows, $(X, \pi; M_X, I_X)$ is an impulsive semidynamical system.

Henceforth, we shall assume that the following conditions hold:

H1) No initial point in (X, π) belongs to the impulsive set M_X and each element of M_X satisfies the condition STC, consequently ϕ_X is continuous on $X \setminus M_X$ (see Theorem 2.1).

H2) $M_X \cap I_X(M_X) = \emptyset$.

H3) For each $x \in X$, the motion $\widetilde{\pi}(x, t)$ is defined for every $t \geq 0$, that is, $[0, +\infty)$ denotes the maximal interval of definition of $\widetilde{\pi}_x$.

Conditions (H1)-(H3) are motivated by several results in the theory of impulsive systems which can be found, in particular, in [1, 12, 15].

Given $A \subset X$ and $\Delta \subset \mathbb{R}_+$, we define $\widetilde{\pi}^+(A) = \bigcup_{x \in A} \widetilde{\pi}^+(x)$ and $\widetilde{\pi}(A, \Delta) = \bigcup_{x \in A, t \in \Delta} \widetilde{\pi}(x, t)$.

If $\widetilde{\pi}^+(A) \subset A$, we say that A is *positively $\widetilde{\pi}-$invariant.*

The *limit set* of $A \subset X$ in $(X, \pi; M_X, I_X)$ is given by

$$\widetilde{L}_X^+(A) = \{y \in X : \text{there exist sequences } \{x_n\}_{n \geq 1} \subset A \text{ and } \{t_n\}_{n \geq 1} \subset \mathbb{R}_+$$

$$\text{such that } t_n \stackrel{n \to +\infty}{\longrightarrow} +\infty \text{ and } \widetilde{\pi}(x_n, t_n) \stackrel{n \to +\infty}{\longrightarrow} y\},$$

the *prolongational limit set* of $A \subset X$ is defined by

$$\widetilde{J}_X^+(A) = \{y \in X : \text{there are sequences } \{x_n\}_{n \geq 1} \subset X \text{ and } \{t_n\}_{n \geq 1} \subset \mathbb{R}_+ \text{ such that}$$

$$\rho_X(x_n, A) \stackrel{n \to +\infty}{\longrightarrow} 0, \quad t_n \stackrel{n \to +\infty}{\longrightarrow} +\infty \quad \text{and} \quad \widetilde{\pi}(x_n, t_n) \stackrel{n \to +\infty}{\longrightarrow} y\}$$

and the *prolongation set* of $A \subset X$ is defined by

$$\widetilde{D}_X^+(A) = \{y \in X : \text{there are sequences } \{x_n\}_{n \geq 1} \subset X \text{ and } \{t_n\}_{n \geq 1} \subset \mathbb{R}_+ \text{ such that}$$

$$\rho_X(x_n, A) \stackrel{n \to +\infty}{\longrightarrow} 0 \text{ and } \widetilde{\pi}(x_n, t_n) \stackrel{n \to +\infty}{\longrightarrow} y\}.$$

If $A = \{x\}$, we set $\widetilde{L}_X^+(x) = \widetilde{L}_X^+(\{x\})$, $\widetilde{J}_X^+(x) = \widetilde{J}_X^+(\{x\})$ and $\widetilde{D}_X^+(x) = \widetilde{D}_X^+(\{x\})$.

Given A and B nonempty bounded subsets of X, we denote by $\beta_X(A, B)$ the *semi-deviation* of A to B, that is, $\beta_X(A, B) = \sup\{\rho_X(a, B) : a \in A\}$.

Next, we present an auxiliary result.

Lemma 3.1. [6, Lemma 3.3] *Given an impulsive semidynamical system $(X, \pi; M_X, I_X)$, assume that $w \in X \setminus M_X$ and $\{z_n\}_{n \geq 1}$ is a sequence in X which converges to w. Then, for any $t \geq 0$ such that $t \neq \sum_{j=0}^{k} \phi_X(w_j^+)$, $k = 0, 1, 2, \ldots$, we have $\widetilde{\pi}(z_n, t) \stackrel{n \to +\infty}{\longrightarrow} \widetilde{\pi}(w, t)$.*

A compact set A in $(X, \pi; M_X, I_X)$ is said to be:

1. *orbitally $\widetilde{\pi}-$stable*, if given $\epsilon > 0$ there is $\delta = \delta(\epsilon) > 0$ such that $\rho_X(x, A) < \delta$ implies $\rho_X(\widetilde{\pi}(x, t), A) < \epsilon$ for all $t \geq 0$;

2. *uniformly $\widetilde{\pi}-$attracting*, if there is $\gamma > 0$ such that $\lim_{t \to +\infty} \sup_{x \in B_X(A, \gamma)} \rho_X(\widetilde{\pi}(x, t), A) = 0$.

Now, we turn our attention to the theory of dissipativity on impulsive semidynamical systems. The study of dissipativity for continuous dynamical systems may be found in [9] and its study for the impulsive case is presented in [7].

An impulsive system $(X, \pi; M_X, I_X)$ is said to be:

3. *point b-dissipative* if there exists a bounded subset $K \subset X \setminus M_X$ such that for every $x \in X$
$$\lim_{t \to +\infty} \rho_X(\widetilde{\pi}(x,t), K) = 0; \tag{3.1}$$

4. *compact b-dissipative* if the convergence in (3.1) takes place uniformly with respect to x on the compact subsets from X;

5. *locally b-dissipative* if for each point $x \in X$ there exists $\delta_x > 0$ such that the convergence (3.1) takes place uniformly with respect to $y \in B_X(x, \delta_x)$;

6. *bounded b-dissipative* if the convergence (3.1) takes place uniformly with respect to x on every bounded subset from X.

There exists a more general definition of $\mathfrak{M}-$dissipativity for impulsive systems, where $\mathfrak{M}$ is a family of subsets from X. This definition is stated in [7] and it says that an impulsive system $(X, \pi; M_X, I_X)$ is $\mathfrak{M}-$*dissipative* if there exists a bounded set $K \subset X \setminus M_X$ such that for every $\epsilon > 0$ and $A \in \mathfrak{M}$ there exists $\ell(\epsilon, A) > 0$ such that $\widetilde{\pi}(A, t) \subset B_X(K, \epsilon)$ for all $t \geq \ell(\epsilon, A)$. In this case, the set K is called an *attractor* for the family $\mathfrak{M}$.

Remark 3.1. A point (compact)(locally)(bounded) $b-$dissipative system is a $\mathfrak{M}-$dissipative system with $\mathfrak{M} = \{\{x\} : x \in X\}$ $(\mathfrak{M} = Comp(X))(\mathfrak{M} = \{B_X(x, \delta_x) : x \in X, \delta_x > 0\})(\mathfrak{M} = B(X))$.

Remark 3.2. In Definitions 3, 4, 5 and 6 above, when K is compact, we say that the impulsive system $(X, \pi; M_X, I_X)$ is *compact* $k-$*dissipative*.

Let $(X, \pi; M_X, I_X)$ be compact $k-$dissipative and K be a nonempty compact set such that $K \cap M_X = \emptyset$ and it is an attractor for all compact subsets of X. The set
$$J_X = \widetilde{L}^+_X(K)$$
is called the *center of Levinson* of the compact $k-$dissipative system $(X, \pi; M_X, I_X)$. Also, it is showed that $J_X = \cap\{\widetilde{\pi}(K, t) : t \geq 0\}$, see Lemma 3.13 in [7].

The set J_X does not depend on the choice of set K which attracts all compact subsets of X and $K \cap M_X = \emptyset$. Also, we have $J_X \cap M_X = \emptyset$. For more details, see [7].

Theorem 3.1. [7, Theorem 3.20] *Let $(X, \pi; M_X, I_X)$ be compact $k-$dissipative and J_X be its center of Levinson. Then*

a) J_X *is a compact positively* $\widetilde{\pi}-$*invariant set;*

b) J_X *is orbitally* $\widetilde{\pi}-$*stable;*

c) J_X *is the attractor of the family of all compacts of* X*;*

d) J_X *is the maximal compact positively* $\widetilde{\pi}-$*invariant set in* $(X, \pi; M_X, I_X)$ *such that* $J_X \subset \widetilde{\pi}(J_X, t)$ *for each* $t \geq 0$.

Define Ω_X by the set $\overline{\cup\{\widetilde{L}^+_X(x) : x \in X\}}$. If $(X, \pi; M, I)$ is compact $k-$dissipative then $\widetilde{L}^+_X(x) \subset J_X$ for all $x \in X$, and therefore $\Omega_X \subset J_X$.

4 The main results

In this section, we present the main results from this paper. Let X and Y be metric spaces with metrics ρ_X and ρ_Y, respectively. Let $(X, \pi; M_X, I_X)$ and $(Y, \sigma; M_Y, I_Y)$ be impulsive semidynamical systems. All the concepts defined in the system $(X, \pi; M_X, I_X)$ are defined in $(Y, \sigma; M_Y, I_Y)$ similarly.

We shall assume that $(X, \pi; M_X, I_X)$ and $(Y, \sigma; M_Y, I_Y)$ satisfy the hypotheses H1), H2) and H3) presented in Section 3.

4.1 Homomorphisms

The concept of a homomorphism between two impulsive systems is defined similarly as in the continuous case.

Definition 4.1. A mapping $h : X \to Y$ is called a homomorphism from the impulsive system $(X, \pi; M_X, I_X)$ with values in $(Y, \sigma; M_Y, I_Y)$, if the mapping h is continuous, surjective and $h(\widetilde{\pi}(x, t)) = \widetilde{\sigma}(h(x), t)$ for all $x \in X$ and for all $t \in \mathbb{R}_+$.

In the sequel, we prove that some topological properties are preserved under homomorphisms. The first result shows that $M_Y \subset h(M_X)$.

Lemma 4.1. *Let* $h : X \to Y$ *be a homomorphism from the system* $(X, \pi; M_X, I_X)$ *to* $(Y, \sigma; M_Y, I_Y)$*. Then* $M_Y \subset h(M_X)$*.*

Proof. Let $y \in M_Y$ be arbitrary. Since y is not an initial point in the continuous system (Y, σ), there is $y_0 \in Y \setminus M_Y$ such that $y = \sigma(y_0, \phi_Y(y_0))$. Note that

$$\widetilde{\sigma}(y_0, t) = \begin{cases} \sigma(y_0, t), & 0 \leq t < \phi_Y(y_0), \\ I_Y(y), & t = \phi_Y(y_0). \end{cases}$$

Let $x \in X$ be such that $h(x) = y_0$. Then

$$h(\widetilde{\pi}(x,t)) = \begin{cases} h(\widetilde{\pi}(x,t)), & 0 \leq t < \phi_Y(y_0), \\ h(\widetilde{\pi}(x,\phi_Y(y_0))), & t = \phi_Y(y_0). \end{cases}$$

Now, we claim that $\phi_Y(y_0) = \sum_{j=-1}^{k} \phi_X(x_j^+)$ for some $k \in \mathbb{N}$, where we set $\phi_X(x_{-1}^+) = 0$. Suppose to the contrary that $\phi_Y(y_0) \neq \sum_{j=-1}^{k} \phi_X(x_j^+)$ for all $k \in \mathbb{N}$. Then there is $k_0 \in \mathbb{N}$ such that

$$\sum_{j=-1}^{k_0-1} \phi_X(x_j^+) < \phi_Y(y_0) < \sum_{j=-1}^{k_0} \phi_X(x_j^+).$$

Thus we can write $\phi_Y(y_0) = \sum_{j=-1}^{k_0-1} \phi_X(x_j^+) + \eta$, with $0 < \eta < \phi_X(x_{k_0}^+)$. Let $\xi > 0$ be such that $\eta - \xi > 0$. Then

$$\widetilde{\pi}(x, \phi_Y(y_0) - \xi) = \pi(x_{k_0}^+, \eta - \xi) \stackrel{\xi \to 0+}{\longrightarrow} \pi(x_{k_0}^+, \eta) = \widetilde{\pi}(x, \phi_Y(y_0)).$$

Since h is continuous we have $\widetilde{\sigma}(y_0, \phi_Y(y_0) - \xi) \stackrel{\xi \to 0+}{\longrightarrow} \widetilde{\sigma}(y_0, \phi_Y(y_0))$. But $\widetilde{\sigma}(y_0, \phi_Y(y_0) - \xi) = \sigma(y_0, \phi_Y(y_0) - \xi) \stackrel{\xi \to 0+}{\longrightarrow} y$. By uniqueness we have

$$y = \widetilde{\sigma}(y_0, \phi_Y(y_0)) \in I_Y(M_Y),$$

which is a contradiction because $y \in M_Y$ and we have hypothesis H2). In conclusion, there is $k \in \mathbb{N}$ such that $\phi_Y(y_0) = \sum_{j=-1}^{k} \phi_X(x_j^+)$. Therefore,

$$y = \lim_{t \to 0+} \widetilde{\sigma}(y_0, \phi_Y(y_0) - t) = \lim_{t \to 0+} h(\widetilde{\pi}(x, \phi_Y(y_0) - t)) = h(x_{k+1}) \in h(M_X)$$

and the proof is complete. □

Remark 4.1. Let $K \subset X$. If $h^{-1}(M_Y) \subset M_X$ then it is not difficult to see that $M_Y \cap h(K) = \emptyset$ whenever $M_X \cap K = \emptyset$.

Proposition 4.1. *Let $h : X \to Y$ be a homomorphism from the system $(X, \pi; M_X, I_X)$ to $(Y, \sigma; M_Y, I_Y)$. If $A \subset X$ is positively $\widetilde{\pi}-$invariant, then $h(A)$ is positively $\widetilde{\sigma}-$invariant.*

Proof. The proof is analogous to the proof of [4, Proposition 3.3]. □

Lemma 4.2. *Let $h : X \to Y$ be a homomorphism from $(X, \pi; M_X, I_X)$ to $(Y, \sigma; M_Y, I_Y)$. The following statements hold:*

a) $h(\widetilde{L}^+_X(K)) \subset \widetilde{L}^+_Y(h(K))$, *for all* $K \subset X$;

b) $h(\Omega_X) \subset \Omega_Y$;

c) *if the set* $A \subset X$ *is compact, then* $h(\widetilde{D}^+_X(A)) \subset \widetilde{D}^+_Y(h(A))$ *and* $h(\widetilde{J}^+_X(A)) \subset \widetilde{J}^+_Y(h(A))$;

d) $h(\widetilde{D}^+_X(\Omega_X)) \subset \widetilde{D}^+_Y(\Omega_Y)$ *and* $h(\widetilde{J}^+_X(\Omega_X)) \subset \widetilde{J}^+_Y(\Omega_Y)$ *provided that* $(X, \pi; M_X, I_X)$ *is point* $k-$*dissipative;*

e) *If* $(X, \pi; M_X, I_X)$ *and* $(Y, \sigma; M_Y, I_Y)$ *are compact* $k-$*dissipative then* $h(J_X) \subset J_Y$.

Proof. a) We may assume that $\widetilde{L}^+_X(K) \neq \emptyset$. Let $y \in h(\widetilde{L}^+_X(K))$. Then there is $z \in \widetilde{L}^+_X(K)$ such that $h(z) = y$. Thus there are sequences $\{a_n\}_{n\geq 1} \subset K$ and $\{t_n\}_{n\geq 1} \subset \mathbb{R}_+$ such that $t_n \stackrel{n\to+\infty}{\longrightarrow} +\infty$ and $\widetilde{\pi}(a_n, t_n) \stackrel{n\to+\infty}{\longrightarrow} z$. Since h is continuous, we have

$$\widetilde{\sigma}(h(a_n), t_n) = h(\widetilde{\pi}(a_n, t_n)) \stackrel{n\to+\infty}{\longrightarrow} h(z) = y.$$

Therefore, $y \in \widetilde{L}^+_Y(h(K))$.

b) Let us assume that $\Omega_X \neq \emptyset$. Let $x \in \Omega_X$. Then there are sequences $\{w_n\}_{n\geq 1} \subset X$ and $\{x_n\}_{n\geq 1} \subset X$ such that $x_n \in \widetilde{L}^+_X(w_n)$, $n = 1, 2, \ldots$, and $x_n \stackrel{n\to+\infty}{\longrightarrow} x$. By item a), $h(\widetilde{L}^+_X(w_n)) \subset \widetilde{L}^+_Y(h(w_n))$ for each $n = 1, 2, \ldots$. Hence, $h(x_n) \in \widetilde{L}^+_Y(h(w_n)) \subset \Omega_Y$ for each $n = 1, 2, 3, \ldots$. Since Ω_Y is closed it follows that $h(x_n) \stackrel{n\to+\infty}{\longrightarrow} h(x) \in \Omega_Y$.

c) Let $x \in \widetilde{D}^+_X(A)$. Since A is compact it follows by [7, Proposition 3.30] that there is $z \in A$ such that $x \in \widetilde{D}^+_X(z)$. Then there are sequences $\{x_n\}_{n\geq 1} \subset X$ and $\{t_n\}_{n\geq 1} \subset \mathbb{R}_+$ such that $x_n \stackrel{n\to+\infty}{\longrightarrow} z$ and $\widetilde{\pi}(x_n, t_n) \stackrel{n\to+\infty}{\longrightarrow} x$. Since h is a homomorphism, we conclude that $h(x_n) \stackrel{n\to+\infty}{\longrightarrow} h(z) \in h(A)$ and

$$\widetilde{\sigma}(h(x_n), t_n) \stackrel{n\to+\infty}{\longrightarrow} h(x).$$

Therefore, $h(x) \in \widetilde{D}^+_Y(h(A))$. Using the same ideas above, we show that $h(\widetilde{J}^+_X(A)) \subset \widetilde{J}^+_Y(h(A))$.

d) Since $(X, \pi; M_X, I_X)$ is point $k-$dissipative we have Ω_X compact. Therefore, the result follows by items b) and c).

e) It follows by item d) above and [7, Theorem 3.36].

□

Next, we present sufficient conditions to obtain the equality of item a) of Lemma 4.2.

Theorem 4.1. *Let* $h : X \to Y$ *be a homomorphism from* $(X, \pi; M_X, I_X)$ *to* $(Y, \sigma; M_Y, I_Y)$. *Let* $K \in Comp(X)$ *be such that* $\widetilde{\pi}^+(K)$ *is relatively compact. Then* $h(\widetilde{L}^+_X(K)) = \widetilde{L}^+_Y(h(K))$.

Proof. It is enough to show that $\widetilde{L}^+_Y(h(K)) \subset h(\widetilde{L}^+_X(K))$, see Lemma 4.2 item a). Given $z \in \widetilde{L}^+_Y(h(K))$ there are sequences $\{x_n\}_{n\geq 1} \subset K$ and $\{t_n\}_{n\geq 1} \subset \mathbb{R}_+$ such that $t_n \overset{n\to+\infty}{\longrightarrow} +\infty$ and $\widetilde{\sigma}(h(x_n), t_n) \overset{n\to+\infty}{\longrightarrow} z$, that is,

$$h(\widetilde{\pi}(x_n, t_n)) \overset{n\to+\infty}{\longrightarrow} z. \tag{4.1}$$

By hypothesis, we may assume without loss of generality that $\widetilde{\pi}(x_n, t_n) \overset{n\to+\infty}{\longrightarrow} \overline{x} \in \widetilde{L}^+_X(K)$. Using the continuity of h and (4.1) we have $z = h(\overline{x}) \in h(\widetilde{L}^+_X(K))$ and the result is proved. □

Corollary 4.1. *If $h : X \to Y$ is a homomorphism from $(X, \pi; M_X, I_X)$ to $(Y, \sigma; M_Y, I_Y)$ and $(X, \pi; M_X, I_X)$ is point $k-$dissipative, then $h(\widetilde{L}^+_X(x)) = \widetilde{L}^+_Y(h(x))$ for all $x \in X$.*

Now, we establish sufficient conditions to obtain the equality $h(\Omega_X) = \Omega_Y$.

Theorem 4.2. *Let $h : X \to Y$ be a homomorphism from $(X, \pi; M_X, I_X)$ to $(Y, \sigma; M_Y, I_Y)$. If $(X, \pi; M_X, I_X)$ is point $k-$dissipative and $h^{-1}(M_Y) \subset M_X$, then $(Y, \sigma; M_Y, I_Y)$ is point $k-$dissipative and $h(\Omega_X) = \Omega_Y$.*

Proof. Let $\epsilon > 0$ and $y \in Y$ be given. Then there exists $x \in X$ such that $h(x) = y$. Since $(X, \pi; M_X, I_X)$ is point $k-$dissipative, there is a nonempty compact set $K \subset X$, $K \cap M_X = \emptyset$, such that

$$\lim_{t\to+\infty} \rho_X(\widetilde{\pi}(x, t), K) = 0. \tag{4.2}$$

Also, we have $h(K) \cap M_Y = \emptyset$ (see Remark 4.1).

We claim that there is a $\delta = \delta(\epsilon) > 0$ such that $h(B_X(K, \delta)) \subset B_Y(h(K), \epsilon)$. Suppose to the contrary that there are $\epsilon_0 > 0$, $\delta_n \overset{n\to+\infty}{\longrightarrow} 0$ ($\delta_n > 0$) and $x_n \in B_X(K, \delta_n)$ such that

$$\rho_Y(h(x_n), h(K)) \geq \epsilon_0 \tag{4.3}$$

for each $n = 1, 2, 3, \ldots$. By the compactness of K, we may assume (taking a subsequence, if necessary) that $x_n \overset{n\to+\infty}{\longrightarrow} \overline{x} \in K$. By (4.3) we get $h(\overline{x}) \notin h(K)$ and it is a contradiction. Hence, there is $\delta = \delta(\epsilon) > 0$ such that $h(B_X(K, \delta)) \subset B_Y(h(K), \epsilon)$.

From (4.2) and for the number $\delta = \delta(\epsilon) > 0$ chosen above, there exists a positive number $\ell = \ell(\epsilon, x) > 0$ such that $\widetilde{\pi}(x, t) \subset B_X(K, \delta)$ for all $t \geq \ell$. Since h is a homomorphism, we conclude that

$$\widetilde{\sigma}(y, t) \subset h(B_X(K, \delta)) \subset B_Y(h(K), \epsilon) \quad \text{for all} \quad t \geq \ell.$$

Therefore, $(Y, \sigma; M_Y, I_Y)$ is point $k-$dissipative and $h(K)$ is its compact attractor.

Now, let us prove that $h(\Omega_X) = \Omega_Y$. By Lemma 4.2 item b) we have $h(\Omega_X) \subset \Omega_Y$. Let us show the another set inclusion. Let $y \in \Omega_Y$. Then there are sequences $\{y_n\}_{n\geq 1} \subset Y$ and $\{\widetilde{y}_n\}_{n\geq 1} \subset Y$ with $y_n \in \widetilde{L}^+_Y(\widetilde{y}_n)$, $n = 1, 2, \ldots$, such that

$$y_n \overset{n\to+\infty}{\longrightarrow} y. \tag{4.4}$$

Since $Y = h(X)$, there exists $\widetilde{x}_n \in X$ such that $\widetilde{y}_n = h(\widetilde{x}_n)$ for each $n = 1, 2, \ldots$. By Corollary 4.1, $h(\widetilde{L}^+_X(\widetilde{x}_n)) = \widetilde{L}^+_Y(\widetilde{y}_n)$. So, there is $x_n \in \widetilde{L}^+_X(\widetilde{x}_n) \subset \Omega_X$ for which $h(x_n) = y_n$, $n = 1, 2, \ldots$. Again, we may assume that $x_n \overset{n\to+\infty}{\longrightarrow} x \in \Omega_X$ since Ω_X is compact (because $(X, \pi; M_X, I_X)$ is point $k-$dissipative). Hence, using equation (4.4) we have $y = h(x) \in h(\Omega_X)$. Therefore, the proof is complete. □

Theorem 4.3 give us sufficient conditions to show that $h(J_X) = J_Y$.

Theorem 4.3. *Let $h : X \to Y$ be a homomorphism from $(X, \pi; M_X, I_X)$ to $(Y, \sigma; M_Y, I_Y)$ such that $h^{-1}(M_Y) \subset M_X$. If h is an open map and $(X, \pi; M_X, I_X)$ is compact $k-$dissipative, then $(Y, \sigma; M_Y, I_Y)$ is compact $k-$dissipative and $h(J_X) = J_Y$.*

Proof. We claim that $h(J_X)$ is an attractor for all compact subsets from Y. In fact, since $M_X \cap J_X = \emptyset$ we have $M_Y \cap h(J_X) = \emptyset$ (see Remark 4.1). Now, given $\epsilon > 0$ one can obtain a $\delta = \delta(\epsilon) > 0$ such that $h(B_X(J_X, \delta)) \subset B_Y(h(J_X), \epsilon)$. Let $A \in Comp(Y)$. For each point $y \in A$, there is $x \in X$ such that $h(x) = y$. By [7, Lemma 3.39] there exist $\gamma_x = \gamma(x, \epsilon) > 0$ and $\ell(x, \epsilon) > 0$ such that $\widetilde{\pi}(B_X(x, \gamma_x), t) \subset B_X(J_X, \delta)$ for all $t \geq \ell(x, \epsilon)$. Since h is a homomorphism, we conclude that

$$\widetilde{\sigma}(h(B_X(x, \gamma_x)), t) \subset B_Y(h(J_X), \epsilon) \tag{4.5}$$

for all $t \geq \ell(x, \epsilon)$.

Note that $\bigcup_{y\in A} \{h(B_X(x, \gamma_x)) : x \in X \cap h^{-1}(y)\}$ is an open covering of A because h is an open map. By the compactness of A we can obtain a finite sub-covering $\{h(B_X(x_i, \gamma_{x_i})) : i = 1, \ldots, m\}$, that is,

$$A \subset \bigcup_{i=1}^{m} h(B_X(x_i, \gamma_{x_i})),$$

with $h(x_i) = y_i \in A$, $i = 1, 2, \ldots, m$. From the last inclusion and (4.5) we get

$$\widetilde{\sigma}(A, t) \subset B_Y(h(J_X), \epsilon)$$

for all $t \geq \max\{\ell(x_i, \epsilon) : i = 1, \ldots, m\}$. Then, $\lim_{t\to+\infty} \beta_Y(\widetilde{\sigma}(A, t), h(J_X)) = 0$ and $(Y, \sigma; M_Y, I_Y)$ is compact $k-$dissipative.

Now, we show that $h(J_X) = J_Y$. Since J_Y is the least compact positively $\widetilde{\pi}-$invariant set attracting all compacts from Y (see [7, Theorem 3.21]), we conclude that $J_Y \subset h(J_X)$. The other set inclusion follows by Lemma 4.2 item e). The proof is complete. □

If the homomorphism h is an open mapping and $h^{-1}(M_Y) \subset M_X$ then it takes local $k-$dissipative systems in local $k-$dissipative systems, see the next result. Note that Theorems 4.3 and 4.4 hold when h is an isomorphism.

Theorem 4.4. *Let $h : X \to Y$ be a homomorphism from $(X, \pi; M_X, I_X)$ to $(Y, \sigma; M_Y, I_Y)$ and assume that $h^{-1}(M_Y) \subset M_X$. If h is an open map and $(X, \pi; M_X, I_X)$ is locally $k-$dissipative then $(Y, \sigma; M_Y, I_Y)$ is locally $k-$dissipative.*

Proof. According to Theorem 4.3 and [7, Lemma 3.12] the system $(Y, \sigma; M_Y, I_Y)$ is compact $k-$dissipative and $h(J_X) = J_Y$. By [7, Theorem 3.48] we need to prove that the center of Levinson J_Y of $(Y, \sigma; M_Y, I_Y)$ is uniformly $\widetilde{\sigma}-$attracting. Given $\epsilon > 0$ there is a $\delta = \delta(\epsilon) > 0$ such that $h(B_X(J_X, \delta)) \subset B_Y(J_Y, \epsilon)$. By [7, Theorem 3.48] the center of Levinson J_X is uniformly $\widetilde{\pi}-$attracting, that is, there exists $\gamma > 0$ such that

$$\lim_{t \to +\infty} \beta_X(\widetilde{\pi}(B_X(J_X, \gamma), t), J_X) = 0.$$

Let $T = T(\epsilon) > 0$ be such that $\widetilde{\pi}(B_X(J_X, \gamma), t) \subset B_X(J_X, \delta)$ for all $t \geq T$. Since h is a homomorphism one can conclude that

$$\widetilde{\sigma}(h(B_X(J_X, \gamma)), t) \subset B_Y(J_Y, \epsilon) \tag{4.6}$$

for all $t \geq T$. Note that $V = h(B_X(J_X, \gamma)) \supset J_Y$ is an open set in Y because h is an open mapping. Thus there is $\nu > 0$ such that $B_Y(J_Y, \nu) \subset V$. Hence, by (4.6) we conclude that $\widetilde{\sigma}(B_Y(J_Y, \nu), t) \subset B_Y(J_Y, \epsilon)$ for all $t \geq T$. Therefore, the system $(Y, \sigma; M_Y, I_Y)$ is locally $k-$dissipative. □

In the sequel, we define the concept of sections for homomorphisms between the impulsive systems $(X, \pi; M_X, I_X)$ and $(Y, \sigma; M_Y, I_Y)$.

Definition 4.2. A mapping $\varphi : Y \to X$ is called a continuous section of the homomorphism h from $(X, \pi; M_X, I_X)$ to $(Y, \sigma; M_Y, I_Y)$ if φ is continuous and $h \circ \varphi = id_Y$, where $id_Y : Y \to Y$ is the identity operator.

Definition 4.3. A continuous section $\varphi : Y \to X$ of the homomorphism h from $(X, \pi; M_X, I_X)$ to $(Y, \sigma; M_Y, I_Y)$ is called invariant if $\varphi(\widetilde{\sigma}(y, t)) = \widetilde{\pi}(\varphi(y), t)$ for all $y \in Y$ and for all $t \in \mathbb{R}_+$.

Next, we show that point, compact and local $k-$dissipativity are preserved by homomorphisms with invariant continuous section. In Theorem 4.5 we drop out the condition that h is open as presented in Theorems 4.3 and 4.4.

Theorem 4.5. *Let $h : X \to Y$ be a homomorphism from $(X, \pi; M_X, I_X)$ to $(Y, \sigma; M_Y, I_Y)$ such that $h^{-1}(M_Y) \subset M_X$ and $\varphi : Y \to X$ be an invariant continuous section of h. The following statements hold:*

a) if $(X, \pi; M_X, I_X)$ is point $k-$dissipative, then $(Y, \sigma; M_Y, I_Y)$ is point $k-$dissipative, $h(\Omega_X) = \Omega_Y$ and $h(\widetilde{D}^+_X(\Omega_X)) = \widetilde{D}^+_Y(\Omega_Y)$ $(h(\widetilde{J}^+_X(\Omega_X)) = \widetilde{J}^+_Y(\Omega_Y))$;

b) *if* $(X,\pi;M_X,I_X)$ *is compact* $k-$*dissipative, then* $(Y,\sigma;M_Y,I_Y)$ *is compact* $k-$*dissipative and* $h(J_X)=J_Y$*;*

c) *if* $(X,\pi;M_X,I_X)$ *is locally* $k-$*dissipative, then* $(Y,\sigma;M_Y,I_Y)$ *is locally* $k-$*dissipative.*

Proof. First, note that $\varphi: Y \to X$ is a homomorphism because φ is continuous and $\varphi(\widetilde{\sigma}(y,t)) = \widetilde{\pi}(\varphi(y),t)$ for all $y\in Y$ and for all $t\in\mathbb{R}_+$.

a) By Theorem 4.2 we have $(Y,\sigma;M_Y,I_Y)$ is point $k-$dissipative and $h(\Omega_X)=\Omega_Y$. By Lemma 4.2 we have $h(\widetilde{D}^+_X(\Omega_X))\subset\widetilde{D}^+_Y(\Omega_Y)$. Let us show the other set inclusion. Since $\varphi: Y\to X$ is a homomorphism, then using Lemma 4.2 we obtain $\varphi(\widetilde{D}^+_Y(\Omega_Y))\subset\widetilde{D}^+_X(\Omega_X)$. Consequently, $\widetilde{D}^+_Y(\Omega_Y)=h\circ\varphi(\widetilde{D}^+_Y(\Omega_Y))\subset h(\widetilde{D}^+_X(\Omega_X))$. The proof of the equality $h(\widetilde{J}^+_X(\Omega_X))=\widetilde{J}^+_Y(\Omega_Y)$ follows in the same way.

b) It follows by item $a)$ that the system $(Y,\sigma;M_Y,I_Y)$ is point $k-$dissipative. According to [7, Theorem 3.43] it is enough to show that $\widetilde{D}^+_Y(\Omega_Y)\cap M_Y=\emptyset$ and that $\widetilde{\sigma}^+(A)$ is relatively compact for all compact $A\subset Y$. Since $h^{-1}(M_Y)\subset M_X$, $\widetilde{D}^+_X(\Omega_X)\cap M_X=\emptyset$ (by [7, Theorem 3.42]) and $h(\widetilde{D}^+_X(\Omega_X))=\widetilde{D}^+_Y(\Omega_Y)$, we have $\widetilde{D}^+_Y(\Omega_Y)\cap M_Y=\emptyset$. Now, let $A\in Comp(Y)$ and consider a sequence $\{\widetilde{y}_n\}_{n\geq 1}\subset\widetilde{\sigma}^+(A)$. Then there are sequences $\{y_n\}_{n\geq1}\subset A$ and $\{t_n\}_{n\geq1}\subset\mathbb{R}_+$ such that $\widetilde{y}_n=\widetilde{\sigma}(y_n,t_n)$. In virtue of the invariance of φ we have $\varphi(\widetilde{\sigma}(y_n,t_n))=\widetilde{\pi}(\varphi(y_n),t_n)$ for each $n=1,2,\ldots$. Since $\varphi(A)\subset X$ is compact and $\varphi(y_n)\in\varphi(A)$, $n=1,2,\ldots$, we use the compact $k-$dissipativity of $(X,\pi;M_X,I_X)$ and we obtain

$$\varphi(\widetilde{\sigma}(y_n,t_n))=\widetilde{\pi}(\varphi(y_n),t_n)\overset{n\to+\infty}{\longrightarrow} x\in X.$$

Then,

$$\widetilde{\sigma}(y_n,t_n)\overset{n\to+\infty}{\longrightarrow} h(x),$$

since $h\circ\varphi=id_Y$. Therefore, $\{\widetilde{y}_n\}_{n\geq1}$ is convergent and $\widetilde{\sigma}^+(A)$ is relatively compact. Consequently, the system $(Y,\sigma;M_Y,I_Y)$ is compact $k-$dissipative by [7, Theorem 3.43]. The equality $h(J_X)=J_Y$ follows by item $a)$ and [7, Theorem 3.36].

c) If we show that the center of Levinson $J_Y=h(J_X)$ is uniformly $\widetilde{\sigma}-$attracting, then by item $b)$ and [7, Theorem 3.48] we get the result. Let $\epsilon>0$ be given. Then there is $\delta=\delta(\epsilon)>0$ such that $h(B_X(J_X,\delta))\subset B_Y(J_Y,\epsilon)$ because J_X is compact and h is continuous. Since $(X,\pi;M_X,I_X)$ is locally $k-$dissipative it follows by [7, Theorem 3.48] that there is $\gamma>0$ such that $\lim\limits_{t\to+\infty}\beta_X(\widetilde{\pi}(B_X(J_X,\gamma),t),J_X)=0$. Then there exists $T=T(\epsilon)>0$ such that

$$\widetilde{\pi}(B_X(J_X,\gamma),t)\subset B_X(J_X,\delta) \tag{4.7}$$

for all $t\geq T$.

Now take $\nu > 0$ such that $\varphi(B_Y(J_Y,\nu)) \subset B_X(\varphi(J_Y),\gamma)$. But $\varphi(J_Y) \subset J_X$ as φ is a homomorphism, we have item $d)$ from Lemma 4.2 and Theorem 3.36 from [7]. Then

$$\varphi(B_Y(J_Y,\nu)) \subset B_X(J_X,\gamma). \tag{4.8}$$

By (4.7) and (4.8), we have $\widetilde{\pi}(\varphi(B_Y(J_Y,\nu)),t) \subset B_X(J_X,\delta)$ for all $t \geq T$. Hence,

$$h(\widetilde{\pi}(\varphi(B_Y(J_Y,\nu)),t)) \subset B_Y(J_Y,\epsilon) \tag{4.9}$$

for all $t \geq T$. Note that $h(\widetilde{\pi}(\varphi(y),t)) = \widetilde{\sigma}(h(\varphi(y)),t) = \widetilde{\sigma}(y,t)$ for all $y \in Y$ and for all $t \in \mathbb{R}_+$. Thus, from the inclusion (4.9), we conclude that

$$\widetilde{\sigma}(B_Y(J_Y,\nu),t) \subset B_Y(J_Y,\epsilon)$$

for all $t \geq T$. Therefore, the system $(Y,\sigma;M_Y,I_Y)$ is locally $k-$dissipative.

□

Definition 4.4. Let h be a homomorphism from $(X,\pi;M_X,I_X)$ to $(Y,\sigma;M_Y,I_Y)$. A set $A \subset X$ is called uniformly stable in the positive direction with respect to the homomorphism h, if for all $\epsilon > 0$ there exists $\delta = \delta(\epsilon,A) > 0$ such that for all $x_1,x_2 \in A$ with $h(x_1) = h(x_2)$ the inequality $\rho_X(x_1,x_2) < \delta$ implies $\rho_X(\widetilde{\pi}(x_1,t),\widetilde{\pi}(x_2,t)) < \epsilon$ for every $t \geq 0$. The impulsive system $(X,\pi;M_X,I_X)$ is called uniformly stable (in the positive direction) with respect to the homomorphism h on compact subsets from X, if every compact set $A \in Comp(X)$ is uniformly stable in the positive direction with respect to h.

Contrary to Theorems 4.2, 4.3 and 4.5, the next result presents conditions for the system $(X,\pi;M_X,I_X)$ to be point (compact) $k-$dissipative provided $(Y,\sigma;M_Y,I_Y)$ is point (compact) $k-$dissipative.

Theorem 4.6. *Let h be a homomorphism from $(X,\pi;M_X,I_X)$ to $(Y,\sigma;M_Y,I_Y)$ satisfying the following conditions:*

i) $h(M_X) \subset M_Y$*;*

ii) there is a continuous invariant section $\varphi : Y \to X$ of the homomorphism h;

iii) $\lim\limits_{t\to+\infty} \rho_X(\widetilde{\pi}(x_1,t),\widetilde{\pi}(x_2,t)) = 0$ *for all $x_1,x_2 \in X$ with $h(x_1) = h(x_2)$;*

iv) the impulsive system $(X,\pi;M_X,I_X)$ is uniformly stable in the positive direction with respect to h on compact subsets from X.

Then the following hold:

a) if $(Y,\sigma;M_Y,I_Y)$ is point $k-$dissipative then $(X,\pi;M_X,I_X)$ is point $k-$dissipative. Moreover, Ω_X and Ω_Y are homeomorphic;

b) if $(Y,\sigma;M_Y,I_Y)$ is compactly $k-$dissipative then $(X,\pi;M_X,I_X)$ is also compactly $k-$dissipative and, moreover, J_X and J_Y are homeomorphic.

Proof. $a)$ Since $(Y,\sigma;M_Y,I_Y)$ is point $k-$dissipative, then Ω_Y is nonempty and compact. Consequently, $\varphi(\Omega_Y)\subset\Omega_X$ (see Lemma 4.2) is nonempty and compact. Note that $\varphi(\Omega_Y)\cap M_X=\emptyset$ because $\Omega_Y\cap M_Y=\emptyset$ and $h(M_X)\subset M_Y$.

We want to show that $\Omega_X=\varphi(\Omega_Y)$ to get the result. Then we need to prove that $\Omega_X\subset\varphi(\Omega_Y)$. In fact, let $x\in X$ be arbitrary and $y=h(x)$. By condition $iii)$ we have

$$\lim_{t\to+\infty}\rho_X(\widetilde{\pi}(x,t),\widetilde{\pi}(\varphi(y),t))=0 \tag{4.10}$$

as $h(x)=y=h(\varphi(y))$.

Note that $\widetilde{\sigma}^+(y)$ is relatively compact since $(Y,\sigma;M_Y,I_Y)$ is point $k-$dissipative. Thus, $B=\varphi(\overline{\widetilde{\sigma}^+(y)})$ is compact and by (4.10) we have

$$\lim_{t\to+\infty}\rho_X(\widetilde{\pi}(x,t),B)=0 \tag{4.11}$$

which implies that $\widetilde{L}^+_X(x)\neq\emptyset$.

Now, we claim that $\widetilde{L}^+_X(x)\subset\varphi(\Omega_Y)$. In fact, given $z\in\widetilde{L}^+_X(x)$ then there is a sequence $\{t_n\}_{n\geq1}\subset\mathbb{R}_+$ such that $t_n\overset{n\to+\infty}{\longrightarrow}+\infty$ and

$$\widetilde{\pi}(x,t_n)\overset{n\to+\infty}{\longrightarrow}z. \tag{4.12}$$

Since $\rho_X(\widetilde{\pi}(\varphi(y),t_n),z)\leq\rho_X(\widetilde{\pi}(\varphi(y),t_n),\widetilde{\pi}(x,t_n))+\rho_X(\widetilde{\pi}(x,t_n),z)$, it follows by (4.10) and (4.12) that $\lim_{n\to+\infty}\widetilde{\pi}(\varphi(y),t_n)=z$. This implies that

$$\lim_{n\to+\infty}\varphi(\widetilde{\sigma}(y,t_n))=z.$$

We may assume that $\widetilde{\sigma}(y,t_n)\overset{n\to+\infty}{\longrightarrow}b\in\Omega_Y$ because $(Y,\sigma;M_Y,I_Y)$ is point $k-$dissipative. Thus $z=\varphi(b)\in\varphi(\Omega_Y)$. Hence, $\widetilde{L}^+_X(x)\subset\varphi(\Omega_Y)$.

As x was taking arbitrary we get $\widetilde{L}^+_X(x)\subset\varphi(\Omega_Y)$ for all $x\in X$. Consequently, we obtain $\Omega_X\subset\varphi(\Omega_Y)$. Hence, $\Omega_X=\varphi(\Omega_Y)$ is compact in X and $\Omega_X\cap M_X=\emptyset$. Now, we just note that $\lim_{t\to+\infty}\rho_X(\widetilde{\pi}(x,t),\Omega_X)=0$ for all $x\in X$. Therefore, the system $(X,\pi;M_X,I_X)$ is point $k-$dissipative. Since $\varphi:\Omega_Y\to\Omega_X$ is surjective and $h\circ\varphi=Id_Y$, we have Ω_Y and Ω_X are homeomorphic.

$b)$ Let $(Y,\sigma;M_Y,I_Y)$ be compact $k-$dissipative. By item $a)$ the impulsive system $(X,\pi;M_X,I_X)$ is point $k-$dissipative. Let $A=\varphi(J_Y)$. By [7, Corollary 3.37] and Lemma 4.2 we get $A=\varphi(\widetilde{D}^+_Y(\Omega_Y))\subset\widetilde{D}^+_X(\Omega_X)$. Observe that A is positively

$\widetilde{\pi}-$invariant, because J_Y is $\widetilde{\sigma}-$invariant and φ is an invariant section. Moreover, $A \cap M_X = \emptyset$ since $J_Y \cap M_Y = \emptyset$ and $h(M_X) \subset M_Y$. Also, we have

$$\Omega_X = \varphi(\Omega_Y) \subset \varphi(\widetilde{D}^+_Y(\Omega_Y)) = \varphi(J_Y) = A.$$

In the sequel, we show that the set A is orbitally $\widetilde{\pi}-$stable. In fact, suppose to the contrary that there are $\epsilon_0 > 0$, $x_n \overset{n\to+\infty}{\longrightarrow} x_0 \in A$ and $t_n \overset{n\to+\infty}{\longrightarrow} +\infty$ ($t_n > 0$) (because $A \cap M_X = \emptyset$ and A is positively $\widetilde{\pi}-$invariant) such that

$$\rho_X(\widetilde{\pi}(x_n, t_n), A) \quad \geq \epsilon_0 \tag{4.13}$$

for each $n = 1, 2, \ldots$. Note that $y_n = h(x_n) \overset{n\to+\infty}{\longrightarrow} h(x_0) = y_0 \in h(A) = h(\varphi(J_Y)) = J_Y$. We may assume that the sequence $\{\widetilde{\sigma}(y_n, t_n)\}_{n\geq 1}$ is convergent because $(Y, \sigma; M_Y, I_Y)$ is compact $k-$dissipative. Let $y = \lim\limits_{t\to+\infty} \widetilde{\sigma}(y_n, t_n)$. Then $y \in J_Y$ and

$$\varphi(y) = \lim_{n\to+\infty} \varphi(\widetilde{\sigma}(y_n, t_n)) = \lim_{n\to+\infty} \widetilde{\pi}(\varphi(y_n), t_n). \tag{4.14}$$

Since $\varphi : J_Y \to A = \varphi(J_Y)$ is surjective and $h \circ \varphi = Id_Y$, it follows that $\varphi : J_Y \to A$ is a homeomorphism and $(\varphi \circ h)(x) = x$ for all $x \in A$ and, consequently, $\varphi(h(x_n)) \overset{n\to+\infty}{\longrightarrow} \varphi(h(x_0)) = x_0 \in A$. But $\{x_n\}_{n\geq 1}$ also converges to x_0, then

$$\lim_{n\to+\infty} \rho_X(x_n, \varphi(h(x_n)) = 0. \tag{4.15}$$

Let $K = A \bigcup \overline{\{x_n\}_{n\geq 1}} \bigcup \overline{\{\varphi(h(x_n))\}_{n\geq 1}}$. Since the system $(X, \pi; M_X, I_X)$ is uniformly stable in the positive direction with respect to the homomorphism h on compact subsets, given $\epsilon > 0$, there exists $\delta = \delta(\epsilon, K) > 0$ such that for each $z_1, z_2 \in K$ with $h(z_1) = h(z_2)$ and $\rho_X(z_1, z_2) < \delta$ we have $\rho_X(\widetilde{\pi}(z_1, t), \widetilde{\pi}(z_2, t)) < \frac{\epsilon}{2}$ for every $t \geq 0$. Now, the equality (4.15) implies that there is $n_1 \in \mathbb{N}$ such that $\rho_X(x_n, \varphi(h(x_n))) < \delta$ for all $n > n_1$ and consequently,

$$\rho_X(\widetilde{\pi}(x_n, t), \widetilde{\pi}(\varphi(h(x_n)), t)) < \frac{\epsilon}{2} \qquad \text{for all } t \geq 0 \text{ and } \quad \text{for all } n > n_1.$$

In particular,

$$\rho_X(\widetilde{\pi}(x_n, t_n), \widetilde{\pi}(\varphi(h(x_n)), t_n)) < \frac{\epsilon}{2} \qquad \text{for all } n > n_1. \tag{4.16}$$

On the other hand, from equality (4.14) one can obtain $n_2 \in \mathbb{N}$ such that

$$\rho_X(\widetilde{\pi}(\varphi(h(x_n)), t_n), \varphi(y)) < \frac{\epsilon}{2} \qquad \text{for all } n > n_2. \tag{4.17}$$

Take $n_0 = \max\{n_1, n_2\}$. Using (4.16) and (4.17) for $n > n_0$, we get

$$\begin{aligned}\rho_X(\widetilde{\pi}(x_n,t_n),\varphi(y)) &\le \rho_X(\widetilde{\pi}(x_n,t_n),\widetilde{\pi}(\varphi(h(x_n)),t_n)) + \rho_X(\widetilde{\pi}(\varphi(h(x_n)),t_n),\varphi(y)) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,\end{aligned}$$

that is, $\lim\limits_{n\to+\infty} \widetilde{\pi}(x_n,t_n) = \varphi(y) \in A$ which contradicts (4.13). Thus A is orbitally $\widetilde{\pi}-$stable. Then by [7, Theorem 3.41] we conclude that $(X,\pi;M_X,I_X)$ is compactly $k-$dissipative and $J_X \subset A = \varphi(J_Y) \subset \widetilde{D}_X^+(\Omega_X)$. According to [7, Corollary 3.37] we have $J_X = \widetilde{D}_X^+(\Omega_X)$, therefore, $J_X = \varphi(J_Y)$. Thus, φ is a homeomorphism from J_Y onto J_X. The lemma is proved.

□

4.2 Non-autonomous systems with convergence

In this section we present the concept of non-autonomous systems with convergence for the impulsive case.

Definition 4.5. The triple $\langle (X,\pi;M_X,I_X),(Y,\sigma;M_Y,I_Y),h\rangle$, where h is a homomorphism from the system $(X,\pi;M_X,I_X)$ to $(Y,\sigma;M_Y,I_Y)$, is called a non-autonomous impulsive semidynamical system. The impulsive system $(Y,\sigma;M_Y,I_Y)$ is called a factor of the impulsive system $(X,\pi;M_X,I_X)$ by the homomorphism h.

Let us show how to construct a non-autonomous impulsive system in the sense as Definition 4.5 from a non-autonomous impulsive differential system. In fact, consider the following system

$$\begin{cases} u' = f(t,u), \\ I : M \to \mathbb{R}^n, \end{cases} \tag{4.18}$$

where $f \in C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$, $M \subset \mathbb{R}^n$ is an impulsive set and the continuous map I is an impulse function such that $I(M)\cap M = \emptyset$. Along to the system (4.18), we consider its $H-$class, that is, the family of systems

$$\begin{cases} v' = g(t,v), \\ I : M \to \mathbb{R}^n, \end{cases} \tag{4.19}$$

where $g \in H(f) = \overline{\{f_\tau : \tau \in \mathbb{R}\}}$, $f_\tau(t,u) = f(t+\tau,u)$ for all $\tau\in\mathbb{R}$ and $(t,u)\in\mathbb{R}\times\mathbb{R}^n$ and f is regular, that is, for every equation $v' = g(t,v)$ (without impulses) and for every system (4.19) the conditions of existence, uniqueness and extendability on $\mathbb{R}_+$ are fulfilled.

Denote by $\psi(\cdot,v,g)$ the solution of the equation $v' = g(t,v)$ passing through the point $v\in\mathbb{R}^n$ at the initial moment $t=0$. It is well-known (see [9] for instance) that the continuous mapping

$$\psi : \mathbb{R}_+\times\mathbb{R}^n\times H(f) \to \mathbb{R}^n,$$

satisfies the following conditions:

i) $\psi(0, v, g) = v$ for all $v \in \mathbb{R}^n$ and for all $g \in H(f)$;

ii) $\psi(t, \psi(\tau, v, g), g_\tau) = \psi(t + \tau, v, g)$ for every $v \in \mathbb{R}^n$, $g \in H(f)$ and $t, \tau \in \mathbb{R}_+$.

Moreover, the mapping $\pi : \mathbb{R}^n \times H(f) \times \mathbb{R}_+ \to \mathbb{R}^n \times H(f)$ given by

$$\pi(v, g, t) = (\psi(t, v, g), g_t), \tag{4.20}$$

defines a continuous semidynamical system in $\mathbb{R}^n \times H(f)$.

Denote $\mathbb{R}^n \times H(f)$ by X and define the function $\phi_X : X \to (0, +\infty]$ by

$$\phi_X(v, g) = \begin{cases} s, & \text{if } \psi(s, v, g) \in M \text{ and } \psi(t, v, g) \notin M \text{ for } 0 < t < s, \\ +\infty, & \text{if } \psi(t, v, g) \notin M \text{ for all } t > 0. \end{cases}$$

Now, we define $M_X = M \times H(f)$ and $I_X : M_X \to X$ by $I_X(v, g) = (I(v), g)$. Then $(X, \pi; M_X, I_X)$ is an impulsive semidynamical system on X, where the impulsive trajectory of a point $(v, g) \in X$ is an $X-$valued function $\widetilde{\pi}(v, g, \cdot)$ defined inductively as described in the following lines.

Let $(v, g) \in X$ be given. If $\phi_X(v, g) = +\infty$ then $\widetilde{\pi}(v, g, t) = \pi(v, g, t)$ for all $t \in \mathbb{R}_+$. However, if $\phi_X(v, g) = s_0$, that is, $\psi(s_0, v, g) = v_1 \in M$ and $\psi(t, v, g) \notin M$ for $0 < t < s_0$, then we define $\widetilde{\pi}(v, g, \cdot)$ on $[0, s_0]$ by

$$\widetilde{\pi}(v, g, t) = \begin{cases} \pi(v, g, t), & 0 \leq t < s_0, \\ (v_1^+, g_{s_0}), & t = s_0, \end{cases}$$

where $v_1^+ = I(v_1)$.

Since $s_0 < +\infty$, the process now continues from (v_1^+, g_{s_0}) onwards. If $\phi_X(v_1^+, g_{s_0}) = +\infty$ then we define $\widetilde{\pi}(v, g, t) = \pi(v_1^+, g_{s_0}, t - s_0)$ for $s_0 \leq t < +\infty$. If $\phi_X(v_1^+, g_{s_0}) = s_1$, that is, $\psi(s_1, v_1^+, g_{s_0}) = v_2 \in M$ and $\psi(t - s_0, v_1^+, g_{s_0}) \notin M$ for $s_0 < t < s_0 + s_1$, then we define $\widetilde{\pi}(v, g, \cdot)$ on $[s_0, s_0 + s_1]$ by

$$\widetilde{\pi}(v, g, t) = \begin{cases} \pi(v_1^+, g_{s_0}, t - s_0), & s_0 \leq t < s_0 + s_1, \\ (v_2^+, g_{s_0+s_1}), & t = s_0 + s_1, \end{cases}$$

where $v_2^+ = I(v_2)$, and so on.

Let $(H(f), \sigma)$ be the continuous semidynamical system on $H(f)$ given by $\sigma(g, t) = g_t$ for all $g \in H(f)$ and for all $t \geq 0$. Then

$$\langle (X, \pi; M_X, I_X), (H(f), \sigma), h \rangle,$$

where h is the projection on the second coordinate, is a non-autonomous impulsive semidynamical system associated to the system (4.18). We remark that hypotheses H1) and H2) are not necessary for the system on $H(f)$ since $M_Y = \emptyset$.

Analogous to the continuous case, we define the concepts of dissipativity and center of Levinson for non-autonomous semidynamical systems with impulses. See the next lines.

Definition 4.6. The non-autonomous impulsive semidynamical system

$$\langle (X, \pi; M_X, I_X), (Y, \sigma; M_Y, I_Y), h \rangle \tag{4.21}$$

is said to be point (compact, local, bounded) $b-$dissipative if the autonomous impulsive semidynamical system $(X, \pi; M_X, I_X)$ possesses this property. Analogously, the system (4.21) is said to be $k-$dissipative if the system $(X, \pi; M_X, I_X)$ is $k-$dissipative.

Definition 4.7. Let system (4.21) be compact $k-$dissipative and J_X be the center of Levinson of $(X, \pi; M_X, I_X)$. The set J_X is said to be the Levinson's center of the non-autonomous system (4.21).

Next, we define the concept of convergence for non-autonomous systems with impulses.

Definition 4.8. The non-autonomous system (4.21) is said to be convergent if the following conditions hold:

i) the systems $(X, \pi; M_X, I_X)$ and $(Y, \sigma; M_Y, I_Y)$ are compactly $k-$dissipative;

ii) the set $J_X \cap X_y$ contains no more than one point for all $y \in J_Y$, where $X_y = X \cap h^{-1}(y)$.

From Theorem 4.6 we have the following result.

Theorem 4.7. *Under the conditions of Theorem 4.6, if* $(Y, \sigma; M_Y, I_Y)$ *is compact* $k-$*dissipative then the system* (4.21) *is convergent.*

Proof. It is enough to note that $h : J_X \to J_Y$ is a homeomorphism. □

In the sequel, we prove some auxiliary results. These results deal with impulsive negative semisolutions. Given a continuous semidynamical system (X, π), we say that a negative semisolution through a point $x \in X$ is a continuous function $\varphi_x : I_x \to X$ defined on an interval $I_x \subset (-\infty, 0]$ with $0 \in I_x$ satisfying the properties $\varphi_x(0) = x$ and $\pi(\varphi_x(t), s) = \varphi_x(t + s)$ for all $t \in I_x$ and for all $s \in \mathbb{R}_+$ such that $t + s \in I_x$. See [3] for instance.

The theory of impulsive negative semisolutions is constructed in [1]. Given a negative semisolution φ_x through a point $x \in X \backslash M_X$, it is proved in [1] that there is a corresponding mapping $\widetilde{\varphi}_x : I_x \to X$ through the point $x \in X \backslash M_X$, defined in some interval $I_x \subset (-\infty, 0]$ with $0 \in I_x$, such that $\widetilde{\varphi}_x(0) = x$ and $\widetilde{\pi}(\widetilde{\varphi}_x(t), s) = \widetilde{\varphi}_x(t+s)$ for all $t \in I_x$ and $s \in [0, +\infty)$ such that $t+s \in I_x$. The mapping $\widetilde{\varphi}_x$ is called an *impulsive negative semisolution* through the point x.

An *orbit* through $x \in X \setminus M_X$ in $(X, \pi; M_X, I_X)$ with respect to an impulsive negative semisolution $\widetilde{\varphi}_x$ defined on I_x will be given by

$$\widetilde{\pi}_{\varphi_x}(x) = \widetilde{\varphi}_x(I_x) \cup \widetilde{\pi}^+(x).$$

Lemma 4.3. *Let $h : X \to Y$ be a homomorphism from $(X, \pi; M_X, I_X)$ to $(Y, \sigma; M_Y, I_Y)$ and $x \in X \setminus M_X$. If there exists an impulsive negative semisolution through x then $h(x) \notin M_Y$.*

Proof. Let $\widetilde{\varphi}_x$ be an impulsive negative semisolution through x defined on $I_x \subset (-\infty, 0]$. Let $t \in I_x$, $t \neq 0$ and $\widetilde{\varphi}_x(t) = x_1$. Then $\widetilde{\pi}(x_1, -t) = x$ and

$$\widetilde{\sigma}(h(x_1), -t) = h(\widetilde{\pi}(x_1, -t)) = h(x).$$

Since $I_Y(M_Y) \cap M_Y = \emptyset$ and $t \neq 0$, then we have $h(x) \notin M_Y$. □

Lemma 4.4. *Let $h : X \to Y$ be a homomorphism from $(X, \pi; M_X, I_X)$ to $(Y, \sigma; M_Y, I_Y)$ and $x \in X \setminus M_X$. If there exists an impulsive negative semisolution through x defined on $(-\infty, 0]$ then there will be an impulsive negative semisolution $\widetilde{\Gamma}_{h(x)}$ through $h(x)$ defined on $(-\infty, 0]$.*

Proof. Let $x \in X \setminus M_X$ and $h(x) = y$. By Lemma 4.3 we have $y \notin M_Y$. By hypothesis, there is an impulsive negative semisolution $\widetilde{\varphi}_x$ through x defined on $(-\infty, 0]$. Define the mapping $\widetilde{\Gamma}_y : (-\infty, 0] \to Y$ by $\widetilde{\Gamma}_y(t) = h(\widetilde{\varphi}_x(t))$ for all $t \leq 0$. Then,

i) $\widetilde{\Gamma}_y(0) = h(\widetilde{\varphi}_x(0)) = h(x) = y$;

ii) $\widetilde{\sigma}(\widetilde{\Gamma}_y(t), s) = \widetilde{\sigma}(h(\widetilde{\varphi}_x(t)), s) = h(\widetilde{\pi}(\widetilde{\varphi}_x(t), s)) = h(\widetilde{\varphi}_x(t + s)) = \widetilde{\Gamma}_y(t + s)$, for all $t \in (-\infty, 0]$ and $s \in [0, +\infty)$ such that $t + s \leq 0$.

Hence, $\widetilde{\Gamma}_y : (-\infty, 0] \to Y$ is an impulsive negative semisolution through y in $(Y, \sigma; M_Y, I_Y)$ defined on $(-\infty, 0]$. □

A mapping $\widetilde{\varphi} : I_x \to X$ is called a *maximal impulsive negative semisolution* through $x \in X \setminus M_X$ if there is not any impulsive negative semisolution through x, $\widetilde{\varphi}_1 : J_x \to X$, such that $I_x \subset J_x$, $J_x \neq I_x$ and $\widetilde{\varphi}_1|_{I_x} = \widetilde{\varphi}$. Moreover, a point $x \in X \setminus M_X$ is a called a *point with maximal negative unicity* if there is a unique maximal impulsive negative semisolution through x in X.

Lemma 4.5. *Let $h : X \to Y$ be a homomorphism from $(X, \pi; M_X, I_X)$ to $(Y, \sigma; M_Y, I_Y)$ and $K \subset X \setminus M_X$ be a set such that $\widetilde{\pi}(K, t) = K$ for all $t \geq 0$. Assume that each point in $h(K) \setminus M_Y$ is a point with maximal negative unicity. If $x_1, x_2 \in K$ with $h(x_1) = h(x_2)$ then there are impulsive negative semisolutions $\widetilde{\varphi}_{x_1}$ and $\widetilde{\varphi}_{x_2}$ through x_1 and x_2, respectively, defined on $(-\infty, 0]$ such that $\widetilde{\pi}_{\varphi_{x_1}}(x_1), \widetilde{\pi}_{\varphi_{x_2}}(x_2) \subset K$, $h(\widetilde{\pi}(x_1, t)) = h(\widetilde{\pi}(x_2, t))$ for all $t \geq 0$ and $h(\widetilde{\varphi}_{x_1}(t)) = h(\widetilde{\varphi}_{x_2}(t))$ for all $t \leq 0$.*

Proof. Let $x_1, x_2 \in K$ with $h(x_1) = h(x_2) = y$. Then $x_1, x_2 \in \widetilde{\pi}(K, 1)$ and there exist $x^1_{-1}, x^2_{-1} \in K$ such that

$$\widetilde{\pi}(x^1_{-1}, 1) = x_1 \quad \text{and} \quad \widetilde{\pi}(x^2_{-1}, 1) = x_2.$$

Again, since $x^1_{-1}, x^2_{-1} \in K$ there exist $x^1_{-2}, x^2_{-2} \in K$ such that $\widetilde{\pi}(x^1_{-2}, 1) = x^1_{-1}$ and $\widetilde{\pi}(x^2_{-2}, 1) = x^2_{-1}$. Inductively, we can construct sequences $\{x^1_{-n}\}_{n\geq 1}, \{x^2_{-n}\}_{n\geq 1}$ in K such that $\widetilde{\pi}(x^i_{-n-1}, 1) = x^i_{-n}$ for all natural $n \geq 0$ with $x^i_0 = x_i$, $i = 1, 2$. Note that $\widetilde{\pi}(x^i_{-n}, n) = x_i$ for all $n = 1, 2, \ldots$, $i = 1, 2$. Then we can define

$$\widetilde{\varphi}_{x_i}(t) = \widetilde{\pi}(x^i_{-n}, t + n) \ \text{ if } \ t \in [-n, -n+1], \ n = 1, 2, \ldots,$$

which is an impulsive negative semisolution through x_i defined on $(-\infty, 0]$, $i = 1, 2$. Thus, by construction, we have $\widetilde{\pi}_{\varphi_{x_i}}(x_i) = \widetilde{\varphi}_{x_i}((-\infty, 0]) \cup \widetilde{\pi}^+(x_i) \subset K$ for each $i = 1, 2$.

Note that $h(x^1_{-n}) = h(x^2_{-n})$ for all $n = 0, 1, 2, \ldots$, because if for some $n_0 \in \mathbb{N}$ we have $h(x^1_{-n_0}) \neq h(x^2_{-n_0})$ then

$$\widetilde{\sigma}(h(x^1_{-n_0}), n_0) = h(\widetilde{\pi}(x^1_{-n_0}, n_0)) = h(x_1) = y = h(x_2) = h(\widetilde{\pi}(x^2_{-n_0}, n_0)) = \widetilde{\sigma}(h(x^2_{-n_0}), n_0).$$

But $y \notin M_Y$, see Lemma 4.3, and $(Y, \sigma; M_Y, I_Y)$ possesses a unique maximal impulsive negative semisolution through each point in $h(K) \setminus M_Y$. We get a contradiction. Hence, if $t \in [-n, -n+1]$, $n = 1, 2, \ldots$, we have

$$h(\widetilde{\varphi}_{x_1}(t)) = h(\widetilde{\pi}(x^1_{-n}, t+n)) = \widetilde{\sigma}(h(x^1_{-n}), t+n)) =$$

$$= \widetilde{\sigma}(h(x^2_{-n}), t+n)) = h(\widetilde{\pi}(x^2_{-n}, t+n)) = h(\widetilde{\varphi}_{x_2}(t)).$$

Thus, $h(\widetilde{\varphi}_{x_1}(t)) = h(\widetilde{\varphi}_{x_2}(t))$ for all $t \leq 0$. On the other hand, since $h(x_1) = h(x_2)$ it follows that $h(\widetilde{\pi}(x_1, t)) = h(\widetilde{\pi}(x_2, t))$ for all $t \geq 0$. □

Given $A \subset X$, define $A \otimes A = \{(x_1, x_2) : \, x_1, x_2 \in A, h(x_1) = h(x_2)\}$.

Lemma 4.6. *Let $h : X \to Y$ be a homomorphism from $(X, \pi; M_X, I_X)$ to $(Y, \sigma; M_Y, I_Y)$ and $K \subset X \setminus M_X$ be a set such that $\widetilde{\pi}(K, t) = K$ for all $t \geq 0$. Assume that each point in $h(K) \setminus M_Y$ is a point with maximal negative unicity. If*

$$\lim_{t\to+\infty} \sup_{(a,b)\in K\otimes K} \rho_X(\widetilde{\pi}(a, t), \widetilde{\pi}(b, t)) = 0 \tag{4.22}$$

then the set $K \cap h^{-1}(y)$ contains one single point for all $y \in h(K)$.

Proof. Clearly $K \cap h^{-1}(y) \neq \emptyset$ for $y \in h(K)$. Also, by the initial part of the proof of Lemma 4.5 and Lemma 4.3 we have $h(K) \cap M_Y = \emptyset$.

Suppose to the contrary that there are $y \in h(K)$ and $x_1, x_2 \in K \cap h^{-1}(y)$ with $x_1 \neq x_2$. By Lemma 4.5, there are impulsive negative semisolutions $\widetilde{\varphi}_{x_1}$ and $\widetilde{\varphi}_{x_2}$ through x_1 and x_2, respectively, such that $\widetilde{\varphi}_{x_1}((-\infty, 0]) \subset K$, $\widetilde{\varphi}_{x_2}((-\infty, 0]) \subset K$ and

$$h(\widetilde{\varphi}_{x_1}(-t)) = h(\widetilde{\varphi}_{x_2}(-t)) \quad \text{for all} \quad t \geq 0.$$

By (4.22), given $0 < \epsilon < \dfrac{\rho_X(x_1, x_2)}{2}$ one can obtain a number $\ell = \ell(\epsilon) > 0$ such that

$$\rho_X(\widetilde{\pi}(a, t), \widetilde{\pi}(b, t)) < \epsilon,$$

for all $t \geq \ell$ and for all $(a, b) \in K \otimes K$. Hence, for $t \geq \ell$ we get

$$\epsilon < \rho_X(x_1, x_2) = \rho_X(\widetilde{\pi}(\widetilde{\varphi}_{x_1}(-t), t), \widetilde{\pi}(\widetilde{\varphi}_{x_2}(-t), t)) < \epsilon,$$

which is a contradiction. □

Theorem 4.8 presents sufficient conditions to obtain convergence.

Theorem 4.8. *Let $(X, \pi; M_X, I_X)$ and $(Y, \sigma; M_Y, I_Y)$ be compact $k-$dissipative systems. Assume that each point in $h(J_X)$ is a point with maximal negative unicity. If* (4.22) *holds for $K = J_X$ then the system* (4.21) *is convergent.*

Proof. The proof follows by Lemma 4.6 and Theorem 3.1. □

We have the following result using Theorem 4.5 and Theorem 4.8.

Corollary 4.2. *Assume that* (4.21) *is such that $(X, \pi; M_X, I_X)$ is compactly $k-$dissipative, each point in $h(J_X)$ is a point with maximal negative unicity and $h^{-1}(M_Y) \subset M_X$. Let $\varphi : Y \to X$ be an invariant continuous section of h. If* (4.22) *holds for $K = J_X$ then the system* (4.21) *is convergent.*

Next, we define the concept of orbits which are asymptotically stable with respect to non-autonomous systems. We will show later that convergent systems produce asymptotically stable orbits under additional hypotheses.

Definition 4.9. A positive orbit through a point $p \in X \setminus M_X$ is said to be asymptotically stable with respect to the system $\langle (X, \pi; M_X, I_X), (Y, \sigma; M_Y, I_Y), h\rangle$ if the following conditions hold:

i) for all $\epsilon > 0$ there exists a $\delta(p, \epsilon) > 0$ such that if $\rho_X(x, p) < \delta$ with $x \in X$ and $h(x) = h(p)$ then $\rho_X(\widetilde{\pi}(x, t), \widetilde{\pi}(p, t)) < \epsilon$ for all $t \geq 0$;

ii) there is $\gamma(p) > 0$ such that if $\rho_X(x, p) < \gamma(p)$ with $x \in X$ and $h(x) = h(p)$ then $\lim_{t \to +\infty} \rho_X(\widetilde{\pi}(x, t), \widetilde{\pi}(p, t)) = 0$.

Theorem 4.9 give us sufficient conditions to assure that every positive orbit through a point in $X \setminus M_X$ is asymptotically stable with respect to the system (4.21).

Theorem 4.9. *Assume that the system* (4.21) *is such that $(Y, \sigma; M_Y, I_Y)$ is compact $k-$dissipative and $h^{-1}(M_Y) = M_X$. Assume that $\widetilde{\pi}^+(K)$ is relatively compact for every $K \in Comp(X)$. If the set $h^{-1}(J_Y) \cap X_y$ contains only one point for each $y \in J_Y$, then $\widetilde{\pi}^+(x)$ is asymptotically stable with respect to the system* (4.21) *for all $x \in X \setminus M_X$.*

Proof. First, let us show that condition *i)* from Definition 4.9 holds. Suppose to the contrary that there are $p_0 \in X \backslash M_X$, $\epsilon_0 > 0$, $p_n \stackrel{n\to+\infty}{\longrightarrow} p_0$ ($\{p_n\}_{n\geq 1} \subset X$ and $h(p_n) = h(p_0)$) and $\{t_n\}_{n\geq 1} \subset \mathbb{R}_+$ such that

$$\rho_X(\widetilde{\pi}(p_n, t_n), \widetilde{\pi}(p_0, t_n)) \geq \epsilon_0 \tag{4.23}$$

for each $n = 1, 2, 3, \ldots$.

By hypothesis, we may assume

$$\widetilde{\pi}(p_n, t_n) \stackrel{n\to+\infty}{\longrightarrow} a \quad \text{and} \quad \widetilde{\pi}(p_0, t_n) \stackrel{n\to+\infty}{\longrightarrow} b. \tag{4.24}$$

Case 1: $\{t_n\}_{n\geq 1} \subset \mathbb{R}_+$ admits a convergent subsequence.

Let us assume without loss of generality that $t_n \stackrel{n\to+\infty}{\longrightarrow} r$ for some $r \geq 0$.

If $r \neq \sum_{j=0}^{k} \phi_X((p_0)_j^+)$ for all $k \in \mathbb{N}$, then by the proof of [7, Lemma 3.7] we can conclude that

$$\widetilde{\pi}(p_0, t_n) \stackrel{n\to+\infty}{\longrightarrow} \widetilde{\pi}(p_0, r) = b \quad \text{and} \quad \widetilde{\pi}(p_n, t_n) \stackrel{n\to+\infty}{\longrightarrow} \widetilde{\pi}(p_0, r) = a,$$

that is, $a = b$ and it contradicts (4.23) as $n \to +\infty$.

If $r = \sum_{j=0}^{k} \phi_X((p_0)_j^+)$ for some $k \in \mathbb{N}$, then using the proof of [7, Lemma 3.7] again, we have either $b = (p_0)_{k+1}$ or $b = (p_0)_{k+1}^+$ and either $a = (p_0)_{k+1}$ or $a = (p_0)_{k+1}^+$.

- If either $a = b = (p_0)_{k+1}$ or $a = b = (p_0)_{k+1}^+$, we get a contradiction using (4.23) as $n \to +\infty$.
- If $a = (p_0)_{k+1} \in M_X$ and $b = (p_0)_{k+1}^+ \notin M_X$, we also have a contradiction because $h(b) \notin M_Y$ and $h(a) \in M_Y$ since $h^{-1}(M_Y) = M_X$, and

$$h(a) = \lim_{t\to+\infty} \widetilde{\sigma}(h(p_n), t_n) = \lim_{t\to+\infty} \widetilde{\sigma}(h(p_0), t_n) = h(b).$$

- If $a = (p_0)_{k+1}^+ \notin M_X$ and $b = (p_0)_{k+1} \in M_X$, we have again a contradiction by the previous case.

Case 2: $t_n \stackrel{n\to+\infty}{\longrightarrow} +\infty$.

From (4.23) and (4.24) we have $a \neq b$. But, on the other hand,

$$h(a) = \lim_{t\to+\infty} \widetilde{\sigma}(h(p_n), t_n) = \lim_{t\to+\infty} \widetilde{\sigma}(h(p_0), t_n) = h(b). \tag{4.25}$$

Taking $\overline{y} = h(a) = h(b)$, it follows by (4.25) that $\overline{y} \in J_Y$. Then $a, b \in h^{-1}(J_Y) \cap X_{\overline{y}}$ with $a \neq b$ which contradicts the hypothesis.

In both cases above we obtain a contradiction. Therefore, item $i)$ from Definition 4.9 holds.

To finish the proof of this result, we need to show that condition $ii)$ from Definition 4.9 holds. Note that we can use the proof of Case 2 above to conclude this part.

Hence, $\widetilde{\pi}^+(x)$ is asymptotically stable with respect to the system (4.21) for all $x \in X \setminus M_X$. $\square$

Assume that $(X, \pi; M_X, I_X)$ is compact $k-$dissipative in Theorem 4.9. We may apply the proof of Theorem 4.9 to show that condition $i)$ from Definition 4.9 holds. We just note that the existence of the limits in (4.24) follows by the compact $k-$dissipativity of $(X, \pi; M_X, I_X)$.

In order to see that condition $ii)$ from Definition 4.9 holds, it is enough to use the proof of Theorem 4.9. In this case, since $t_n \overset{n\to+\infty}{\longrightarrow} +\infty$ we observe that $b \in J_X$ and $a \in \widetilde{L}^+_X(K) \subset J_X$, where $K = \{p_n\}_{n\geq 1} \cup \{p_0\}$ and $a \neq b$. Thus, by the proof of Case 2, we get $a, b \in J_X \cap X_{\overline{y}}$. Hence, if (4.21) is convergent then we get $a = b$, which is a contradiction. This shows the following result.

Theorem 4.10. *If* (4.21) *is convergent with* $h^{-1}(M_Y) = M_X$ *then* $\widetilde{\pi}^+(x)$ *is asymptotically stable with respect to the system* (4.21) *for all* $x \in X \setminus M_X$.

Next, we show that the concept of asymptotic stability is uniform on compact fibers.

Theorem 4.11. *Let* $K \in Comp(X)$ *be such that* $K \cap M_X = \emptyset$. *Assume that* $\widetilde{\pi}^+(p)$ *is asymptotically stable with respect to the system* (4.21) *for each* $p \in K$. *Then given* $\epsilon > 0$ *and* $y \in h(K)$ *there exists a* $\delta = \delta(\epsilon, K_y) > 0$ *such that for* $x_1, x_2 \in K_y$ *satisfying* $\rho_X(x_1, x_2) < \delta$ *we have* $\rho_X(\widetilde{\pi}(x_1, t), \widetilde{\pi}(x_2, t)) < \epsilon$ *for all* $t \geq 0$.

Proof. Let $\epsilon > 0$ and $y \in h(K)$ be given. There are $p \in K$ such that $h(p) = y$ and $\delta_p = \delta(p, \epsilon) > 0$ such that if $\rho_X(x, p) < \delta_p$ with $x \in X$ and $h(x) = h(p)$ then $\rho_X(\widetilde{\pi}(x, t), \widetilde{\pi}(p, t)) < \dfrac{\epsilon}{2}$ for all $t \geq 0$. By compactness, there are $p_1, \ldots, p_n \in K_y$ such that

$$K_y \subset B_X\left(p_1, \frac{\delta_{p_1}}{2}\right) \cup \ldots \cup B_X\left(p_n, \frac{\delta_{p_n}}{2}\right),$$

where $\delta_{p_j} = \delta(p_j, \epsilon) > 0$ comes from the asymptotic stability of p_j, $j = 1, 2, \ldots, n$. Take $\delta = \min\{\delta_{p_1}, \ldots, \delta_{p_n}\}$.

Let $x_1, x_2 \in K_y$ be such that $\rho_X(x_1, x_2) < \dfrac{\delta}{2}$. Note that there is $j \in \{1, \ldots, n\}$ such that $x_1 \in B_X\left(p_j, \dfrac{\delta_{p_j}}{2}\right)$. Then $\rho_X(x_2, p_j) \leq \rho_X(x_2, x_1) + \rho_X(x_1, p_j) < \delta_{p_j}$. Hence, $x_1, x_2 \in B_X(p_j, \delta_{p_j})$ with $h(x_1) = h(x_2) = h(p_j) = y$. Consequently,

$$\rho_X(\widetilde{\pi}(x_1, t), \widetilde{\pi}(x_2, t)) \leq \rho_X(\widetilde{\pi}(x_1, t), \widetilde{\pi}(p_j, t)) + \rho_X(\widetilde{\pi}(p_j, t), \widetilde{\pi}(x_2, t)) < \epsilon,$$

for all $t \geq 0$. $\square$

In [8], the concept of minimality is defined in the following way: a set $A \subset X$ is minimal in $(X, \pi; M_X, I_X)$ if $A \setminus M_X \neq \emptyset$, A is closed, $A \setminus M_X$ is positively $\widetilde{\pi}-$invariant and A does not contain any proper subset satisfying these three properties.

Assuming that the center of Levinson J_Y of $(Y, \sigma; M_Y, I_Y)$ is minimal, we get the following result.

Theorem 4.12. *Let* (4.21) *be such that* $h(M_X) = M_Y$ *and* $(Y, \sigma; M_Y, I_Y)$ *be compact* $k-$*dissipative with* J_Y *minimal. Assume that* $\widetilde{\pi}^+(K)$ *is relatively compact for every* $K \in Comp(X)$ *and* $h^{-1}(J_Y) \cap X_y$ *has only one point for all* $y \in J_Y$. *Then* (4.21) *is convergent.*

Proof. Let $K \in Comp(X)$. By Lemma 4.2 and by the compactness of $h(K)$ we have

$$h(\widetilde{L}^+_X(K)) \subset \widetilde{L}^+_Y(h(K)) \subset J_Y.$$

Thus, $h(\widetilde{L}^+_X(K)) \cap M_Y = \emptyset$. Consequently, $\widetilde{L}^+_X(K) \cap M_X = \emptyset$ since $h(M_X) = M_Y$. This shows that $\widetilde{L}^+_X(K)$ is positively $\widetilde{\pi}-$invariant, and consequently, $h(\widetilde{L}^+_X(K))$ is positively $\widetilde{\sigma}-$invariant, see Proposition 4.1. Also, as $\widetilde{\pi}^+(K)$ is relatively compact, we have $\widetilde{L}^+_X(K)$ nonempty and compact. Then by minimality of J_Y we get $h(\widetilde{L}^+_X(K)) = J_Y$. By [8, Theorem 4.6] we can write $J_Y = \widetilde{L}^+_Y(y)$ for all $y \in J_Y$. Choose $y_0 \in J_Y$. Then

$$h(\widetilde{L}^+_X(K)) = \widetilde{L}^+_Y(y_0).$$

Take $x_0 \in X$ such that $h(x_0) = y_0$. By Theorem 4.1 we have $h(\widetilde{L}^+_X(x_0)) = \widetilde{L}^+_Y(y_0)$ and then $\widetilde{L}^+_X(x_0) \subset h^{-1}(J_Y)$.

Now, let $y \in \widetilde{L}^+_Y(y_0) = J_Y$. From hypothesis there is only one $x \in h^{-1}(J_Y) \cap X_y$. It shows that h is injective from $h^{-1}(J_Y)$ to $\widetilde{L}^+_Y(y_0)$. Since $\widetilde{L}^+_X(K) \subset h^{-1}(J_Y)$ we conclude that $\widetilde{L}^+_X(K) = \widetilde{L}^+_X(x_0)$ for all $K \in Comp(X)$. Hence, $(X, \pi; M_X, I_X)$ is compact $k-$dissipative.

Let J_X be the center of Levinson of $(X, \pi; M_X, I_X)$, $y \in J_Y$ and $a, b \in J_X \cap X_y$. Since $J_X \subset h^{-1}(J_Y)$ we have $J_X \cap X_y \subset h^{-1}(J_Y) \cap X_y$ and we conclude the result. □

4.3 Tests for convergence

In this section, we present tests to obtain convergence. We use Lyapunov functions to achieve the results.

Definition 4.10. A function $V : X \otimes X \to \mathbb{R}_+$ is called $\otimes-$continuous on $X \otimes X$, if given a sequence $\{(x^1_n, x^2_n)\}_{n \geq 1} \subset X \otimes X$ such that $x^i_n \overset{n \to +\infty}{\longrightarrow} x^i$ $(i = 1, 2)$ then $V(x^1_n, x^2_n) \overset{n \to +\infty}{\longrightarrow} V(x^1, x^2)$.

Lemma 4.7. *Let* $h : X \to Y$ *be a homomorphism from* $(X, \pi; M_X, I_X)$ *to* $(Y, \sigma; M_Y, I_Y)$. *Assume that* $(X, \pi; M_X, I_X)$ *is compact* $k-$*dissipative and there exists a* $\otimes-$*continuous function* $V : X \otimes X \to \mathbb{R}_+$ *satisfying the following properties:*

a) V *is positively defined, that is,* $V(x_1, x_2) = 0$ *if and only if* $x_1 = x_2$*;*

b) $V(\widetilde{\pi}(x_1, t), \widetilde{\pi}(x_2, t)) \leq V(x_1, x_2)$ *for all* $t \geq 0$ *and* $(x_1, x_2) \in X \otimes X$*.*

Then $(X, \pi; M_X, I_X)$ *is uniformly stable in the positive direction with respect to the homomorphism* h *on compact subsets from* X*.*

Proof. Suppose to the contrary that there exist $\epsilon_0 > 0$, $K_0 \in Comp(X)$, sequences $\delta_n \overset{n\to+\infty}{\longrightarrow} 0$, $\{x_n^1\}_{n\geq 1}, \{x_n^2\}_{n\geq 1} \subset K_0$ $(h(x_n^1) = h(x_n^2))$ and $\{t_n\}_{n\geq 1} \subset \mathbb{R}_+$ such that

$$\rho_X(x_n^1, x_n^2) < \delta_n \quad \text{and} \quad \rho_X(\widetilde{\pi}(x_n^1, t_n), \widetilde{\pi}(x_n^2, t_n)) \geq \epsilon_0 \tag{4.26}$$

for each $n = 1, 2, \ldots$. Since the system $(X, \pi; M_X, I_X)$ is compact $k-$dissipative, we may assume that the sequences $\{\widetilde{\pi}(x_n^1, t_n)\}_{n\geq 1}$ and $\{\widetilde{\pi}(x_n^2, t_n)\}_{n\geq 1}$ are convergent. Let $\bar{x}_i = \lim\limits_{n\to+\infty} \widetilde{\pi}(x_n^i, t_n)$, $i = 1, 2$. Note that

$$\begin{aligned} h(\bar{x}_1) = h(\lim_{n\to+\infty} \widetilde{\pi}(x_n^1, t_n)) = \lim_{n\to+\infty} \widetilde{\sigma}(h(x_n^1), t_n) = \\ = \lim_{n\to+\infty} \widetilde{\sigma}(h(x_n^2), t_n) = h(\lim_{n\to+\infty} \widetilde{\pi}(x_n^2, t_n)) = h(\bar{x}_2). \end{aligned}$$

From (4.26) we may assume that $\lim\limits_{n\to+\infty} x_n^1 = \lim\limits_{n\to+\infty} x_n^2 = \bar{x} \in K_0$ and

$$0 \leq V(\bar{x}_1, \bar{x}_2) = \lim_{n\to+\infty} V(\widetilde{\pi}(x_n^1, t_n), \widetilde{\pi}(x_n^2, t_n)) \leq \lim_{n\to+\infty} V(x_n^1, x_n^2) = V(\bar{x}, \bar{x}) = 0,$$

which implies that $\bar{x}_1 = \bar{x}_2$. This contradicts (4.26) as $n \to +\infty$. Thus the result is proved. □

Lemma 4.8. *Let* $h : X \to Y$ *be a homomorphism from* $(X, \pi; M_X, I_X)$ *to* $(Y, \sigma; M_Y, I_Y)$*. Assume that* $(X, \pi; M_X, I_X)$ *is compact* $k-$*dissipative and that there is a* $\otimes-$*continuous function* $V : X \otimes X \to \mathbb{R}_+$ *satisfying the following properties:*

a) V *is positively defined;*

b) $V(\widetilde{\pi}(x_1, t), \widetilde{\pi}(x_2, t)) \leq V(x_1, x_2)$ *for all* $t \geq 0$ *and* $(x_1, x_2) \in X \otimes X$*;*

c) $V(\widetilde{\pi}(x_1, t), \widetilde{\pi}(x_2, t)) = V(x_1, x_2)$ *for all* $t \geq 0$ *if and only if* $x_1 = x_2$*.*

Then $\lim\limits_{t\to+\infty} \rho_X(\widetilde{\pi}(x_1, t), \widetilde{\pi}(x_2, t)) = 0$ *for all* $(x_1, x_2) \in X \otimes X$*.*

Proof. Suppose to the contrary that there exist $y_0 \in Y$, $\bar{x}_1, \bar{x}_2 \in X_{y_0}$, $\epsilon_0 > 0$ and $t_n \overset{n\to+\infty}{\longrightarrow} +\infty$ $(t_n > 0)$ such that

$$\rho_X(\widetilde{\pi}(\bar{x}_1, t_n), \widetilde{\pi}(\bar{x}_2, t_n)) \geq \epsilon_0 \quad \text{for each} \quad n = 1, 2, \ldots. \tag{4.27}$$

Define the function $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ by $\xi(t) = V(\widetilde{\pi}(\bar{x}_1, t), \widetilde{\pi}(\bar{x}_2, t))$. By condition $b)$ we get ξ is non-increasing and $\xi(t) \leq V(\bar{x}_1, \bar{x}_2)$ for all $t \geq 0$. Thus $\lim\limits_{t\to+\infty} \xi(t) = V_0 \geq 0$, that is,

$$\lim_{t\to+\infty} V(\widetilde{\pi}(\bar{x}_1, t), \widetilde{\pi}(\bar{x}_2, t)) = V_0. \tag{4.28}$$

In the other hand, since $(X, \pi; M_X, I_X)$ is compact $k-$dissipative, we have $\widetilde{L}^+_X(x) \neq \emptyset$ and $\widetilde{L}^+_X(x) \subset J_X$ for all $x \in X$. This implies that $\widetilde{L}^+_X(x) \cap M_X = \emptyset$ for all $x \in X$ because $J_X \cap M_X = \emptyset$. Moreover, we may assume that the sequences $\{\widetilde{\pi}(\bar{x}_1, t_n)\}_{n\geq 1}$ and $\{\widetilde{\pi}(\bar{x}_2, t_n)\}_{n\geq 1}$ are convergent. Let $\lim\limits_{n\to+\infty} \widetilde{\pi}(\bar{x}_1, t_n) = \bar{p}$ and $\lim\limits_{n\to+\infty} \widetilde{\pi}(\bar{x}_2, t_n) = \bar{q}$. Since V is $\otimes-$continuous, we conclude that $V(\widetilde{\pi}(\bar{x}_1, t_n), \widetilde{\pi}(\bar{x}_2, t_n)) \overset{n\to+\infty}{\longrightarrow} V(\bar{p}, \bar{q})$ and $V(\bar{p}, \bar{q}) = V_0$. Observe that $\bar{p} \notin M_X$ and $\bar{q} \notin M_X$, because $\bar{p} \in \widetilde{L}^+_X(\bar{x}_1)$, $\bar{q} \in \widetilde{L}^+_X(\bar{x}_2)$ and $\widetilde{L}^+_X(x) \cap M_X = \emptyset$ for all $x \in X$.

As $\widetilde{L}^+_X(x) \cap M_X = \emptyset$ for all $x \in X$ we have $\widetilde{L}^+_X(\bar{x}_1)$ and $\widetilde{L}^+_X(\bar{x}_2)$ positively $\widetilde{\pi}-$invariant, see [7, Lemma 3.5]. Using Lemma 3.1, we obtain for each $t \geq 0$ the following convergences

$$\widetilde{\pi}(\bar{x}_1, t_n + t) \overset{n\to+\infty}{\longrightarrow} \widetilde{\pi}(\bar{p}, t) = \pi(\bar{p}, t) \quad \text{and} \quad \widetilde{\pi}(\bar{x}_2, t_n + t) \overset{n\to+\infty}{\longrightarrow} \widetilde{\pi}(\bar{q}, t) = \pi(\bar{q}, t). \tag{4.29}$$

Then $V(\widetilde{\pi}(\bar{x}_1, t_n + t), \widetilde{\pi}(\bar{x}_2, t_n + t)) \overset{n\to+\infty}{\longrightarrow} V(\widetilde{\pi}(\bar{p}, t), \widetilde{\pi}(\bar{q}, t))$. By (4.28) we have

$$V(\widetilde{\pi}(\bar{p}, t), \widetilde{\pi}(\bar{q}, t)) = V_0 = V(\bar{p}, \bar{q}) \quad \text{for all} \quad t \geq 0.$$

Therefore, by condition $c)$ we have $\bar{p} = \bar{q}$ which contradicts (4.27) as $n \to +\infty$. □

The next result presents sufficient conditions to obtain convergence via Lyapunov functions.

Theorem 4.13. *Let* (4.21) *be such that* $(X, \pi; M_X, I_X)$ *is compact* $k-$*dissipative and* $h^{-1}(M_Y) = M_X$. *Let* $\varphi : Y \to X$ *be a continuous invariant section of* h. *Assume that there is a* $\otimes-$*continuous function* $V : X \otimes X \to \mathbb{R}_+$ *satisfying the following properties:*

a) V *is positively defined;*

b) $V(\widetilde{\pi}(x_1, t), \widetilde{\pi}(x_2, t)) \leq V(x_1, x_2)$ *for all* $t \geq 0$ *and* $(x_1, x_2) \in X \otimes X$*;*

c) $V(\widetilde{\pi}(x_1, t), \widetilde{\pi}(x_2, t)) = V(x_1, x_2)$ *for all* $t \geq 0$ *if and only if* $x_1 = x_2$.

Then system (4.21) *is convergent.*

Proof. The proof follows by Theorem 4.5, Theorem 4.7, Lemma 4.7 and Lemma 4.8. □

Theorem 4.14. *Let* (4.21) *be such that* $(X, \pi; M_X, I_X)$ *is compact* $k-$*dissipative and* $h^{-1}(M_Y) = M_X$. *Let* $\varphi : Y \to X$ *be a continuous invariant section of* h. *Assume that there is a* $\otimes-$*continuous function* $V : X \otimes X \to \mathbb{R}_+$ *satisfying the following properties:*

a) V is positively defined;

b) $V(\widetilde{\pi}(x_1,t),\widetilde{\pi}(x_2,t)) < V(x_1,x_2)$ for all $t > 0$ and $(x_1,x_2) \in X \otimes X \setminus \Delta_X$, where $\Delta_X = \{(x,x) : x \in X\}$.

Then the system (4.21) *is convergent.*

Proof. The proof of this theorem is an immediate consequence of Theorem 4.13. It is sufficient to note that the item $b)$ of this theorem implies items $b)$ and $c)$ of Theorem 4.13. □

Another sufficient condition for convergence is given in the next theorem.

Theorem 4.15. *Let* (4.21) *be such that* $(X,\pi;M_X,I_X)$ *and* $(Y,\sigma;M_Y,I_Y)$ *are compact* $k-$*dissipative. Assume that each point in* $h(J_X)$ *is a point with maximal negative unicity. Let* $V : X\otimes X \to \mathbb{R}_+$ *be a* $\otimes-$*continuous function on* $J_X \otimes J_X$ *satisfying the following properties:*

a) V is positively defined;

b) $V(\widetilde{\pi}(x_1,t),\widetilde{\pi}(x_2,t)) \le \omega(V(x_1,x_2),t)$ for all $t \ge 0$ and $(x_1,x_2) \in J_X \otimes J_X$, where $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a non-decreasing function in the first variable and $\omega(x,t) \to 0$, as $t \to +\infty$, for every $x \in \mathbb{R}_+$.

Then the system (4.21) *is convergent.*

Proof. By Theorem 4.8, it is sufficient to show that

$$\lim_{t\to+\infty} \sup_{(x_1,x_2)\in J_X\otimes J_X} \rho_X(\widetilde{\pi}(x_1,t),\widetilde{\pi}(x_2,t)) = 0. \tag{4.30}$$

In order to do that, we claim that

$$\lim_{t\to+\infty} \sup_{(x_1,x_2)\in J_X\otimes J_X} V(\widetilde{\pi}(x_1,t),\widetilde{\pi}(x_2,t)) = 0. \tag{4.31}$$

Indeed, since V is $\otimes-$continuous on the compact set $J_X \otimes J_X$, then there are $\alpha > 0$ and $(\bar{x}_1,\bar{x}_2) \in J_X \otimes J_X$ such that $V(x_1,x_2) \le V(\bar{x}_1,\bar{x}_2) = \alpha$ for all $(x_1,x_2) \in J_X \otimes J_X$. Consequently,

$$V(\widetilde{\pi}(x_1,t),\widetilde{\pi}(x_2,t)) \le \omega(V(x_1,x_2),t) \le \omega(\alpha,t) \tag{4.32}$$

for all $t \ge 0$. Since $\omega(\alpha,t) \to 0$ as $t \to +\infty$ then inequality (4.32) implies (4.31).

Now, we show that (4.31) implies (4.30). Suppose to the contrary that there exist $\epsilon_0 > 0$, $\{x_n^1\}_{n\ge1}, \{x_n^1\}_{n\ge2} \subset J_X$ with $h(x_n^1) = h(x_n^2)$ and $t_n \overset{n\to+\infty}{\longrightarrow} +\infty$ such that

$$\rho_X(\widetilde{\pi}(x_n^1,t_n),\widetilde{\pi}(x_n^2,t_n)) \ge \epsilon_0, \tag{4.33}$$

for each $n = 1, 2, \ldots$. We may assume that the sequences $\{\widetilde{\pi}(x_n^1, t_n)\}_{n\geq 1}$ and $\{\widetilde{\pi}(x_n^2, t_n)\}_{n\geq 1}$ are convergent because $(X, \pi; M_X, I_X)$ is compact $k-$dissipative. Let $\bar{x}_i = \lim\limits_{n\to+\infty} \widetilde{\pi}(x_n^i, t_n)$, $i = 1, 2$. From the inequality (4.33), we have $\bar{x}_1 \neq \bar{x}_2$.

On the other hand, by the inequality (4.32) we have

$$\lim_{n\to+\infty} V(\widetilde{\pi}(x_n^1, t_n), \widetilde{\pi}(x_n^2, t_n)) \leq \lim_{n\to+\infty} \omega(\alpha, t_n) = 0.$$

Since J_X is positively $\widetilde{\pi}-$invariant (see Theorem 3.1) and V is continuous on $J_X \otimes J_X$, we have $V(\bar{x}_1, \bar{x}_2) = \lim\limits_{n\to+\infty} V(\widetilde{\pi}(x_n^1, t_n), \widetilde{\pi}(x_n^2, t_n)) = 0$, that is, $\bar{x}_1 = \bar{x}_2$ which is a contradiction. Therefore, the result follows by Theorem 4.8. □

Now, we present a converse result. We give sufficient conditions to obtain a Lyapunov function for convergent non-autonomous systems with impulses.

Theorem 4.16. *Let system* (4.21) *be convergent and assume that*

$$\lim_{t\to+\infty} \rho_X(\widetilde{\pi}(x_1, t), \widetilde{\pi}(x_2, t)) = 0 \quad \textit{for all} \quad (x_1, x_2) \in X \otimes X.$$

Assume there is $\beta > 0$ such that $\phi_X(x) \geq \beta$ for all $x \in I_X(M_X)$. Then there is a function $V : X \otimes X \to \mathbb{R}_+$ $\otimes-$continuous at every point $(x_1, x_2) \in X \otimes X$ such that $x_1, x_2 \notin M_X$ which satisfies the following conditions:

a) V is positively defined;

b) $V(\widetilde{\pi}(x_1, t), \widetilde{\pi}(x_2, t)) \leq V(x_1, x_2)$ for all $t \geq 0$ and $(x_1, x_2) \in X \otimes X$;

c) $V(\widetilde{\pi}(x_1, t), \widetilde{\pi}(x_2, t)) = V(x_1, x_2)$ for all $t \geq 0$ if and only if $x_1 = x_2$.

Proof. Define the mapping $V : X \otimes X \to \mathbb{R}_+$ by

$$V(x_1, x_2) = \sup\{\rho_X(\widetilde{\pi}(x_1, t), \widetilde{\pi}(x_2, t)) : \, t \geq 0\}, \quad (x_1, x_2) \in X \otimes X. \tag{4.34}$$

It is clear that V is positively defined. Let $(x_1, x_2) \in X \otimes X$ and $t \geq 0$, then

$$V(\widetilde{\pi}(x_1, t), \widetilde{\pi}(x_2, t)) = \sup\{\rho_X(\widetilde{\pi}(x_1, t+s), \widetilde{\pi}(x_2, t+s)) : \, s \geq 0\} =$$

$$= \sup\{\rho_X(\widetilde{\pi}(x_1, s), \widetilde{\pi}(x_2, s)) : \, s \geq t\} \leq \sup\{\rho_X(\widetilde{\pi}(x_1, s), \widetilde{\pi}(x_2, s)) : \, s \geq 0\} = V(x_1, x_2).$$

For item *c*) it is enough to justify the sufficient condition. Suppose that $V(\widetilde{\pi}(x_1, t), \widetilde{\pi}(x_2, t)) = V(x_1, x_2)$ for all $t \geq 0$ and $x_1 \neq x_2$. Take $\delta = V(x_1, x_2) > 0$. By hypothesis, there exists $t_0 = t_0(\delta) > 0$ such that $\rho_X(\widetilde{\pi}(x_1, t), \widetilde{\pi}(x_2, t)) < \frac{\delta}{2}$ for all $t \geq t_0$. Hence,

$$V(\widetilde{\pi}(x_1, t_0), \widetilde{\pi}(x_2, t_0)) = \sup\{\rho_X(\widetilde{\pi}(x_1, t), \widetilde{\pi}(x_2, t)) : \, t \geq t_0\} \leq \frac{\delta}{2} < V(x_1, x_2),$$

which is a contradiction. Hence, V satisfies the conditions $a), b)$ and $c)$.

Now, we need to prove the $\otimes-$continuity of V.

First, we claim that for every compact $K \subset X$

$$\lim_{t\to+\infty} \sup_{(x_1,x_2)\in K\otimes K} \rho_X(\widetilde{\pi}(x_1,t),\widetilde{\pi}(x_2,t)) = 0. \tag{4.35}$$

In fact, suppose to the contrary that there are $\epsilon_0 > 0$, $K_0 \in Comp(X)$, $t_n \overset{n\to+\infty}{\longrightarrow} +\infty$ and sequences $\{x_n^1\}_{n\geq 1}, \{x_n^2\}_{n\geq 1} \subset K_0$ with $h(x_n^1) = h(x_n^2)$ such that

$$\rho_X(\widetilde{\pi}(x_n^1,t_n),\widetilde{\pi}(x_n^2,t_n)) \geq \epsilon_0 \quad \text{for each} \quad n = 1,2,\ldots. \tag{4.36}$$

Since K_0 is compact and $(X,\pi;M_X,I_X)$ is compact $k-$dissipative, we may assume without loss of generality that the sequences $\{\widetilde{\pi}(x_n^1,t_n)\}_{n\geq 1}$ and $\{\widetilde{\pi}(x_n^2,t_n)\}_{n\geq 1}$ are convergent. Let $\bar{x}_i = \lim\limits_{n\to+\infty} \widetilde{\pi}(x_n^i,t_n)$ for $i = 1,2$. Note that $\bar{x}_1, \bar{x}_2 \in \widetilde{L}_X^+(K_0) \subset J_X$, $h(\bar{x}_1) = h(\bar{x}_2) \in J_Y$ (see Lemma 4.2) and $\bar{x}_1 = \bar{x}_2$ which contradicts (4.36) as $n \to +\infty$. Thus the equality (4.35) is proved.

Let $x_1, x_2 \in X \setminus M_X$ and $\{(x_n^1,x_n^2)\}_{n\geq 1} \subset X\otimes X$ be a sequence such that $x_n^i \overset{n\to+\infty}{\longrightarrow} x_i$ $(i = 1,2)$. First, we assume that $x_1 \neq x_2$. Set $\epsilon = \frac{1}{2}\rho_X(x_1,x_2) > 0$ and $K = \{x_n^1,x_n^2\}_{n\geq 1} \cup \{x_1,x_2\}$. By (4.35) there is $T = T(K,\epsilon) > 0$ such that $\rho_X(\widetilde{\pi}(a,t),\widetilde{\pi}(b,t)) < \epsilon$ for all $a,b \in K$, $h(a) = h(b)$, and for all $t \geq T$.

Given $x \in X$ and $j \in \mathbb{N}$, we define

$$D(x_j^+) = \begin{cases} [0,\phi_X(x_j^+)], & \text{if } \phi_X(x_j^+) < +\infty, \\ [0,+\infty), & \text{if } \phi_X(x_j^+) = +\infty. \end{cases}$$

Since $\phi_X(x) \geq \beta$ for all $x \in I_X(M_X)$, the impulsive semitrajectory of x_i admits a finite number of discontinuities in $[0,T]$, we say m_i jumps, $i = 1,2$. Thus there are $k_i = k_i(x_i) \in \{0,1,2,\ldots,m_i\}$ and $\tau_i \in D((x_i)_{k_i}^+) \cap [0,T]$, $i = 1,2$, such that

$$V(x_1,x_2) = \rho_X(\pi((x_1)_{k_1}^+,\tau_1),\pi((x_2)_{k_2}^+,\tau_2)).$$

By the convergence $x_n^i \overset{n\to+\infty}{\longrightarrow} x_i$ we may assume that x_n^i admits the same number of discontinuities of x_i in $[0,T]$, for each $n \in \mathbb{N}$, $i = 1,2$. Thus, there are $\ell_n^i \in \{0,1,2,\ldots,m_i\}$ and $\tau_n^i \in D((x_n^i)_{\ell_n^i}^+) \cap [0,T]$ such that

$$V(x_n^1,x_n^2) = \rho_X(\pi((x_n^1)_{\ell_n^1}^+,\tau_n^1),\pi((x_n^2)_{\ell_n^2}^+,\tau_n^2))$$

for all integers $n \geq 1$. Consequently, for each $i = 1,2$, there is a natural n_0^i such that we can decompose the sequence $\{x_n^i\}_{n\geq n_0^i}$ in the following manner

$$\{x_n^i\}_{n\geq n_0^i} = \{x_{n_{1j}}^i\}_{j\geq 1} \cup \ldots \cup \{x_{n_{rj}}^i\}_{j\geq 1},$$

where $\{x^i_{n_{lj}}\}_{j\geq 1}$ is a subsequence of $\{x^i_n\}_{n\geq n^i_0}$, $\ell^i_{n_{lj}} = \ell^i_{n_l} \in \{0,1,2,\ldots,m_i\}$ for all integers $j \geq 1$, $1 \leq l \leq r$ and $1 \leq r \leq m_i + 1$.

Consider an arbitrary subsequence $\{x^i_{n_{lj}}\}_{j\geq 1}$, where $l \in \{1,2,\ldots,r\}$ and $1 \leq r \leq m^i + 1$, $i = 1,2$. For convenience, we denote $\{x^i_{n_{lj}}\}_{j\geq 1}$ by $\{x^i_{n_j}\}_{j\geq 1}$ and $\ell^i_{n_{l_j}} = \ell_i$. Then

$$V(x^1_{n_j}, x^2_{n_j}) = \rho_X(\pi((x^1_{n_j})^+_{\ell_1}, \tau^1_{n_j}), \pi((x^2_{n_j})^+_{\ell_2}, \tau^2_{n_j}))$$

for all integers $j \geq 1$. Now, we may assume without loss of generality that $\tau^i_{n_j} \stackrel{j\to+\infty}{\longrightarrow} \nu_i \in D((x_i)^+_{\ell_i}) \cap [0,T]$. Note that

$$V(x^1_{n_j}, x^2_{n_j}) \stackrel{j\to+\infty}{\longrightarrow} \rho_X(\pi((x_1)^+_{\ell_1}, \nu_1), \pi((x_2)^+_{\ell_2}, \nu_2)).$$

Let us show that $V(x_1,x_2) = \rho_X(\pi((x_1)^+_{\ell_1}, \nu_1), \pi((x_2)^+_{\ell_2}, \nu_2)$. In fact, we can choose a sequence $\{\beta^i_m\}_{m\geq 1}$ such that $\beta^i_m \in D((x_i)^+_{k_i}) \cap [0,T]$, $m \geq 1$, $\beta^i_m \stackrel{m\to+\infty}{\longrightarrow} \tau_i$ and β^i_m is not an impulsive point for x_i, $i = 1,2$. By definition of V we get

$$V(x^1_{n_j}, x^2_{n_j}) = \rho_X(\pi((x^1_{n_j})^+_{\ell_1}, \tau^1_{n_j}), \pi((x^2_{n_j})^+_{\ell_2}, \tau^2_{n_j})) \geq \rho_X(\widetilde{\pi}(x^1_{n_j}, t), \widetilde{\pi}(x^2_{n_j}, t))$$

for all $t \geq 0$. Let $t_m = \sum_{j=-1}^{k_1-1} \phi((x_1)^+_j) + \beta^1_m = \sum_{j=-1}^{k_2-1} \phi((x_2)^+_j) + \beta^2_m$, where $\phi((x_1)^+_{-1}) = \phi((x_2)^+_{-1}) = 0$. As $\beta^i_m \in D((x_i)^+_{k_i}) \cap [0,T]$ is not an impulsive point of x_i for each $m = 1,2,\ldots$, then it follows by Lemma 3.1 that

$$\rho_X(\widetilde{\pi}(x^1_{n_j}, t_m), \widetilde{\pi}(x^2_{n_j}, t_m)) \stackrel{j\to+\infty}{\longrightarrow} \rho_X(\widetilde{\pi}(x_1, t_m), \widetilde{\pi}(x_2, t_m)) = \rho_X(\pi((x_1)^+_{k_1}, \beta^1_m), \pi((x_2)^+_{k_2}, \beta^2_m)).$$

Thus $\lim_{j\to+\infty} V(x^1_{n_j}, x^2_{n_j}) \geq \rho_X(\pi((x_1)^+_{k_1}, \beta^1_m), \pi((x_2)^+_{k_2}, \beta^2_m))$, that is,

$$\rho_X(\pi((x_1)^+_{\ell_1}, \nu_1), \pi((x_2)^+_{\ell_2}, \nu_2)) \geq \rho_X(\pi((x_1)^+_{k_1}, \beta^1_m), \pi((x_2)^+_{k_2}, \beta^2_m))$$

for each $m = 1,2,\ldots$. When $m \to +\infty$ we obtain,

$$\rho_X(\pi((x_1)^+_{\ell_1}, \nu_1), \pi((x_2)^+_{\ell_2}, \nu_2)) \geq \rho_X(\pi((x_1)^+_{k_1}, \tau_1), \pi((x_2)^+_{k_2}, \tau_2)) = V(x_1, x_2).$$

But $V(x_1,x_2) \geq \rho_X(\pi((x_1)^+_{\ell_1}, \nu_1), \pi((x_2)^+_{\ell_2}, \nu_2)$. Then,

$$V(x_1,x_2) = \rho_X(\pi((x_1)^+_{\ell_1}, \nu_1), \pi((x_2)^+_{\ell_2}, \nu_2).$$

Hence, $V(x^1_{n_{lj}}, x^2_{n_{lj}}) \stackrel{j\to+\infty}{\longrightarrow} V(x_1,x_2)$ for all $l = 1,2,\ldots,r$, and consequently

$$V(x^1_n, x^2_n) \stackrel{n\to+\infty}{\longrightarrow} V(x_1,x_2).$$

Now, suppose that $x_1 = x_2 = x$. Note that $V(x,x) = 0$. If $\{(x^1_n, x^2_n)\}_{n\geq 1} \subset X \otimes X$ is a sequence such that $x^i_n \stackrel{n\to+\infty}{\longrightarrow} x$ $(i = 1,2)$, we need to note that

$$V(x^1_n, x^2_n) = \rho_X(\pi((x^1_n)^+_{\ell^1_n}, \tau^1_n), \pi((x^2_n)^+_{\ell^2_n}, \tau^2_n))$$

with $\ell^1_n = \ell^2_n$ and $\tau^1_n = \tau^2_n$, $n = 1,2,\ldots$. It shows that $V(x^1_n, x^2_n) \stackrel{n\to+\infty}{\longrightarrow} 0 = V(x,x)$.

Therefore, V is $\otimes-$continuous on $(X \setminus M_X)\otimes(X \setminus M_X)$. $\square$

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