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NLS-LIKE EQUATIONS IN BOUNDED DOMAINS: PARABOLIC APPROXIMATION PROCEDURE

ALEXANDRE N. CARVALHO
JAN W. CHOLEWA

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Abstract. The article is devoted to semilinear Schrödinger equations in bounded domains. A unified semigroup approach is applied following a concept of Trotter-Kato approximations. Critical exponents in L^2 and H^1 are exhibited and global solutions are constructed for nonlinearities satisfying even a certain critical growth condition.

1. Introduction

We study a family of initial-boundary value problems of the form

$$\begin{cases}(\pm\sqrt{1-\eta^2}i-\eta)u_t+\Delta u+f(x,u)=0,\ t>0,\ x\in\Omega,\\ u_{|\partial\Omega}=0,\ t>0,\quad u(0,x)=u_0(x),\ x\in\Omega,\end{cases}\tag{1.1}$$

where $\eta\in[0,1]$ plays a role of a parameter, Ω is a smooth bounded domain in $\mathbb{R}^N$ and $f:\Omega\times\mathbb{C}\to\mathbb{C}$ is a continuous map satisfying a suitable growth condition.

The equations in (1.1) contain as the limiting problems some well known models. Namely, $\eta\to 1$ leads to a parabolic equation

$$u_t=\Delta u+f(x,u)\tag{1.2}$$

in $\mathbb{R}^+\times\Omega$ and when $\eta\to 0$ one gets a semilinear Schrödinder equation

$$iu_t+\Delta u+f(x,u)=0\tag{1.3}$$

in $\mathbb{R}^+\times\Omega$ or in $\mathbb{R}^-\times\Omega$ respectively. Here note that changing $u(t;x)$ into $u(-t;x)$ leads from the equation (1.3) in $\mathbb{R}^-\times\Omega$ to $-iu_t+\Delta u+f(x,u)=0$ in $\mathbb{R}^+\times\Omega$ and that the latter equation can be viewed as the limit of (1.1) with the minus sign as $\eta\to 0$.

Nonlinear Schrödinger like problems have brought a lot of attention in recent years and much progress has been achieved. See, for example, [6, 8, 13, 16, 18, 19] and references therein, which nonetheless are merely samples of the rich literature devoted to this subject.

It can be seen that the properties of Schrödinger's equation fall into the theory of both parabolic and hyperbolic equations. Concerning related tools of the theory, Strichartz's estimates and some natural conservation laws, like energy and charge conservation properties, are especially useful.

When Ω is a bounded domain, which situation we consider here, the essential particularity is that Strichartz's estimates are not applicable (see [6, Remark 2.7.3]) although, on the other hand, they are crucial in the analysis of the Cauchy problem in the whole of $\mathbb{R}^N$ (see [6] for

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an extensive results and further comments in that matter). Consequently, the analysis of the NLS like equations in bounded domains is not so complete as in the case of $\Omega = \mathbb{R}^N$, especially for nonlinearities satisfying a critical growth condition.

Note that in the profound studies [6, § 3.3, 3.4] existential results where obtained in $H^1_0(\Omega)$ via regularization of the nonlinear term f in the case when f behaved subcritically; namely for $N > 3$ the image $f(H^1_0(\Omega))$ was contained in $L^{q'}(\Omega)$ for some $q' \in [2, \frac{2N}{N-2})$. Consequently, f was then well defined from $H^1_0(\Omega)$ into $(H^{\frac{N}{2}-\frac{N}{q'}}(\Omega))'$, the latter space being an intermediate space between $(H^1_0(\Omega))'$ and $H^1_0(\Omega)$.

In this article we describe complementary regularization procedure, relying on regularization of the linear main part operator, which in a natural way reveals the critical exponents. This leads to the approximation of solutions of the Dirichlet initial-boundary value problem for the equation (1.3) by solutions of $(1.1)_{\eta\in(0,1]}$. Such parabolic approximation is then advantageous in the consideration of (1.3) as on the one hand one can obtain global solutions of (1.3) under some mild assumptions on the nonlinear term and, on the other, one can even consider some situation when f takes $H^1_0(\Omega)$ into $(H^1_0(\Omega))'$ but the image $f(H^1_0(\Omega))$ is not contained in any intermediate space between $(H^1_0(\Omega))'$ and $H^1_0(\Omega)$. Within this approach one can thus handle nonlinearities, which behave in a critical manner (see Examples 2.3 and 2.4 v) given below, for which Theorem 2.3 can be applied).

A brief description of this work is as follows. In Section 2 below we tersely describe the main results. Moreover, we exhibit critical exponents and give examples of some typical nonlinearities, involving even a critically growing one, to which the results are applicable. The results are then proved in the following two sections. Section 3 deals with approximate problems $(1.1)_{\eta\in(0,1]}$ and Section 4 is devoted to the limit equation (1.3). Some auxiliary results are included in the Appendix.

2. Notation and main results

We will use Lebesgue's spaces $H^s_p(\Omega)$, $s \in \mathbb{R}$, $p \in [1,\infty)$ as in [20], where $H^s_p(\Omega) = (H^{-s}_{p'}(\Omega))'$ for $s < 0$. Some of these spaces will involve zero trace boundary condition in which case, following [20], we let $H^s_{p,\{Id\}}(\Omega) := \{\varphi \in H^s(\Omega) : \varphi_{|\partial\Omega} = 0 \text{ for } s - \frac{1}{p} > 0\}$. When $s < 0$ we denote $H^s_{p,\{Id\}}(\Omega) := (H^{-s}_{p',\{Id\}}(\Omega))'$. If $p = 2$ it is typical to write $H^s(\Omega)$ instead of $H^s_2(\Omega)$. Actually, to keep the notation short, given a smooth bounded domain $\Omega \subset \mathbb{R}^N$, we will omit from now on the dependence on Ω denoting $H^s_p(\Omega) =: H^s_p$, $H^s_2(\Omega) =: H^s$, $H^s_{p,\{Id\}}(\Omega) =: \dot{H}^s_p$ and $H^s_{2,\{Id\}}(\Omega) =: \dot{H}^s$. We also write $L^p(\Omega) =: L^p$.

To express our results better let us consider the negative Laplacian operator

$$A_p = -\Delta \text{ in } L^p,\, p \in (1,\infty), \tag{2.1}$$

with the domain $D(A_p) = \dot{H}^2_p$ and let $\theta_\eta = Arg(\eta + \sqrt{1-\eta^2}i)$. Then

$$\eta + \sqrt{1-\eta^2}i = e^{i\theta_\eta},\ \eta \in [0,1]$$

and the first equation in (1.1) rewrites as

$$u_t + e^{\pm i\theta_\eta} A_p u = e^{\pm i\theta_\eta} f(\cdot, u),\ t > 0,\ \theta_\eta \in [0, \frac{\pi}{2}]. \tag{2.2}$$

Concerning the linear main part operator in (2.2),

$$e^{\pm i\theta_\eta} A_p =: \mathcal{A}^{\pm}_{p,\eta},\ \eta \in [0,1],\ p \in (1,\infty), \tag{2.3}$$

Stone's theorem (see [16, Theorem 1.10.8]) implies the following result in the limit case $\eta = 0$.

Proposition 2.1. *$\mathcal{A}^{\pm}_{2,0} = \pm i A_2$ generates a C^0 group of unitary operators on L^2.*

On the other hand, if $\eta \in (0,1]$, the corresponding linear semigroup will be analytic.

Proposition 2.2. *Given $p \in (1,\infty)$ and $\eta \in (0,1]$ the operator $\mathcal{A}^{\pm}_{p,\eta}$ in (2.3) is a negative generator of a C^0 analytic semigroup in L^p.*

Actually, for any $\eta \in [0,1]$, $\mathcal{A}^{\pm}_{2,\eta}$ is a maximal accretive operator with zero in the resolvent set. This, due to [16, Theorem 1.4.3], [2, §III.4.7.3(b)] and [20, §1.15.3] (see also [15]), leads to the following result.

Proposition 2.3. *Given $\eta \in [0,1]$ $-\mathcal{A}^{\pm}_{2,\eta}$ is an infinitesimal generator of the semigroup of contractions in L^2. Furthermore, $\mathcal{A}^{\pm}_{2,\eta}$ has bounded imaginary powers[1] and for $\alpha \in (0,1)$ the domains of fractional powers $D((\mathcal{A}^{\pm}_{2,\eta})^\alpha)$ are characterized independently of $\eta \in [0,1]$ as the complex interpolation spaces $[L^2, \dot{H}^2]_\alpha$.*

Due to Proposition 2.2, if $\eta \in (0,1]$ then (2.2) can be treated with the aid of the analytic semigroup theory.

Concerning properties of a nonlinear right hand side in (2.2) we associate with f the operator f^e, where

$$f^e(u)(x) = f(x, u(x)) \ \text{ a.e. in } \Omega \tag{2.4}$$

for any measurable $u : \Omega \to \mathbb{C}$, and consider the following hypothesis $\mathcal{H}^k_p$ relative to the phase space of initial data $\dot{H}^k_p$ with $k = 1$ or $k = 0$ respectively.

Hypothesis $\mathcal{H}^k_p$. *Let $k \in \{0,1\}$ be given and $p \in (1,\infty)$. We assume that there are constants $c > 0$, $\varepsilon \in (0, \frac{1}{\rho})$, $\gamma = \gamma(\varepsilon)$ satisfying $\rho\varepsilon \le \gamma < 1$, $\gamma \le 1 - \frac{k}{2}$, such that*

$$f^e \in C(H^{k+2\varepsilon}_p, H^{-(2-k-2\gamma)}_p) \tag{2.5}$$

and there also exist certain constants $\zeta > 0$, $C_\zeta > 0$ such that for all $v, w \in H^{k+2\varepsilon}_p$

$$\|f^e(v) - f^e(w)\|_{H^{-(2-k-2\gamma)}_p} \le c\|v - w\|_{H^{k+2\varepsilon}_p}\Big(C_\zeta + \zeta\|v\|^{\rho-1}_{H^{k+2\varepsilon}_p} + \zeta\|w\|^{\rho-1}_{H^{k+2\varepsilon}_p}\Big). \tag{2.6}$$

Theorem 2.1. *Let $k \in \{0,1\}$ be given. If $\eta \in (0,1]$ and hypothesis $\mathcal{H}^k_p$ is satisfied then (2.2) is locally well posed in $\dot{H}^k_p$.*

Furthermore, if u is a solution of $(2.2)_{|_{\eta\in(0,1]}}$ through initial condition $u_0 \in \dot{H}^k_p$ defined on a maximal interval of existence $[0, \tau_{u_0})$ and if one of the following conditions holds
(c_1) hypothesis $\mathcal{H}^k_p$ holds with $\gamma > \rho\varepsilon$
(c_2) hypothesis $\mathcal{H}^k_p$ holds with $\gamma = \rho\varepsilon$ and arbitrarily small $\zeta > 0$

[1] We remark that $\mathcal{A}^{\pm}_{2,\eta}$ is of the class $\mathcal{B}IP(1, \frac{\pi}{2})$ for each $\eta \in (0,1]$.

then u satisfies the blow up $\dot{H}^k_p$ alternative, that is,

$$\text{either } \tau_{u_0} = \infty \text{ or otherwise } \limsup_{t\to\tau_{u_0}^-} \|u\|_{\dot{H}^k_p} = \infty. \tag{2.7}$$

Theorem 2.1 leads in a natural way to the consideration of critical exponents $\rho_c(k,p)$, which describe the maximal growth of the nonlinear term allowed for the local well posedness of $(2.2)_{\eta\in(0,1]}$ in $\dot{H}^k_p$ with $k=1$ or $k=0$ respectively.

Proposition 2.4. *Let $k \in \{0,1\}$ be given. Hypothesis $\mathcal{H}^k_p$ holds if f satisfies*

$$\exists_{c>0}\ \forall_{z_1,z_2\in\mathbb{C}}\ |f(z_1)-f(z_2)| \le c|z_1-z_2|(1+|z_1|^{\rho-1}+|z_2|^{\rho-1}) \tag{2.8}$$

with any $\rho > 1$ when $k=1$ and $N \le p$, and with

$$1 < \rho \le 1 + \frac{2p}{N-kp} =: \rho_c(k,p) \quad \text{when } k=0 \text{ or when } k=1 \text{ and } N>p. \tag{2.9}$$

Furthermore, γ in hypothesis $\mathcal{H}^k_p$ can be chosen strictly bigger than $\rho\varepsilon$ unless $\rho = \rho_c(k,p)$ in which case $\gamma = \rho\varepsilon$.

In Proposition 2.4 no growth restriction is actually needed in the case when $k=1$ and $N<p$ whereas when $k=1$ and $1<p=N$ one can even consider the exponential growth due to Trudinger's inequality [1, §8.25].

Concerning the critical exponent $\rho_c(k,p)$ the following version of the above proposition holds (see [4, Lemma 3.2]).

Proposition 2.5. *Let $k \in \{0,1\}$ be given. Hypothesis $\mathcal{H}^k_p$ holds with arbitrarily small $\zeta > 0$ provided that $f(z) = h(|z|)$ for some differentiable real map $h : \mathbb{R} \to \mathbb{R}$ satisfying*

$$\lim_{|s|\to\infty} \frac{|h'(s)|}{|s|^{\rho_c(k,p)-1}} = 0, \quad \text{where } N>p. \tag{2.10}$$

Note that hypothesis $\mathcal{H}^k_p$ is also satisfied by multiplication operators Q_V associated with external potentials V, where Q_V is defined for any measurable function $\phi : \Omega \to \mathbb{C}$ by

$$Q_V(\phi)(x) = V(x)\phi(x) \quad \text{a.e. in } \Omega. \tag{2.11}$$

Proposition 2.6. *If $V : \Omega \to \mathbb{R}$ is a potential of the class L^r and $r > \frac{N}{2}$, $r \ge 1$ then $f^e(u) = Q_V(u)$ satisfies hypothesis $\mathcal{H}^k_p$ with $k=1$ and $k=0$ respectively. Furthermore, γ in hypothesis $\mathcal{H}^k_p$ can be chosen strictly bigger than $\rho\varepsilon$.*

For global solvability of approximate equations we restrict our consideration to Hilbert phase spaces as we need to rely on the a priori bounds on the solutions in L^2 and in H^1 respectively. This is the reason why, although local existence of solutions of approximate equations $(2.2)_{\eta\in(0,1]}$ can be obtained in more general spaces, global existence will be limited to the Hilbert setting.

We remark that if

$$Im(f(x,u)\bar{u}) = 0 \quad \text{a.e. in } \Omega, \tag{2.12}$$

then, formally, the limit problem $(1.1)_{\eta=0}$ has the charge conservation property

$$\|u\|_{L^2} = \|u_0\|_{L^2}. \tag{2.13}$$

If f can be viewed as the gradient of some suitable functional F, more precisely if

$$F' = f, \quad \text{where} \quad F \in C^1(\dot{H}^1, \mathbb{R}) \quad \text{and} \quad \langle F'(u), v\rangle_{\dot{H}^{-1},\dot{H}^1} = Re \int_\Omega f(\cdot, u)\bar{v}, \tag{2.14}$$

then, formally, the limit problem $(1.1)_{\eta=0}$ has the energy conservation property

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - F(u) = E(u_0). \tag{2.15}$$

To obtain existential results in $\dot{H}^1$ we will assume both (2.12) and (2.14). On the other hand, working with initial conditions $u_0 \in L^2$ we will assume (2.12) but not (2.14). In both situations we will use the structure condition

$$f(x, u)\bar{u} \le C(x)|u|^2 + D(x)|u| \quad \text{for a.e.} \quad x \in \Omega, \tag{2.16}$$

with certain

$$C \in L^r, \ r > \frac{N}{2}, \ r \ge 1 \quad \text{and} \quad D \in L^s, \ s \ge \max\{\frac{2N}{N+2}, 1\}. \tag{2.17}$$

Also in some cases we can assume either (2.16)-(2.17) or

$$f(x, u)\bar{u} \le C(x)|u|^2 + D(x) \quad \text{for a.e.} \quad x \in \Omega, \tag{2.18}$$

with

$$C \in L^r, \ r > \frac{N}{2}, \ r \ge 1 \quad \text{and} \quad D \in L^1 \tag{2.19}$$

(see Lemma 3.1 and Remark 3.1).

Note that since (2.17) allows D to be of the class L^s with $s < 2$ then (2.16)-(2.17) do not imply (2.18)-(2.19) in general. Also note that we neither assume that $C(\cdot)$ in (2.16) or (2.18) is negative nor that the bottom spectrum of $-\Delta - C(\cdot)Id$ is positive.

Concerning approximate problems we have the following well posedness result.

Theorem 2.2. *Let $k \in \{0, 1\}$ be given. Suppose that hypothesis $\mathcal{H}_2^k$ holds with $\gamma > \rho\varepsilon$ or that it holds with $\gamma = \rho\varepsilon$ and arbitrarily small $\zeta > 0$. Suppose also that*
i) (2.12), (2.16)-(2.17) (alternatively (2.12), (2.18)-(2.19)) are satisfied if $k = 0$,
ii) (2.12), (2.14), (2.16)-(2.17) hold if $k = 1$.
Then, the problems $(2.2)_{\eta\in(0,1]}$ are globally well posed in $\dot{H}^k$.

Sample nonlinearities can be mentioned for which the assumptions of Theorem 2.2 hold.

Example 2.1. *i) Typical nonlinearities which satisfy (2.12), (2.16)-(2.17) are*

$$f(x, u) = -a(x)|u|^{\rho-1}u + b(x)|u|^{\tilde{\rho}-1}u + V(x)u \tag{2.20}$$

where $\rho > \tilde{\rho} > 1$ and a, b, V are real valued functions such that $a \ge 0$ and $|b|^{\frac{\rho}{\rho-\tilde{\rho}}} a^{-\frac{\tilde{\rho}}{(\rho-\tilde{\rho})}} \in L^s$, $V \in L^r$ for some $s \ge \max\{\frac{2N}{N+2}, 1\}$, $r > \frac{N}{2}$, $r \ge 1$. Here note that (2.16)-(2.17) hold with $C = V$ and D being a multiple of $|b|^{\frac{\rho}{\rho-\tilde{\rho}}} a^{-\frac{\tilde{\rho}}{(\rho-\tilde{\rho})}}$ as one can write $b|u|^{\tilde{\rho}+1} = a^{1-\theta}|u|^{\tilde{\rho}+1-\theta}\frac{b}{a^{1-\theta}}|u|^\theta$ with $\theta = \frac{\rho-\tilde{\rho}}{\rho}$ and use Young's inequality.
ii) If in addition $1 < \tilde{\rho} < \rho < \rho_c(k, 2)$ and $a, b \in L^\infty$ then nonlinearities entering the sum in (2.20) satisfy hypothesis $\mathcal{H}_2^k$ with $\gamma > \rho\varepsilon$ for both $k = 0$ and $k = 1$ (see Propositions 2.4, 2.6) and in the case $k = 1$ they also have property (2.14) (see [6, Propositions 3.2.2 and 3.2.5]*).*

Example 2.2. *Some nonlocal nonlinearities satisfying (2.12), (2.16)-(2.17) can be considered; like*

$$f(u) = -u\left|\int_\Omega A_p^{-1}u\right|,$$

for which hypothesis $\mathcal{H}_p^k$ holds with $k=0$ (see Section 3.3).

Example 2.3. *If $h:\mathbb{R}\to[0,\infty)$ is a differentiable function satisfying (2.10) with $(k,p)=(1,2)$, $h(0)=0$ and if $f_h:\mathbb{C}\to\mathbb{C}$ is an extension of h such that*

$$f_h(z) = \begin{cases} \frac{z}{|z|}h(|z|), \ z\in\mathbb{C}\setminus\{0\}, \\ 0, \ z=0, \end{cases}$$

then given a nonnegative $a\in L^\infty$ the nonlinearity in (2.21) below,

$$f(x,z) = -a(x)f_h(z), \ (x,z)\in\Omega\times\mathbb{C}, \tag{2.21}$$

satisfies (2.12), (2.14), (2.16)-(2.17), exhibits a critical growth $\rho=\rho_c(k,p)_{|_{k=1,p=2}}=\frac{N+2}{N-2}$ $(N\geq 3)$ and fulfils hypothesis $\mathcal{H}^k_{p_{|_{k=1,p=2}}}$ with $\gamma=\rho\varepsilon$ and arbitrarily small $\zeta>0$ (see Section 3.3.4).

Focusing on the case $p=2$ we now consider the solutions of $(2.2)_{\eta\in(0,1]}$ as approximate solutions of

$$\begin{cases} iu_t+\Delta u+f(x,u)=0, \ t\in\mathbb{R}, \ x\in\Omega, \\ u_{|\partial\Omega}=0, \ t>0, \quad u(0,x)=u_0(x), \ x\in\Omega. \end{cases} \tag{2.22}$$

This latter problem involves the equation (1.3) obtained by passing in (2.2) to a limit as $\eta\to 0$.

In the linear case, that is when $f=0$ in (2.2), the following result holds, which in turn comes back to the Trotter-Kato approximation theorem.

Proposition 2.7. *For each $\eta\in[0,1]$ the resolvent set of $-\mathcal{A}^\pm_{2,\eta}$ contains $\lambda\geq 0$ and the associated resolvent operators converge in $\mathcal{L}(L^2)$, that is*

$$R(\lambda,-\mathcal{A}^\pm_{2,\eta}) \stackrel{\mathcal{L}(L^2)}{\longrightarrow} R(\lambda,-\mathcal{A}^\pm_{2,0}) \quad as \quad \eta\to 0. \tag{2.23}$$

The associated linear semigroups converge as well. Actually, given any $u_0\in L^2$ and a bounded time interval J we have, uniformly for $t\in J$,

$$e^{-\mathcal{A}^\pm_{2,\eta}t}u_0 \stackrel{L^2}{\longrightarrow} e^{-\mathcal{A}^\pm_{2,0}t}u_0 \quad as \quad \eta\to 0. \tag{2.24}$$

In what follows we will show that such an approximation procedure also applies in a nonlinear case. Given $k\in\{0,1\}$ and passing to the limit as $\eta\to 0$ we will assume the following *k-condition* which is satisfied in many situations as shown in Example 2.4 below.

Definition 2.1. *Let $k\in\{0,1\}$ be given. We say that k-condition holds if*

$$\|f^e(v)\|_{H^{k-2}} \leq g_1(\|v\|_{H^k}), \ v\in H^k \tag{2.25}$$

and

$$\|f^e(v)-f^e(w)\|_{H^{-2}} \leq g_2(\|v\|_{H^1}, \ \|w\|_{H^1}, \|v-w\|_{H^{k-2}}), \ v,w\in H^k, \tag{2.26}$$

where $g_1 = g_1(y_1)$, $g_2 = g_2(y_1, y_2, y_3)$ *are nonnegative functions such that* g_1 *is bounded on bounded subsets of* $[0,\infty)$ *and* $\lim_{y_3\to 0} g(y_1,y_2,y_3) = 0$ *uniformly for* (y_1,y_2) *in bounded subsets of* $[0,\infty)\times[0,\infty)$.

Example 2.4. *i) If* f *satisfies (2.8) with* $\rho \in (1, \frac{N+2}{N-2})$ *when* $N \geq 3$, *or with any* $\rho > 1$ *when* $N = 1, 2$, *then for* $\theta = \frac{(\rho-1)(N-2)}{4}$

$$\|f(v) - f(w)\|_{H^{-1}} \leq \bar{c}\,\|v-w\|_{H^{-1}}^{1-\theta}\,\|v-w\|_{H^1}^{\theta}\left(1 + \|v\|_{H^1}^{\rho-1} + \|w\|_{H^1}^{\rho-1}\right),\ v, w \in H^1, \quad (2.27)$$

(see the proof of [5, Lemma 2.2]*) so that* k*-condition in Definition 2.1 holds with* $k = 1$. *Actually, no growth restriction is needed when* $k = 1$ *and* $N = 1$ *whereas in the case* $k = 1$ *and* $N = 2$ *the growth can even be exponential.*
ii) If V *is an external potential as in Proposition 2.6, then* $f^e(u) = Vu$ *satisfies* k*-condition with* $k = 1$ *(see Subsection 3.3.2).*
iii) Due to i)-ii) above typical nonlinearities as in Example 2.1 satisfy k*-condition with* $k = 1$.
iv) A nonlocal nonlinearity in Example 2.2 satisfies k*-condition with* $k = 0$ *(see Section 3.3.3).*
v) A critically growing nonlinearity defined in (2.21) of Example 2.3 satisfies k*-condition with* $k = 1$ *(see Section 3.3.4).*

We will look for a solution of (2.22) satisfying variation of constants formula as in Definition 2.2 below.

Definition 2.2. *Let* J *be an interval of* $\mathbb{R}$, $0 \in J$, $k \in \{0, 1\}$ *and* f^e *be a map from* $\dot{H}^k$ *into* $\dot{H}^{k-2}$. *Given* $u_0 \in \dot{H}^k$ *we say that* $u : J \to \dot{H}^k$ *is a mild* $\dot{H}^k$ *solution of (2.22) on the interval* J *if and only if* $u(0) = u_0$, u *is weakly continuous from* J *into* $\dot{H}^k$, $f^e(u)$ *is weakly continuous from* J *into* $\dot{H}^{k-2}$ *and*

$$u(t) = e^{-iA_2 t}u_0 + i\int_0^t e^{-iA_2(t-s)} f^e(u(s))ds,\ t \in J, \quad (2.28)$$

holds in $\dot{H}^{k-2}$.

Note that in (2.28) the nonlinear term $f^e(u(s))$ will in general belong to a function space $\dot{H}^s$ for some $s < 0$. Hence the linear semigroup appearing therein has to be suitably extended from L^2 to these larger spaces in which we follow the ideas of [2]. Namely, combining Proposition 2.3 together with [2, Theorem §V.1.5.4, Theorem §V.2.1.3, Remark V.2.1.4 and formula (V.2.2.1)] we have the following result.

Proposition 2.8. *Given* $\eta \in [0, 1]$, *a closed extension of* $-\mathcal{A}^{\pm}_{2,\eta}$ *(which we denote the same) is an infinitesimal generator of the semigroup of contractions in* $\dot{H}^\sigma$ *for any* $\sigma \in [-2, 0]$ *and the resolvent set of* $-\mathcal{A}^{\pm}_{2,\eta}$ *in* $\dot{H}^\sigma$ *coincides with the one in* L^2.

Using the concept of parabolic approximation we then prove the existence of global solutions of (2.22).

Theorem 2.3. *Let* $k \in \{0, 1\}$ *be given. Suppose that* k*-condition holds as in Definition 2.1. Assume also that hypothesis* $\mathcal{H}^k_2$ *holds either with* $\gamma > \rho\varepsilon$ *or with* $\gamma = \rho\varepsilon$ *and arbitrarily small* $\zeta > 0$, *and that*
i) (2.12), (2.16)-(2.17) (alternatively (2.12), (2.18)-(2.19)) are satisfied if $k = 0$,

ii) (2.12), (2.14), (2.16)-(2.17) hold if $k = 1$.

Then, given $u_0 \in \dot{H}^k$, (2.22) has a mild $\dot{H}^k$ solution u on the interval $(-\infty, \infty)$ and

$$u \in C(\mathbb{R}, \dot{H}^s) \cap L^\infty_{loc}(\mathbb{R}, \dot{H}^k), \ \ s < k. \tag{2.29}$$

Note that Theorem 2.3 applies with $k = 1$ to subcritical nonlinearities (2.20) as in Example 2.1 ii) and with $k = 0$ to a sample nonlinearity in Example 2.2. On the other hand note that Theorem 2.3 applies with $k = 1$ to critically growing nonlinearities as in Example 2.3 (see Example 2.4 iii)-v)).

With additional assumptions one can obtain further properties of the limit solution as in Propositions 2.9, 2.10 below (see also Remarks 4.2, 4.3).

Proposition 2.9. *If C in (2.16) or (2.18) is such that the bottom spectrum of the operator $A_C = -\Delta - C(\cdot)I$ in L^2 is strictly positive then in Theorem 2.3 we will also have that*

$$u \in L^\infty(\mathbb{R}, \dot{H}^k). \tag{2.30}$$

Proposition 2.10. *If the assumptions of Theorem 2.3 hold with $k = 1$ then*
i) (charge conservation) the solution will satisfy the equality (2.13),
ii) (energy inequality) if, in addition,

$$|F(v) - F(w)| \le g(\|v\|_{H^1}, \|w\|_{H^1}, \|v - w\|_{H^s}), \ \ v, w \in H^1, \tag{2.31}$$

holds for some $s < 1$ with a certain function $g = g(y_1, y_2, y_3)$ such that $\lim_{y_3 \to 0} g(y_1, y_2, y_3) = 0$ uniformly for (y_1, y_2) in bounded subsets of $[0, \infty) \times [0, \infty)$ then

$$E(u) \le E(u_0), \tag{2.32}$$

iii) (uniqueness) the solution will even be unique if similarly as in [6, Corollary 3.3.11] *one assumes that given $r > 0$ there exists $L(r) > 0$ such that*

$$\|f(v) - f(w)\|_{L^2} \le L(r)\|v - w\|_{L^2} \ \ \textit{whenever} \ \ \|v\|_{H^1} \le r, \|w\|_{H^1} \le r. \tag{2.33}$$

The above mentioned results will be proved in the following two sections.

3. Solutions of (1.1) with $\eta \in (0, 1]$

In this section we consider (2.2) with η strictly positive, i.e. with θ_η strictly less than $\frac{\pi}{2}$.

3.1. **Generalities concerning operators $\mathcal{A}^\pm_{p,\eta} := e^{\pm i\theta_\eta} A_p$ with $\eta \in (0, 1]$.** Given $p \in (1, \infty)$, $\eta \in (0, 1]$ and $Re(\lambda) \le 0$ we have that $\tilde{\lambda} = \lambda e^{\pm i\theta_\eta} \in \rho(A_p)$ because $\sigma(A_p)$ consists of strictly positive eigenvalues separated from zero. Therefore the sector $\mathcal{S}_{\theta_\eta} = \{\tilde{\lambda} \in \mathbb{C} : \frac{\pi}{2} - \theta_\eta \le |\arg(\tilde{\lambda})| \le \pi, \tilde{\lambda} \ne 0\}$ is in the resolvent set of A_p and combining this with [17, Theorem 2] we get $|\tilde{\lambda}| \|(\tilde{\lambda} Id - A_p)^{-1}\|_{L^2(\Omega)} \le M_{\theta_\eta}$ for $\tilde{\lambda} \in \mathcal{S}_{\theta_\eta}$. Hence we have

$$|\lambda| \|(\lambda e^{\pm i\theta_\eta} Id - A_p)^{-1}\phi\|_{L^p(\Omega)} = |\lambda| \|(\lambda Id - \mathcal{A}^\pm_{\eta,p})^{-1}\phi\|_{L^p(\Omega)} \le M_{\theta_\eta}, \ Re(\lambda) \le 0, \tag{3.1}$$

which proves Proposition 2.2; in particular, $\mathcal{A}^\pm_{p,\eta}$ is a sectorial operator in $X_p := L^p$.

Given $\eta \in (0, 1]$ the initial boundary value problem for the approximate equations (2.2) can be thus viewed as an abstract Cauchy problem

$$\frac{du}{dt} + \mathcal{A}^\pm_{p,\eta} u = f^e_{\eta,\pm}(u), \ t > 0, \quad u(0) = u_0, \tag{3.2}$$

with

$$\mathcal{A}^{\pm}_{p,\eta} = e^{\pm i\theta_\eta} A_p \quad \text{ and } \quad f^e_{\eta,\pm} = e^{\pm i\theta_\eta} f^e.$$

By [20, Theorem 4.9.1] (see also [11]) for each $\eta \in (0,1]$ $\mathcal{A}^{\pm}_{p,\eta}$ possesses bounded imaginary powers, that is,

$$\exists_{\epsilon>0} \sup_{s\in[-\epsilon,\epsilon]} \|(\mathcal{A}^{\pm}_{p,\eta})^{is}(t)\|_{L(X_p)} < \infty, \tag{3.3}$$

and the domains of fractional powers $X^\alpha_{p,\eta,\pm} = D((\mathcal{A}^{\pm}_{p,\eta})^\alpha)$ coincide with the fractional power spaces $X^\alpha_p = D(A^\alpha_p)$ associated to A_p, namely

$$X^\alpha_{p,\eta,\pm} = [L^p, \dot{H}^2]_\alpha = X^\alpha_p, \ \alpha \in (0,1), \ p \in (1,\infty), \ \eta \in (0,1]. \tag{3.4}$$

Although we have assumed that Ω is a smooth domain let us remark that [20, Theorem 4.9.1] requires $\partial\Omega$ to be of the class C^∞. The latter can be weakened following [11], where in the case of the second order operators it is required that $\partial\Omega$ is of the class C^2.

On the other hand note that the discussion concerning boundedness of imaginary powers can be avoided if $p = 2$ as in Proposition 2.3 or if one considers the interpolation scale instead of the fractional power scale. We do not pursue this here focusing on the main aspects of the parabolic approximation procedure, thus using fractional powers as the natural tools of the theory.

Following [2] operators $\mathcal{A}^{\pm}_{p,\eta}$ can be considered as closed operators on the extrapolated space $X^{-1}_{p,\eta,\pm}$ being the completion of the normed space $(L^p, \|(\mathcal{A}^{\pm}_{p,\eta})^{-1}\cdot\|_{L^p})$. We then have

$$X^{-1}_{p,\eta,\pm} = (X^1_{p',\eta,\pm})' = (\dot{H}^2_{p'})' = X^{-1}_p,$$

where p, p' are Hölder's conjugate exponents and $X^{-1}_p =: Y_p$ denotes the extrapolated space of (X_p, A_p). Also, the closed extension of $\mathcal{A}^{\pm}_{p,\eta}$ to $X^{-1}_{p,\eta,\pm}$ (for which we use the same notation) has the same resolvent set as $\mathcal{A}^{\pm}_{p,\eta}$ in $X_{p,\eta,\pm}$, belongs to a class of linear isomorphisms from $X_{p,\eta,\pm}$ into $X^{-1}_{p,\eta,\pm}$ and generates a strongly continuous analytic semigroup (see [2, Theorem V.2.1.3]). Thus, letting

$$Y_{p,\eta,\pm} = X^{-1}_{p,\eta,\pm} = X^{-1}_p =: Y_p,$$

we associate with $(Y_{p,\eta,\pm}, \mathcal{A}^{\pm}_{p,\eta})$ the fractional power scale $\{Y^\alpha_{p,\eta,\pm} : \alpha \geqslant 0\}$ and obtain by duality argument (see [20, §1.11.3]) that

$$Y^\alpha_{p,\eta,\pm} = [(X^1_{p',\eta,\pm})', (L^{p'})']_\alpha = (X^{1-\alpha}_{p',\eta,\pm})' = (X^{1-\alpha}_{p'})' =: Y^\alpha_p, \ \alpha \in (0,1), \ \eta \in (0,1]. \tag{3.5}$$

On the other hand

$$Y^1_{p,\eta,\pm} = (\mathcal{A}^{\pm}_{p,\eta})^{-1}(Y_{p,\eta,\pm}) = X_{p,\eta,\pm}, \quad Y^2_{p,\eta,\pm} = (\mathcal{A}^{\pm}_{\eta})^{-2}(Y_{p,\eta,\pm}) = (\mathcal{A}^{\pm}_{p,\eta})^{-1}(X_{p,\eta,\pm}) = X^1_{p,\eta,\pm}$$

and via (3.4)

$$Y^{1+\alpha}_{p,\eta,\pm} = [L^p, \dot{H}^2_p]_\alpha = X^\alpha_p =: Y^{1+\alpha}_p, \ \alpha \in (0,1), \ p \in (1,\infty), \ \eta \in (0,1). \tag{3.6}$$

Consequently, for each $p \in (1,\infty)$, $\alpha \in [0,1]$ we have

$$Y^{1+\alpha}_p \hookrightarrow H^{2\alpha}_p, \qquad Y^\alpha_p \hookleftarrow H^{-2(1-\alpha)}_p \tag{3.7}$$

and

$$
\begin{aligned}
&H^{2\alpha}_p(\Omega) \hookrightarrow L^s,\ \alpha\in[0,1],\ 2\alpha-\frac{N}{p}\geq -\frac{N}{s},\ s\geq 1,\\
&H^{-2(1-\alpha)}_p \hookleftarrow L^\sigma,\ \alpha\in[0,1),\ \frac{Np}{N+2(1-\alpha)p}\leq\sigma,\ \sigma>1.
\end{aligned}
\tag{3.8}
$$

3.2. **Proof of Theorem 2.1.** In the proof of local well posedness of $(2.2)_{|_{\eta\in(0,1]}}$ in $\dot H^k_p$ we rely on the formulation of the problem as in (3.2) and on the approach developed in [3]. Hence all what needs to be shown is that, under hypothesis $\mathcal{H}^k_p$ the Lipschitz type condition

$$
\|f^e(v)-f^e(w)\|_{Y_p^{\gamma+\frac{k}{2}}}\leq c\|v-w\|_{Y_p^{1+\varepsilon+\frac{k}{2}}}\left(1+\|v\|^{\rho-1}_{Y_p^{1+\varepsilon+\frac{k}{2}}}+\|w\|^{\rho-1}_{Y_p^{1+\varepsilon+\frac{k}{2}}}\right),\ v,w\in Y_p^{1+\varepsilon},\tag{3.9}
$$

holds for $k=1$ and $k=0$ respectively with certain constants $c>0$, $\varepsilon\in(0,\frac{1}{\rho})$ and

$$
\gamma=\gamma(\varepsilon)\in[\rho\varepsilon,1),\quad \gamma\leq 1-\frac{k}{2}.\tag{3.10}
$$

With the set up as in Section 3.1 condition (3.9) follows from (3.7) and from hypothesis $\mathcal{H}^k_p$. Then [3, Corollary 1] ensures that (2.2) is locally well posed in $\dot H^k_p$.

When (c_1) or (c_2) holds we also have the blow up alternative (2.7) (see [3, 4]).

Following [3, 14] we additionally have that for γ as in (3.9)-(3.10) and for any $\theta\in[0,1)$ the solution u constructed above satisfies

$$
u\in C([0,\tau_{u_0}),Y_p^{1+\frac{k}{2}})\cap C((0,\tau_{u_0}),Y_p^{\gamma+\frac{k}{2}+1})\cap C^1((0,\tau_{u_0}),Y_p^{\gamma+\frac{k}{2}+\theta}),\tag{3.11}
$$

that is,

$$
u\in C([0,\tau_{u_0}),\dot H^k_p)\cap C((0,\tau_{u_0}),\dot H^{2\gamma+k}_p)\cap C^1((0,\tau_{u_0}),\dot H^\sigma_p),\ \sigma<2\gamma+k.\tag{3.12}
$$

3.3. **Sample nonlinearities.** We exhibit here properties of sample nonlinearities which appeared in Section 1.

3.3.1. *Critical exponents: proofs of Propositions 2.4 and 2.5.* We first prove Proposition 2.4 starting from the situation when either $k=0$ or $k=1$ and $N>p$ (see (3.15)).

Using (2.8) and (3.8) we have

$$
\begin{aligned}
&\|f^e(v)-f^e(w)\|_{H_p^{-2(1-\gamma-\frac{k}{2})}}\leq c'\|f^e(v)-f^e(w)\|_{L^{\frac{Np}{N+2(1-\gamma-\frac{k}{2})p}}}\\
&\leq c'\|c|v-w|(1+|v|^{\rho-1}+|w|^{\rho-1})\|_{L^{\frac{Np}{N+2(1-\gamma-\frac{k}{2})p}}}\\
&\leq c''\int_\Omega\left(|v-w|^{\frac{Np}{N+2(1-\gamma-\frac{k}{2})p}}\left(1+|v|^{\frac{Np(\rho-1)}{N+2(1-\gamma-\frac{k}{2})p}}+|w|^{\frac{Np(\rho-1)}{N+2(1-\gamma-\frac{k}{2})p}}\right)dx\right)^{\frac{N+2(1-\gamma-\frac{k}{2})p}{Np}}
\end{aligned}
\tag{3.13}
$$

provided that $\frac{Np}{N+2(1-\gamma-\frac{k}{2})p}>1$ and $1-\gamma-\frac{k}{2}\geq 0$, which translate into the condition

$$
1-\frac{k}{2}\geq\gamma>\frac{N+2p-Np-kp}{2p}=:\tilde\gamma.\tag{3.14}
$$

Applying next Hölder's inequality with exponents $\frac{N+2(1-\gamma-\frac{k}{2})p}{N-(2\varepsilon+k)p}$, $\frac{N+2(1-\gamma-\frac{k}{2})p}{2(1-\gamma+\varepsilon)p}$, thus assuming

$$N > (2\varepsilon + k)p, \tag{3.15}$$

we obtain

$$\|f^e(v) - f^e(w)\|_{H_p^{-2(1-\gamma-\frac{k}{2})}} \le \tilde{c}\|v-w\|_{L^{\frac{Np}{N-(2\varepsilon+k)p}}} \left(1 + \|v\|^{\rho-1}_{L^{\frac{N(\rho-1)}{2(1-\gamma+\varepsilon)}}} + \|w\|^{\rho-1}_{L^{\frac{N(\rho-1)}{2(1-\gamma+\varepsilon)}}}\right).$$

Due to (3.8), the right hand side above can be bounded by the right hand side of (2.6) provided that $1 \le \frac{N(\rho-1)}{2(1-\gamma+\varepsilon)} \le \frac{Np}{N-(2\varepsilon+k)p}$, for which we need

$$\overline{\gamma} := \frac{(2\varepsilon p + kp - N)(\rho-1) + 2(1+\varepsilon)p}{2p} \ge \gamma \ge 1 + \varepsilon - \frac{N(\rho-1)}{2} =: \underline{\gamma}. \tag{3.16}$$

Evidently $\overline{\gamma} > \underline{\gamma} > 0$ and for $\rho \in (1, \frac{N-kp+2p}{N-kp}]$, $\varepsilon > 0$ we also have $\overline{\gamma} \ge \varepsilon\rho$. Furthermore, $1 > \overline{\gamma}$ holds if $\varepsilon \in (0, \frac{(N-kp)(\rho-1)}{2p\rho})$ and $\overline{\gamma} > \tilde{\gamma}$ if $\varepsilon > \frac{N(\rho-p)}{2p\rho} - \frac{k}{2}$. This ensures that the set of admissible triples $(\rho, \varepsilon, \gamma)$ is nonempty and contains $(\rho, \varepsilon, \gamma)$ such that $\rho \in (1, \frac{N-kp+2p}{N-kp}]$, $\varepsilon \in (\max\{0, \frac{N(\rho-p)}{2p\rho} - \frac{k}{2}\}, \frac{(N-kp)(\rho-1)}{2p\rho})$ and $\gamma \in [\rho\varepsilon, \overline{\gamma}] \cap [\underline{\gamma}, \overline{\gamma}] \cap (\tilde{\gamma}, \overline{\gamma}] =: \mathcal{I}(\varepsilon)$, $\gamma \le 1 - \frac{k}{2}$.

For admissible $(\rho, \varepsilon, \gamma)$ the left hand side inequality in (3.16) imply $\rho \le \frac{N+2p-kp-2p\gamma}{N-2p\varepsilon-kp}$ and hence, since $\gamma \ge \rho\varepsilon$, we get $\rho \le \frac{N+2p-kp-2p\rho\varepsilon}{N-2p\varepsilon-kp}$, which holds if and only if $\rho \le \frac{N-kp+2p}{N-kp} = \rho_c(k)$.

We remark that $\rho = \rho_c(k)$ cannot be attained for any $\gamma > \varepsilon\rho_c(k)$ and $\rho = \rho_c(k)$ necessitates that $\gamma = \varepsilon\rho_c(k)$, in which case we have $\overline{\gamma} = \varepsilon\rho_c(k)$. That is, if $\rho = \rho_c(k)$ then $\mathcal{I}(\varepsilon) = \{\varepsilon\rho_c(k)\}$.

The above analysis also shows that $\gamma = \gamma(\varepsilon)$ can be chosen less or equal than $1 - \frac{k}{2}$ and arbitrarily less than $\overline{\gamma}(\rho, \varepsilon) = \overline{\gamma}(\rho, \frac{(N-kp)(\rho-1)}{2p\rho}) = 1$. Therefore, if $k = 1$ then $\gamma = \gamma(\varepsilon)$ can be chosen less or equal $\frac{1}{2}$ whereas if $k = 0$ then $\gamma = \gamma(\varepsilon)$ can be chosen arbitrarily less than 1.

In the remaining case when $k = 1$ and $N \le p$ we have that $H_p^{k+2\varepsilon} \hookrightarrow L^\infty$. Hence after using in (3.13) Hölder's inequality with any conjugate exponents $\mu, \mu' > 1$ we will have the right hand side bounded by the right hand side of (2.6). In this latter case hypothesis $\mathcal{H}_p^k$ is thus easily satisfied and any triple $(\rho, \varepsilon, \gamma)$ such that $\rho > 1$, $\varepsilon \in (0, \frac{1}{2\rho}]$ and $\gamma \in [\varepsilon\rho, \frac{1}{2}]$ is admissible triple.

Having proved Proposition 2.4 we now observe that if (2.10) is assumed then

$$\forall_{\zeta>0}\ \exists_{C_\zeta>0}\ \forall_{s\in\mathbb{R}}\ |h'(s)| \le (C_\zeta + \zeta|s|^{\rho_c(k,p)-1}). \tag{3.17}$$

Consequently, under the assumptions of Proposition 2.5 we have that

$$\forall_{\zeta>0}\ \exists_{C_\zeta>0}\ \forall_{z_1,z_2\in\mathbb{C}}\ |f(z_1) - f(z_2)| \le |z_1 - z_2|(C_\zeta + \zeta|z_1|^{\rho_c(k,p)-1} + \zeta|z_2|^{\rho_c(k,p)-1}). \tag{3.18}$$

The proof of Proposition 2.5 follows thus the lines of the proof of Proposition 2.4.

3.3.2. *External potentials: proofs of Proposition 2.6 and Remark 2.4 ii).* For Proposition 2.6 we use Lemma A.3 with $\beta = \beta^*(p) - 1$ and $\beta^*(p) := 1 + \left(\frac{N}{2p} - \frac{N}{2r}\right)_-$ to get

$$\|Q_V(v) - Q_V(w)\|_{H_p^{\left(\frac{N}{p}-\frac{N}{r}\right)_-}} \le c\|V\|_{L^r}\|v-w\|_{H_p^{2\alpha}},$$

with α strictly less and arbitrarily close to $\beta^*(p)$. Letting now $\gamma = \frac{1}{2} + \left(\frac{N}{2p} - \frac{N}{2r}\right)_- > 0$ and viewing 2α as a sum $1 + 2\varepsilon$ if $\beta^*(p) > \frac{1}{2}$ or, alternatively, using the embedding $H^{1+\varepsilon}_p \hookrightarrow H^{2\alpha}_p$ if $\beta^*(p) \le \frac{1}{2}$ we obtain

$$\|Q_V(v) - Q_V(w)\|_{H_p^{-(2-k-2\gamma)}} \le c\|v - w\|_{H_p^{k+2\varepsilon}} \tag{3.19}$$

with $k = 1$ and ε strictly less and arbitrarily close to $\frac{1}{2} + \left(\frac{N}{2p} - \frac{N}{2r}\right)_-$.

On the other hand, applying Lemma A.3 with $\beta = \beta^*(p) - 1 - \delta$ we get

$$\|Q_V(v) - Q_V(w)\|_{H_p^{2((\frac{N}{2p}-\frac{N}{2r})_- - \delta)}} \le c\|V\|_{L^r(\Omega)}\|v - w\|_{H^{2\alpha}_p},$$

where α is strictly less and arbitrarily close to $\beta^*(p) - \delta = 1 - \delta + \left(\frac{N}{2p} - \frac{N}{2r}\right)_- > 0$. Viewing now α as ε and letting $\gamma = 1 - \delta + \left(\frac{N}{2p} - \frac{N}{2r}\right)_-$ we obtain (3.19) with $k = 0$, which completes the proof of Proposition 2.6.

Concerning Remark 2.4 ii) we apply Lemma A.3 with $p = 2$, $\beta = -\frac{1}{2} \in (-\beta^*(2), \beta^*(2)-1]$, $\alpha < 1 + \beta = \frac{1}{2}$ and use interpolation inequality (see [20, §4.3.1]) to get

$$\|Q_V(v) - Q_V(w)\|_{H^{-1}} \le c\|V\|_{L^r(\Omega)}\|v - w\|_{H^{2\alpha}} \le \tilde{c}\|v - w\|^{\frac{1}{2}-\alpha}_{H^{-1}}\|v - w\|^{\frac{1}{2}+\alpha}_{H^1},$$

which proves k-condition for $k = 1$.

3.3.3. *Sample nonlocal nonlinearity.* For $f(u) = -u\left|\int_\Omega A_p^{-1}u\right|$ we have

$$\begin{aligned}
\|\psi\left|\int_\Omega A_p^{-1}\psi\right| - \phi\left|\int_\Omega A_p^{-1}\phi\right|\|_{L^p} &\le \|\psi - \phi\|_{L^p}\left|\int_\Omega A_p^{-1}\psi\right| + \|\phi\|_{L^p}\left|\int_\Omega (A_p^{-1}\psi - A_p^{-1}\phi)\right| \\
&\le |\Omega|^{\frac{1}{p'}}\|\psi - \phi\|_{L^p}\|A^{-1}\psi\|_{L^p} + |\Omega|^{\frac{1}{p'}}\|\phi\|_{L^p}\|A_p^{-1}(\psi - \phi)\|_{L^p} \\
&\le c\|\psi - \phi\|_{L^p}(\|\phi\|_{L^p} + \|\psi\|_{L^p}),\ \ \phi, \psi \in L^p,
\end{aligned}$$

which ensures that hypothesis $\mathcal{H}^0_p$ holds with $\rho = 2$ and any $\varepsilon \in (0, \frac{1}{2})$, $\gamma \in (\epsilon\rho, 1)$.

On the other hand we also have

$$\begin{aligned}
\|\psi\left|\int_\Omega A_2^{-1}\psi\right| - \phi\left|\int_\Omega A_2^{-1}\phi\right|\|_{H^{-2}} &\le \|\psi - \phi\|_{H^{-2}}\left|\int_\Omega A_2^{-1}\psi\right| + \|\phi\|_{H^{-2}}\left|\int_\Omega (A_2^{-1}\psi - A_2^{-1}\phi)\right| \\
&\le \|\psi - \phi\|_{H^{-2}}|\Omega|^{\frac{1}{2}}\|A_2^{-1}\psi\|_{L^2} + \|\phi\|_{H^{-2}}|\Omega|^{\frac{1}{2}}\|A_2^{-1}(\psi - \phi)\|_{L^2} \\
&\le \|\psi - \phi\|_{H^{-2}}|\Omega|^{\frac{1}{2}}\|\psi\|_{\dot{H}^{-2}} + \|\phi\|_{\dot{H}^{-2}}|\Omega|^{\frac{1}{2}}\|\psi - \phi\|_{\dot{H}^{-2}} \\
&\le c\|\psi - \phi\|_{H^{-2}}(\|\psi\|_{H^{-2}} + \|\phi\|_{H^{-2}}),
\end{aligned}$$

which proves validity of k-condition with $k = 0$.

3.3.4. *A critically growing map satisfying assumptions of Theorem 2.2 and* 1*-condition.* We exhibit here properties of the map f_h defined in Example 2.3.

First note that condition (2.12) becomes straightforward as $f(x, u)\bar{u} = -a(x)|u|h(|u|)$ and a is real. Since a, h are nonnegative, (2.16)-(2.17) hold even with $C = D = 0$.

Next, as in [6, p. 60], we write

$$|z_1||z_2|(f_h(z_1) - f_h(z_2)) = z_1|z_2|(h(|z_1|) - h(|z_2|)) + (z_1(|z_2| - |z_1|) + |z_1|(z_1 - z_2))(h(|z_2|) - h(0))$$

and using (3.17) we get for any $\zeta > 0$ and some $C_\zeta > 0$ that

$$|z_1||z_2||f_h(z_1) - f_h(z_2)| \le 2|z_1||z_2||z_1 - z_2| \left(C_\zeta + \zeta |z_1|^{\rho_c(1,2)-1} + \zeta |z_2|^{\rho_c(1,2)-1} \right), \ z_1, z_2 \in \mathbb{C}.$$

Consequently, we have

$$\begin{aligned} &\forall_{\zeta>0} \ \exists_{C_\zeta>0} \ \forall_{z_1,z_2\in\mathbb{C}} \ |f(z_1) - f(z_2)| \\ &\qquad\qquad \le c|z_1 - z_2| \left(C_\zeta + \zeta |z_1|^{\rho_c(1,2)-1} + \zeta |z_2|^{\rho_c(1,2)-1} \right), \end{aligned} \tag{3.20}$$

where $c = 2\|a\|_{L^\infty}$, $(k,p) = (1,2)$, $\rho_c(1,2) = \frac{N+2}{N-2}$, $N \ge 3$, and repeating the proof of Proposition 2.4 we conclude that hypothesis $\mathcal{H}_2^1$ holds with $\gamma = \rho_c(1,2)\varepsilon$ and arbitrarily small $\zeta > 0$.

We now define $\mathrm{H}(x,s) = -a(x) \int_0^s h(s)ds$ and consider a functional

$$F : H^1 \to \mathbb{R}, \quad F(\psi) = \int_\Omega \mathrm{H}(x, |\psi(x)|), \quad \psi \in H^1. \tag{3.21}$$

Note that such F is well defined for $\psi \in H^1$ because, due to (3.17) and boundedness of a, $|\mathrm{H}(x,s)|$ is bounded from above by a multiple of $1 + |s|^{\rho_c(1,2)+1}$ whereas $H^1 \hookrightarrow L^{\rho_c(1,2)+1}$. As in the proof of [6, Proposition 3.2.5 (i)] we obtain that for a.e. $x \in \Omega$

$$\frac{1}{t}|\mathrm{H}(x, |u + tv|) - \mathrm{H}(x, |u|) - tRe(-a(x)f_h(u)\bar{v})| \to 0 \ \text{ as } \ t \to 0^+.$$

Hence, using dominated convergence theorem we infer that

$$\frac{1}{t}|F(u + tv) - F(u) - tRe\int_\Omega (-a(x)f_h(u)\bar{v})| \to 0 \ \text{ as } \ t \to 0^+.$$

This with f as in (2.21) reads that

$$\frac{1}{t}|F(u + tv) - F(u) - t\langle f(u), v\rangle_{\dot{H}^{-1},\dot{H}^1}| \to 0 \ \text{ as } \ t \to 0^+,$$

that is, F in (3.21) is Gâteaux differentiable for each $u \in H^1$ and $F' = f$. Since, due to (3.20), $f \in C(H^1, H^{-1})$ we get (2.14).

Concerning validity of k-condition with $k = 1$ we remark that by (3.20) $|f(x,z)|$ is bounded from above by a multiple of $1 + |s|^{\rho_c(1,2)}$. Hence, recalling that $H^1 \hookrightarrow L^{\frac{2N}{N-2}}$, $L^{\frac{2N}{N+2}} \hookrightarrow H^{-1}$ and $\rho_c(1,2)\frac{2N}{N+2} = \frac{2N}{N-2}$, we infer that f^e takes H^1 into H^{-1}. In thus remain to show (2.26) for which we use that $H^2 \hookrightarrow L^{r'}$ and $L^r \hookrightarrow H^{-2}$ for every $r > 1$, $r \ge \frac{2N}{N+4}$.

We fix $r_0 \in (1, \frac{2N}{N+2})$ and letting $q = \frac{N}{2r_0}$, $q' = \frac{N}{N-2r_0}$ we obtain that

$$\begin{aligned} \|f^e(v) - f^e(w)\|_{H^{-2}} &\le C\||v - w| \left(C_\zeta + \zeta|v|^{\frac{4}{N-2}} + \zeta|w|^{\frac{4}{N-2}} \right) \|_{L^{r_0}} \\ &\le \tilde{C}\|v - w\|_{L^{r_0 q'}} \| \left(C_\zeta + \zeta|v|^{\frac{4}{N-2}} + \zeta|w|^{\frac{4}{N-2}} \right) \|_{L^{r_0 q}} \\ &\le \hat{C}\|v - w\|_{H^{2+\frac{N}{2}-\frac{N}{r_0}}} \left(\|C_\zeta\|_{L^{\frac{N}{2}}} + \zeta\|v\|^{\frac{4}{N-2}}_{L^{\frac{2N}{N-2}}} + \zeta\|w\|^{\frac{4}{N-2}}_{L^{\frac{2N}{N-2}}} \right), \end{aligned}$$

where $2 + \frac{N}{2} - \frac{N}{r_0} \in (0,1)$. Since $H^{2+\frac{N}{2}-\frac{N}{r_0}}$ is an intermediate space between spaces H^{-1}, H^1 and $H^1 \hookrightarrow L^{\frac{2N}{N-2}}$ we thus obtain (2.26) with $k = 1$.

3.4. **Proof of Theorem 2.2.** In the following two lemmas we derive a priori bounds on the solutions of $(2.2)_{\eta\in(0,1]}$ in L^2 and in H^1 respectively.

Lemma 3.1. *The solutions of $(2.2)_{\eta\in(0,1]}$ through $\|u_0\|_{L^2}\leq R$, as long as they exist and fulfill (2.12), (2.16)-(2.17) (or alternatively (2.12), (2.18)-(2.19)), satisfy the a priori estimate*

$$\|u\|_{L^2}\leq M(R,t), \tag{3.22}$$

for some $M(R,t)$ which can be chosen independent of $\eta\in(0,1]$.

Proof: If (2.12), (2.16)-(2.17) hold, then multiplying (2.2) by $\bar u$ we obtain from the real parts of the equation that

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2}\leq -\eta\|\nabla u\|^2_{L^2}+\eta\int_\Omega C(x)|u|^2+\eta\int_\Omega D(x)|u|,\ \eta\in(0,1]. \tag{3.23}$$

The integral $\int_\Omega D(x)|u|$ can now be bounded by

$$\|D\|_{L^s}\|u\|_{L^{s'}}\leq c\|D\|_{L^s}(\|\nabla u\|^2_{L^2}+\|u\|^2_{L^2})^{\frac{1}{2}}$$

and hence for any $\mu>0$ and a certain $c_\mu>0$ we have

$$\int_\Omega D(x)|u|\leq \mu(\|\nabla u\|^2_{L^2}+\|u\|^2_{L^2})+c_\mu\|D\|^2_{L^s}. \tag{3.24}$$

Combining (A.5), (3.23) and (3.24) we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2}\leq -\eta(\omega(\mu)-\mu)\|u\|^2_{L^2}+\eta c_\mu\|D\|^2_{L^s},\ \eta\in(0,1],\ \mu\in(0,1). \tag{3.25}$$

Estimating the right hand side of (3.25) by $|\omega(\mu)-\mu|\|u\|^2_{L^2}+c_\mu\|D\|^2_{L^2}$ we get (3.22).

Alternatively, if (2.12), (2.18)-(2.19) are assumed, we obtain (3.22) from

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2}\leq -\eta\|\nabla u\|^2_{L^2}+\eta\int_\Omega C(x)|u|^2+\eta\int_\Omega D(x)\leq -\eta\omega_0\|u\|^2_{L^2}+\eta\|D\|_{L^1}$$

where ω_0 is as in Lemma A.1. □

Lemma 3.2. *The solutions of $(2.2)_{\eta\in(0,1]}$ through $u_0\in\dot H^1$, as long as they exist and fulfil (2.12), (2.14), (2.16)-(2.17) satisfy the a priori estimate*

$$\|u\|_{H^1}\leq E(u_0)+K(\|u_0\|_{L^2},t), \tag{3.26}$$

for some K independent of $\eta\in(0,1]$.

Proof: We multiply $(2.2)_{\eta\in(0,1]}$ by $-e^{\mp i\theta_\eta}\bar u_t$, integrate over Ω and use (2.14) to obtain from the real parts of the equation that

$$\frac{d}{dt}\Big(\frac{1}{2}\int_\Omega|\nabla u|^2-F(u)\Big)=-\eta\|u_t\|^2_{L^2}\leq 0,\ \eta\in(0,1].$$

This yields

$$E(u)=\frac{1}{2}\int_\Omega|\nabla u|^2-F(u)\leq E(u_0) \tag{3.27}$$

whereas using (2.14) and (3.24) with $\mu = \frac{\nu}{4}$ we have

$$
\begin{aligned}
F(u(t)) &= F(0) + \int_0^1 \frac{d}{d\omega}\left(F(\omega u(t))\right) d\omega = F(0) + \int_0^1 \langle F'(\omega u(t)), u(t)\rangle_{\dot H^{-1},\dot H^1} d\omega \\
&= F(0) + \int_0^1 \frac{1}{\omega}\Big(\int_\Omega (f(x,\omega u(t))\overline{\omega u(t)} dx\Big) d\omega \\
&\le F(0) + \int_0^1 \frac{1}{\omega}\Big(\int_\Omega (C(x)|\omega u(t)|^2 + D(x)|\omega u(t)|) dx\Big) d\omega \\
&\le F(0) + (\int_0^1 \omega d\omega)(\int_\Omega C(x)|u(t)|^2 dx) + \frac{\nu}{4}(\|\nabla u\|^2_{L^2} + \|u\|^2_{L^2}) + c_{\nu/4}\|D\|^2_{L^s} \\
&\le F(0) + \frac{1}{2}\int_\Omega C(x)|u(t)|^2 + \frac{\nu}{4}(\|\nabla u\|^2_{L^2} + \|u\|^2_{L^2}) + c_{\nu/4}\|D\|^2_{L^s}.
\end{aligned}
\tag{3.28}
$$

From (3.27), (3.28) we get

$$
\frac{\nu}{4}\int_\Omega |\nabla u|^2 + \frac{1}{2}\left(\int_\Omega (1-\nu)|\nabla u|^2 - C(x)|u|^2\right) - F(0) - c_{\nu/4}\|D\|^2_{L^s} - \frac{\nu}{4}\|u\|^2_{L^2} \le E(u_0)
$$

and using Lemmas A.1, A.2 we obtain

$$
\frac{\nu}{4}\int_\Omega |\nabla u|^2 \le E(u_0) + F(0) + c_{\nu/4}\|D\|^2_{L^s} + \big(\frac{\nu}{4} - \frac{1}{2}\omega(\nu)\big)\|u(t)\|^2_{L^2} \tag{3.29}
$$

where $\nu \in (0,1)$ and $\omega(\nu) \in \mathbb{R}$ satisfies (A.4). The result now follows from (3.22), (3.29). □

Remark 3.1. *In contrary to Lemma 3.1 the proof of Lemma 3.2 indicates that to derive H^1 bound on the solutions condition (2.18) is not as suitable as (2.16).*

Due to the blow up alternative (2.7) and the estimates of Lemmas 3.1, 3.2, Theorem 2.2 is thus proved.

Remark 3.2. *i) Recall that we do not assume in general that C in (2.16) or (2.18) is such that the solutions of the linear problem (3.30) in L^2,*

$$
\begin{cases}
u_t = \Delta u + C(x)u, \ x \in \Omega, \ t > 0, \\
u_{|\partial\Omega} = 0, \ t > 0, \quad u(0,x) = u_0(x), \ x \in \Omega,
\end{cases}
\tag{3.30}
$$

are asymptotically decaying. In particular we do note assume that ω_0 in (A.1) is positive (see [9, Corollary 2.10]*).*
ii) On the other hand, if one assumes that (A.1) is satisfied with $\omega_0 > 0$ then formula (3.25) for some $\mu = \mu_0$ (chosen via Lemma A.2 such that $\omega(\mu_0) - \mu_0 > \frac{\omega_0}{2}$) will read

$$
\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2} \le -\eta\frac{\omega_0}{2}\|u\|^2_{L^2} + \eta c_{\mu_0}\|D\|^2_{L^2}, \ \eta \in (0,1]. \tag{3.31}
$$

In particular, given $\eta \in (0,1]$ we will have the a priori bound of the form

$$
\begin{aligned}
\|u\|^2_{L^2} &\le \|u_0\|^2_{L^2} e^{-\eta\omega_0 t} + \frac{2c_{\mu_0}}{\omega_0}\|D\|^2_{L^2}(1 - e^{-\eta\omega_0 t}) \\
&\le \|u_0\|^2_{L^2} + \frac{2c_{\mu_0}}{\omega_0}\|D\|^2_{L^2}.
\end{aligned}
\tag{3.32}
$$

Furthermore, (3.29) for some $\nu = \nu_0$ (chosen via Lemma A.2 such that $\frac{\nu_0}{4} - \frac{1}{2}\omega(\nu_0) < 0$) will then imply that

$$\frac{\nu_0}{4}\int_\Omega |\nabla u|^2 \leq E(u_0) + F(0) + c_{\nu_0/4}\|D\|^2_{L^s}. \tag{3.33}$$

4. Solutions of (1.1) with $\eta = 0$

We obtain here solutions of the Dirichlet initial-boundary value problem for the equation (1.3) passing to a limit in a sequence of solutions of approximate problems $(1.1)_{\eta\in(0,1]}$.

4.1. **Proof and extension of Proposition 2.7.** For $\eta \in (0,1]$ we have from (3.1) that $[0,\infty) \subset \rho(-\mathcal{A}^{\pm}_{2,\eta})$. We also observe that $[0,\infty) \subset \rho(-\mathcal{A}^{\pm}_{2,0})$ because $i\mathbb{R} \subset \rho(A_2)$. Using that

$$(\lambda Id + \mathcal{A}^{\pm}_{2,\eta})^{-1} = e^{\mp\theta_\eta i}(\lambda e^{\mp\theta_\eta i} Id + A_2)^{-1}, \quad (\lambda Id + \mathcal{A}^{\pm}_{2,0})^{-1} = \mp i(\mp\lambda i Id + A_2)^{-1}$$

and that the resolvent of the operator $-A_2$ is analytic in the resolvent set (thus continuous with respect to the uniform operator topology) we get (2.23). Note that the semigroups in $L^2(\Omega)$ are all of the same type, which follows from the Lumer-Phillips theorem. Thus, applying the Trotter-Kato approximation theorem (see [16]) we get (2.24).

Recall now from Proposition 2.8 that a closed extension of $-\mathcal{A}^{\pm}_{2,\eta}$, which we denote the same, is an infinitesimal generator of the semigroup of contractions in $\dot{H}^\sigma$ for any $\sigma \in [-2,0]$ and the resolvent set of $-\mathcal{A}^{\pm}_{2,\eta}$ in $\dot{H}^\sigma$ coincides with the one in $L^2(\Omega)$.

Thus, with a similar argument as in the proof of Proposition 2.7 we obtain the following convergence result.

Proposition 4.1. *Given $\sigma \in [-2,0]$, the linear semigroups generated by a closed extension of $-\mathcal{A}^{\pm}_{2,\eta}$ to $\dot{H}^\sigma$ are the semigroups of contractions and converge as $\eta \to 0^+$ to a semigroup of contractions generated in $\dot{H}^\sigma$ by a closed extension of $-\mathcal{A}^{\pm}_{2,0}$. Namely, given $\sigma \in [-2,0]$, $u_0 \in \dot{H}^\sigma$ and a bounded time interval J we have that*

$$e^{-\mathcal{A}^{\pm}_{2,\eta}t}u_0 \overset{\dot{H}^\sigma}{\longrightarrow} e^{-\mathcal{A}^{\pm}_{2,0}t}u_0 \quad \text{as} \quad \eta \to 0 \text{ uniformly for } t \in J. \tag{4.1}$$

4.2. **Proof of Theorem 2.3.** Suppose that $k \in \{0,1\}$ is given, $\eta_n \to 0$ and let $u^{\eta_n}_{\pm}$ be the solution of $(2.2)_{\eta=\eta_n}$ through $u_0 \in \dot{H}^k$ as in Theorem 2.1. Rewriting (2.2) as

$$A_2^{-1}u_t = -e^{\pm i\theta_\eta}u + e^{\pm i\theta_\eta}A_2^{-1}f^e(u),$$

we infer from k-condition and Lemmas 3.1, 3.2 that the sequence $\{u^{\eta_n}_{\pm}\}$ is bounded in $W_T = \{\chi \in L^\infty((0,T), Y_2^{1+\frac{k}{2}}) : \dot{\chi} \in L^\infty((0,T), Y_2^{\frac{k}{2}})\}$ for each $T > 0$. Following regularity properties of approximate solutions expressed in (3.11) and applying Arzela-Ascoli theorem (see [12, §7.5] (also [12]) we then have, choosing a subsequence which is still denoted the same, that

$$u^{\eta_n}_{\pm} \to u_{\pm} \text{ in } Y_2^{\frac{k}{2}} \text{ uniformly on } [0,T] \tag{4.2}$$

and

$$u_{\pm} \in C([0,T], Y_2^{\frac{k}{2}}). \tag{4.3}$$

Using properties of weak limits we also infer that

$$u_\pm \in L^\infty((0,T), Y_2^{1+\frac{k}{2}}). \tag{4.4}$$

Furthermore, $u_\pm$ being continuous in $Y_2^{\frac{k}{2}}$ and bounded in $Y_2^{1+\frac{k}{2}}$ have a property that

$$u_\pm \text{ are weakly continuous in } Y_2^{1+\frac{k}{2}} \tag{4.5}$$

(see [19, Lemma II.3.3]), which together with (4.4) imply that

$$\sup_{t\in(0,T)} \|u_\pm(t)\|_{Y_2^{1+\frac{k}{2}}} < \infty. \tag{4.6}$$

With the aid of functions $u_\pm$ we now define u as

$$u(t) = u_+(t) \ \text{ for } t \geq 0 \qquad \text{and} \qquad u(t) = u_-(-t) \ \text{ for } \ t < 0. \tag{4.7}$$

We prove that u satisfies (2.28), for which it suffices to show that $u_\pm$ satisfy

$$u_\pm(t) = e^{-\mathcal{A}^\pm_{2,0}t}u_0 \pm i\int_0^t e^{-\mathcal{A}^\pm_{2,0}(t-s)} f^e(u_\pm(s))ds, \ t\in[0,T]. \tag{4.8}$$

Note that approximate solutions $u_\pm^{\eta_n}$ satisfy variation of constants formula associated with (3.2),

$$u_\pm^{\eta_n}(t) = e^{-\mathcal{A}^\pm_{\eta_n}t}u_0 + \int_0^t e^{-\mathcal{A}^\pm_{\eta_n}(t-s)} f^e_{\eta,\pm}(u_\pm^{\eta_n}(s))ds \tag{4.9}$$

and that, due to Corollary 4.1,

$$e^{-\mathcal{A}^\pm_{\eta_n}t}u_0 \overset{\eta_n\to 0}{\longrightarrow} e^{-\mathcal{A}^\pm_0 t}u_0 \ \text{ in } Y_2^1. \tag{4.10}$$

It thus suffices to ensure that

$$\int_0^t e^{-\mathcal{A}^\pm_{\eta_n}(t-s)} f^e_{\eta,\pm}(u_\pm^{\eta_n}(s))ds \overset{Y_2}{\to} \pm i\int_0^t e^{-\mathcal{A}^\pm_0(t-s)} f^e(u_\pm(s))ds, \ t\in[0,T]. \tag{4.11}$$

Using (4.2), (4.6) and k-condition we have

$$\|f^e_{\eta,\pm}(u_\pm^{\eta_n}(s)) \mp i f^e(u_\pm(s))\|_{Y_2} \to 0 \text{ for } s\in[0,\tau],$$

which ensures by Lebesgue's dominated convergence theorem that

$$\begin{aligned} &\|\int_0^t e^{-\mathcal{A}^\pm_{2,\eta_n}(t-s)}(f^e_{\eta,\pm}(u^{\eta_n}(s)) \mp i f^e(u_\pm(s)))\|_{Y_2} ds \\ &\leq \int_0^t \|(f^e_{\eta,\pm}(u^{\eta_n}(s)) \mp i f^e(u_\pm(s)))\|_{Y_2} ds \to 0. \end{aligned} \tag{4.12}$$

On the other hand, by Corollary 4.1 we have that

$$(e^{-\mathcal{A}^\pm_{2,\eta_n}(t-s)} - e^{-\mathcal{A}^\pm_{2,0}(t-s)})f^e(u_\pm(s)) \overset{Y_2}{\longrightarrow} 0 \ \text{ for } \ s\in[0,T]$$

which gives

$$\int_0^t \|(e^{-\mathcal{A}^\pm_{2,\eta_n}(t-s)} - e^{-\mathcal{A}^\pm_{2,0}(t-s)})f^e(u_\pm(s))\|_{Y_2} ds \to 0. \tag{4.13}$$

Note that all terms under the integrals in (4.12)-(4.13) are bounded in $Y_2^{\frac{k}{2}} \hookrightarrow Y_2$ uniformly for $n \in \mathbb{N}$ and $s \in [0,T]$. This is because the linear semigroups are the semigroups of contractions, approximate solutions $u_{\pm}^{\eta_n}$ and the limit function u are bounded in $Y_2^{1+\frac{k}{2}}$ uniformly for $t \in [0,T]$ and $n \in \mathbb{N}$ and f^e takes bounded sets of $Y_2^{1+\frac{k}{2}}$ into bounded sets of $Y_2^{\frac{k}{2}}$. Condition (4.11) is thus proved.

Due to (4.6) and k-condition $\{f^e(u_{\pm}(t_k))\}$ is a bounded sequence in $Y_2^{\frac{k}{2}}$ and hence weakly converges in $Y_2^{\frac{k}{2}}$. On the other hand, due to (4.3), (4.4) and k-condition, $f^e(u_{\pm}(t_k)) \to f^e(u_{\pm}(t_0))$ in Y_2. Consequently, $f^e(u_{\pm}(t_k)) \to f^e(u_{\pm}(t_0))$ weakly in $Y_2^{\frac{k}{2}}$, that is, $f^e(u_{\pm})$ is weakly continuous from $[0,T]$ into $Y_2^{\frac{k}{2}}$.

Thanks to k-condition and (4.2), (4.4), $e^{-\mathcal{A}^{\pm}_{2,0}(t-s)} f^e(u_{\pm}(s))$ is a continuous function of $s \in [0,T]$ with values in $\dot{H}^{-2}$, bounded in $\dot{H}^{k-2}$. Hence, it is weakly continuous with values in $\dot{H}^{k-2}$ and thus measurable in $\dot{H}^{k-2}$ (see [7, Corollary 1.4.8]). Consequently, due to Bochner's theorem (see [7, Corollary 1.4.14], the integral in (4.8) converges in $\dot{H}^{k-2}$.

Remark 4.1. *i) Since $Y_2^{\sigma+\frac{k}{2}}$ with $\sigma \in (0,1)$ is an intermediate space between spaces $Y_2^{\frac{k}{2}}$, $Y_2^{1+\frac{k}{2}}$ then using (4.2), boundedness of $\{u_{\pm}^{\eta_n}\}$ in $Y_2^{1+\frac{k}{2}}$ and interpolation inequality we get*

$$u_{\pm}^{\eta_n} \overset{Y_2^{\frac{k}{2}+\sigma}}{\to} u_{\pm}(t) \quad \text{for } \sigma \in (0,1) \text{ uniformly on } [0,T]. \tag{4.14}$$

ii) From (4.3), (4.6) and interpolation inequality we have $u_{\pm} \in C([0,T], Y_2^{\frac{k}{2}+\sigma})$ for each $\sigma < k$.
iii) Rewriting (4.8) as $u_{\pm}(t) - e^{-\mathcal{A}^{\pm}_{2,0}t}u_0 = \pm i \int_0^t e^{-\mathcal{A}^{\pm}_{2,0}(t-s)} f^e(u_{\pm}(s))ds,\ t \in [0,T]$ and using (4.10), (4.11), (4.14), we conclude that

$$\int_0^t e^{-\mathcal{A}^{\pm}_{\eta_n}(t-s)} f^e_{\eta,\pm}(u_{\pm}^{\eta_n}(s))ds \overset{Y_2^1}{\to} \pm i \int_0^t e^{-\mathcal{A}^{\pm}_{0}(t-s)} f^e(u_{\pm}(s))ds,\ t \in [0,T]. \tag{4.15}$$

4.3. **Proof of Proposition 2.9.** If the solutions of the linear problem (3.30) in $L^2(\Omega)$ are asymptotically decaying then, due to (3.32), (3.33) we will have in the proof of Theorem 2.3 that $u_{\pm} \in L^{\infty}((0,\infty), Y_2^{1+\frac{k}{2}})$. This and (4.7) will then lead to (2.30).

Remark 4.2. *To satisfy the assumption of Proposition 2.9 for $f(x,u) = -a(x)|u|^{\rho-1}u + b(x)|u|^{\tilde{\rho}-1}u + V(x)u$ with $\rho > \tilde{\rho} > 1$ and a, b, V real valued functions such that $a \geq 0$ and $V \in L^r(\Omega)$ for some $r > \frac{N}{2}$, $r \geq 1$, one can follow a decomposition of the potential in* [10, p. 3528] *writing*

$$V = V_1 + V_2$$

and assuming that the bottom spectrum of $-\Delta - V_1(\cdot)I$ in L^2 is strictly positive. The structure condition (2.16) will then hold with $C = V_1$ and D equal to a multiple of $|b|^{\frac{\rho}{\rho-\tilde{\rho}}} a^{-\frac{\tilde{\rho}}{(\rho-\tilde{\rho})}} + |b|^{\frac{\rho}{\rho-1}} a^{-\frac{1}{(\rho-1)}}$ as $b|u|^{\tilde{\rho}+1} = a^{1-\theta}|u|^{\tilde{\rho}+1-\theta} \frac{b}{a^{1-\theta}}|u|^{\theta}$ with $\theta = \frac{\rho-\tilde{\rho}}{\rho}$, $V_2|u|^2 = a^{1-\mu}|u|^{2-\mu} \frac{V}{a^{1-\mu}}|u|^{\mu}$ with $\mu = \frac{\rho-1}{\rho}$ so that Young's inequality gives the result provided that $|b|^{\frac{\rho}{\rho-\tilde{\rho}}} a^{-\frac{\tilde{\rho}}{(\rho-\tilde{\rho})}} + |b|^{\frac{\rho}{\rho-1}} a^{-\frac{1}{(\rho-1)}} \in L^s$ for some $s \geq \max\{\frac{2N}{N+2}, 1\}$.

4.4. **Proof of Proposition 2.10.** We first prove that the solution constructed for $k=1$ in the proof of Theorem 2.3 will enjoy the conservation of charge property (2.13). Indeed, considering the approximate solutions $u_{\pm}^{\eta_n}$ therein, multiplying $(2.2)_{\eta=\eta_n}$ by $\overline{u_{\pm}^{\eta_n}}$, using (2.12) and taking into account the real parts the equation, we get

$$\frac{1}{2}\frac{d}{dt}\|u_{\pm}^{\eta_n}\|_{L^2}^2+\eta_n\|\nabla u_{\pm}^{\eta_n}\|_{L^2}^2=\eta_n\int_\Omega f(x,u_{\pm}^{\eta_n})\overline{u_{\pm}^{\eta_n}}.$$

After integration with respect to time variable we then have

$$\|u_{\pm}^{\eta_n}\|_{L^2}^2-\|u_0\|_{L^2}^2=2\eta_n\int_0^t\int_\Omega f(x,u_{\pm}^{\eta_n})\overline{u_{\pm}^{\eta_n}}-2\eta_n\int_0^t\|\nabla u_{\pm}^{\eta_n}\|_{L^2}^2 \tag{4.16}$$

where for arbitrarily fixed positive time the right hand side of (4.16) tends to 0 as $\eta_n\to 0^+$ because $\int_0^t\int_\Omega f(x,u_{\pm}^{\eta_n})\overline{u_{\pm}^{\eta_n}}$ and $\int_0^t\|\nabla u_{\pm}^{\eta_n}\|_{L^2}^2$ are bounded uniformly with respect to η_n due to Lemmas 3.1, 3.2 and k-condition. We remark here that once $u_{\pm}^{\eta_n}$ is bounded in H^1 uniformly for the parameter η_n and for t in bounded time intervals, then k-condition ensures that $\|f^e(u_{\pm}^{\eta_n})\|_{H^{-1}}$ is bounded uniformly for η_n and t, which in turn implies such boundedness of $\langle f(u_{\pm}^{\eta_n}),u_{\pm}^{\eta_n}\rangle_{\dot H^{-1},\dot H^1}$. Note that, due to (4.14),

$$u_{\pm}^{\eta_n}\overset{H^s}{\to}u_{\pm}\ \text{ for } s<1 \text{ uniformly on } [0,T]. \tag{4.17}$$

In particular, $u_{\pm}^{\eta_n}\to u_{\pm}$ uniformly on $[0,T]$ in $L^2(\Omega)$ and passing to the limit in (4.16) we conclude via (4.7) that

$$\|u\|_{L^2}^2-\|u_0\|_{L^2}^2=0.$$

Concerning the energy $E(u)=\frac{1}{2}\int_\Omega|\nabla u|^2-F(u)$ note that due (3.27) we have

$$E(u_{\pm}^{\eta_n})=\frac{1}{2}\int_\Omega|\nabla u_{\pm}^{\eta_n}|^2-F(u_{\pm}^{\eta_n})\le E(u_0). \tag{4.18}$$

Now, if F in (2.14) satisfies (2.31), using (4.6), (4.17) and (2.31) we obtain that

$$F(u_{\pm}^{\eta_n})\to F(u_{\pm})$$

for any positive time. Combining this with boundedness of approximate solutions in H^1, weak lower semicontinuity of the norm and (4.7) we get (2.32) from (4.18).

We finally remark that if (2.33) is assumed then the uniqueness result follows by a standard application of Gronwall's lemma (see [6, Corollary 3.3.11] for details).

Remark 4.3. *i) Although (2.31) requires f to be subcritical, it does not necessitate additional restrictions on the growth of typical nonlinearities as in Example 2.1. Indeed, (2.31) holds for $f(x,u)=-a(x)|u|^{\rho-1}u+b(x)|u|^{\tilde\rho-1}u+V(x)u$ as in (2.20) assuming that $a,b\in L^\infty(\mathbb{R}^N)$, $V\in L^r(\mathbb{R}^N)$ are real valued functions, $r>\frac{N}{2}$, $r\ge 1$, $1<\tilde\rho<\rho$ and $\rho<\frac{N+2}{N-2}$ when $N\ge 3$ (see* [6, Lemma 3.3.7]*).*
ii) On the other hand note that (2.33) does cause the growth exponent ρ to satisfy a more restrictive condition than $\rho<\frac{N+2}{N-2}$, that is, than the one associated in Example 2.1 ii) with the case $k=1$, $p=2$, $N\ge 3$.

Appendix A. Auxiliary results

We include here some useful results, which we adapt from [9].

Lemma A.1. *If $C \in L^r$ with some $r > \frac{N}{2}$ and $r \geq 1$ then there exists $\omega_0 \in \mathbb{R}$ such that*

$$\int_\Omega (|\nabla\phi|^2 - C(x)\phi^2) \geq \omega_0 \|\phi\|^2_{L^2}, \quad \phi \in H^1. \tag{A.1}$$

Proof: If r, r' are Hölder's conjugate exponents then $\frac{N}{2} - \frac{N}{2r'} < 1$ and for $s \in (\frac{N}{2} - \frac{N}{2r'}, 1)$ we have

$$\left|\int_\Omega C(x)|\phi|^2\right| \leq \|C\|_{L^r}\|\phi\|^2_{L^{2r'}} \leq \|C\|_{L^r}\|\phi\|^2_{H^s}, \tag{A.2}$$

where, via [20, §4.3.1 and §2.4.2(11)], $\|\phi\|_{H^s} \leq c\|\phi\|^s_{H^1}\|\phi\|^{1-s}_{L^2}$ for $\phi \in H^1$.

Since $\|\phi\|_{H^1} = (\|\nabla\phi\|_{L^2} + \|\phi\|_{L^2})^{\frac{1}{2}}$, given $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that

$$\left|\int_\Omega C(x)|\phi|^2\right| \leq \varepsilon\|\nabla\phi\|^2_{L^2} + (c_\varepsilon + \varepsilon)\|\phi\|^2_{L^2}, \ \phi \in H^1. \tag{A.3}$$

Choosing $\epsilon = 1$ we get (A.1) with $\omega_0 = -(c_1 + 1)$. □

Lemma A.2. *If $C \in L^r$, $r > \frac{N}{2}$, $r \geq 1$ and (A.1) holds for some $\omega_0 \in \mathbb{R}$ then there is a continuous decreasing real valued function $\omega(\nu)$ defined in the interval $[0, 1)$ such that*

$$\lim_{\nu \to 0^+} \omega(\nu) = \omega(0) = \omega_0 \tag{A.4}$$

and for any $\nu \in [0, 1)$ we have

$$\int_\Omega \big((1-\nu)|\nabla\phi|^2 - C(x)\phi^2\big) \geq \omega(\nu)\int_\Omega \phi^2, \quad \phi \in H^1. \tag{A.5}$$

Proof: We write

$$\int_\Omega \big((1-\nu)|\nabla\phi|^2 - C(x)\phi^2\big) = (1-\nu)\int_\Omega |\nabla\phi|^2 - (1-2\nu)\int_\Omega C(x)|\phi|^2 - 2\nu\int_\Omega C(x)|\phi|^2$$

and using (A.3) with $\varepsilon = 1/2$ to estimate the last term in the above equality we get

$$\int_\Omega \big((1-\nu)|\nabla\phi|^2 - C(x)\phi^2\big) \geq (1-2\nu)\int_\Omega \big(|\nabla\phi|^2 - C(x)\phi^2\big) - \nu(2c_{1/2}+1)\|\phi\|^2_{L^2}. \tag{A.6}$$

From (A.6) and (A.1) we obtain (A.5) with $\omega(\nu) = (1-2\nu)\omega_0 - \nu(2c_{1/2}+1)$. □

Lemma A.3. *Suppose that $V \in L^r$ with $r > \frac{N}{2}$, $r \geq 1$, $p \in (1, \infty)$ and let β be any number from the interval $I(p) = (-\beta^*(p'), \beta^*(p) - 1] \subset (-1, 0]$, where*

$$\beta^*(p) := 1 + \left(\frac{N}{2p} - \frac{N}{2r}\right)_- \tag{A.7}$$

and $a_- = \min\{a, 0\}$ denotes the negative part of $a \in \mathbb{R}$.

Then, there is a certain interval $(\alpha_0, 1+\beta)$ such that for any $\alpha \in (\alpha_0, 1+\beta)$, Q_V in (2.11) satisfies

$$Q_V \in \mathcal{L}(H^{2\alpha}_p, H^{2\beta}_p) \quad \text{and} \quad \|Q_V\|_{\mathcal{L}(H^{2\alpha}_p, H^{2\beta}_p)} \leq c\|V\|_{L^r}.$$

Proof: We observe that $\|C\phi\|_{H^{2\beta}_p} = \sup_{\|\psi\|_{H^{-2\beta}_{p'}}=1} |\int_\Omega C\phi\psi|$ and estimate as follows

$$|\int_\Omega C\phi\psi| \le \|C\|_{L^{p_1}}\|\phi\|_{L^{p_2}}\|\psi\|_{L^{p_3}} \le c\|C\|_{L^r}\|\phi\|_{H^{2\alpha}_p}\|\psi\|_{H^{-2\beta}_{p'}},$$

where $p_1 = r$ and parameters p_2, p_3, α will be chosen such that

$$p_2, p_3 \in [r', \infty], \quad \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{r'}, \quad 0 < \alpha < \beta + 1, \quad -1 \le \beta \le 0 \tag{A.8}$$

and

$$\alpha - \frac{N}{2p} \ge -\frac{N}{2p_2}, \quad -\beta - \frac{N}{2p'} \ge -\frac{N}{2p_3}. \tag{A.9}$$

From (A.9) we have $\alpha - \beta - \frac{N}{2} \ge -\frac{N}{2r'}$, which allows us to consider

$$\alpha \in [\beta + \frac{N}{2r}, \beta + 1). \tag{A.10}$$

On the other hand from (A.8), (A.10) we infer that $\frac{N}{2p_3} = \frac{N}{2r'} - \frac{N}{2p_2}$ and $\alpha = \theta(\frac{N}{2r} - 1) + \beta + 1$ for some $\theta \in (0, 1]$, so that (A.9) now reads

$$\theta(\frac{N}{2r} - 1) + \beta + 1 - \frac{N}{2p} \ge -\frac{N}{2p_2}, \quad -\beta - \frac{N}{2p'} \ge \frac{N}{2p_2} - \frac{N}{2r'} \tag{A.11}$$

or, equivalently,

$$-\frac{N}{2p_2} + \frac{N}{2p} - \frac{N}{2r} \ge \beta \ge -\frac{N}{2p_2} + \theta(1 - \frac{N}{2r}) - 1 + \frac{N}{2p}. \tag{A.12}$$

Varying p_2 in $[r', \infty]$ and θ in $(0, 1]$ on the left hand side $L(p_2)$ of (A.12) we can achieve no more than $\frac{N}{2p} - \frac{N}{2r}$, which is the case when $p_2 = \infty$. On the right hand side $R(p_2, \theta)$ in (A.12) we can go down to infimum value $-\frac{N}{2r'} - 1 + \frac{N}{2p} = -1 - \frac{N}{2p'} + \frac{N}{2r}$ which will be achieved for $p_2 = r'$, $\theta = 0$. Note that, by assumption, $R(p_2, \theta)$ is increasing with respect to each variable, $L(p_2) > R(p_2, 0)$ and $L(p_2) = R(p_2, 1)$ for every $p_2 \in [r', \infty]$.

Now, given $\beta \in I(p)$, we have either $L(\infty) \ge \beta \ge L(r')$ (that is $\beta = L(p_2)$ for some $p_2 \in [r', \infty]$) or $L(r') > \beta > R(r', 0)$ (that is $\beta = R(r', \theta)$ for some $\theta \in (0, 1)$) so that in either case $L(p_2) \ge \beta > R(p_2, \theta)$ for some $p_2 \in [r', \infty]$, $\theta \in (0, 1)$. Consequently, (A.12) can be satisfied for some $p_2 \in [r', \infty]$, $\theta \in (0, 1]$ and hence also for some $p_2 \in [r', \infty]$ and each sufficiently small $\theta \in (0, 1]$. This allows us to conclude that, whenever $\beta \in I(p)$, (A.9) can be satisfied with some $p_2, p_3 \in [r', \infty]$ satisfying $\frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{r'}$ and with any $\alpha < \beta + 1$ close enough to $\beta + 1$.

We remark that it is not possible to have both $\beta \notin I(p)$ and (A.8)-(A.9) as, taking into account that $\theta \in (0, 1]$, any such β will then lie outside the range of the left/right hand sides of (A.12). □

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(A. N. Carvalho) Instituto de Ciências Matemáticas e de Computaçao, Universidade de São Paulo-Campus de São Carlos, Caixa Postal 668, 13560-970 São Carlos SP, Brazil
E-mail address: `andcarva@icmc.usp.br`

(J. W. Cholewa) Institute of Mathematics, Silesian University, 40-007 Katowice, Poland
E-mail address: `jan.cholewa@us.edu.pl`