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NORMAL FORM OF REVERSIBLE EQUIVARIANT VECTOR FIELDS

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Normal form of reversible equivariant vector fields

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Abstract

We give an alternative method to obtain normal forms of reversible equivariant vector fields. We adapt the classical method using tools from invariant theory to establish formulae that take symmetries into account as a starting point. Normal forms of two classes of non-resonant and resonant cases are presented, both under a $\mathbf{Z}_2-$action with linearization having a $2-$dimensional nilpotent part and a semisimple part with purely imaginary eigenvalues.

Keywords. Normal form, reversibility, symmetry, homological operator.

AMS classification. 37C80, 34C20, 13A50.

1 Introduction

Normal form theory has been developed as a tool for the local study of the qualitative behavior of vector fields. Its importance in bifurcation theory and many other directions has motivated the development of the subject by several authors by many years, since Poincaré [16], Birkhoff [4], Dulac [6], Belitskii [5], Elphick *et al.* [7] and Takens [17]. The classical method consists of performing changes of coordinates around a singular point that are perturbation of the identity, $\xi = I + \xi_k$, where $k \geq 2$ and ξ_k is a homogeneous polynomial of degree k. The aim is to annihilate as many terms of degree k as possible in the original vector field, obtaining a conjugate vector field written in a simpler and more convenient form. The method developed

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by Belitskii [5] reduces this problem to computing the kernel of the so-called homological operator. This operator is defined on the vector space of homogeneous polynomials of degree k and is associated to the adjoint L^t of the linearization L of the original vector field. This computation, in turn, reduces to finding degree-k polynomial solutions of a PDE. Alternatively, Elphick *et al.* [7] give an algebraic method to obtain the normal form developed by Belitskii by choosing nonlinear terms that are equivariant under a one-parameter group $\mathbf{S}$ given by

$$\mathbf{S} = \overline{\{e^{sL^t}, s \in \mathbb{R}\}}. \tag{1}$$

For the details we refer to [11, Chapter XVI].

When the vector field is reversible equivariant under the action of a group Γ, the occurrence of symmetries and reversing symmetries (the elements of Γ) has a distinct effect not only on the existence and nature of repeating singular points, or on the dynamics, etc, but also on the shape of the coordinate changes. More precisely, the normal form inherits the symmetries and reversing symmetries if the changes of coordinates are equivariant under the whole group Γ. Many authors have used normal form theory to study (occurrence of) limit cycles and family of periodic orbits either in purely reversible vector fields or in reversible equivariants (see, for example [9, 12, 14, 15]). They compute the truncated normal form up to degree k by means of the classical method by Belitskii and then the symmetry conditions are imposed as *a posteriori* step. In those cases, sometimes the normal form is truncated in a low degree because of the difficulties in finding solutions of the associated PDE.

In this paper, we adapt the normal form theory by Belitskii [5] and Elphick [7] for the reversible equivariant context. We use algebraic tools from invariant theory. In this process the group $\mathbf{S}$ given in (1) plays a fundamental role: we show that the $\Gamma-$reversible equivariant normal form comes from the description of the reversible equivariant theory of the semidirect product $\mathbf{S} \rtimes \Gamma$. This is our main result and is presented in Theorem 4.2. After this recognition, we use the results of Antoneli *et al.* [1] as a tool for the computations. Let us remark here that those tools may be applied for compact but also for noncompact Lie groups, once the ring of invariants and the module of equivariants by the group under consideration are finitely generated. The contribution of this result in the general theory is a way to produce the normal form of a reversible equivariant vector field by means of an algebraic method, without passing through a search for solutions of a PDE.

We have organized this paper as follows. In Section 2 we briefly introduce the notation and basic concepts about reversible equivariant theory. In Section 3 we present results in invariant theory of the semidirect product of two arbitrary groups from the invariant theory of each group separately. These results are the grounds for normal form theory that is used in Sections 4 and 5. Section 4 is devoted to a short introduction to the basic concepts of normal form theory and, from that, we introduce the reversible equivariant normal form and prove the main results, Theorem 4.1 and Theorem 4.2. Finally, in Section 5 we apply the results of Section 4 to obtain the normal form for two types of vector fields under $\mathbf{Z}_2-$action. We first consider a vector field whose linearization has a $2-$dimensional nilpotent part and n nonresonant purely imaginary eigenvalues. Then, we treat the case where the linearization has a $2-$dimensional nilpotent part and 2 resonant purely imaginary eigenvalues. This last result generalizes some normal forms with resonance presented by Lima and Teixeira in [12]. A complete classification of nonresonant and resonant vector fields under the action of a group generated by two involutions appears in [13].

2 Preliminaries on invariant theory

Let Γ be a compact Lie group acting linearly on a finite-dimensional real vector space V. Consider a homomorphism

$$\sigma : \Gamma \to \mathbf{Z}_2 = \{-1, 1\}, \tag{2}$$

where $\sigma(\gamma) = 1$ if γ is a symmetry and $\sigma(\gamma) = -1$ if γ is a reversing symmetry. We denote by Γ_+ the group of symmetries of Γ. So, under this formalism, $\Gamma_+ = \ker \sigma$ is a normal subgroup of Γ of index 2. Also, $\Gamma = \Gamma_+ \dot{\cup}\ \delta\Gamma_+$ for an arbitrary reversing symmetry $\delta \in \Gamma$. Throughout this work, δ denotes a reversing symmetry whenever it appears in the text.

A polynomial function $f : V \to \mathbb{R}$ is called $\Gamma-$invariant if

$$f(\gamma x) = f(x),\ \forall \gamma \in \Gamma,\ x \in V,$$

and it is called $\Gamma-$anti-invariant if

$$f(\gamma x) = \sigma(\gamma) f(x),\ \forall \gamma \in \Gamma,\ x \in V.$$

We denote by $\mathcal{P}_V(\Gamma)$ the ring of the $\Gamma-$invariant polynomial functions and by $\mathcal{Q}_V(\Gamma)$ the module of the $\Gamma-$anti-invariant polynomial functions over the ring $\mathcal{P}_V(\Gamma)$.

A polynomial mapping $g : V \to V$ is called $\Gamma-$equivariant if

$$g(\gamma x) = \gamma g(x),\ \forall \gamma \in \Gamma,\ x \in V,$$

and it is called $\Gamma-$reversible-equivariant

$$g(\gamma x) = \sigma(\gamma)\gamma g(x),\ \forall \gamma \in \Gamma,\ x \in V.$$

We denote by $\vec{\mathcal{P}}_V(\Gamma)$ the module of the $\Gamma-$equivariant polynomial mappings and by $\vec{\mathcal{Q}}_V(\Gamma)$ the module of the $\Gamma-$reversible-equivariant polynomial mappings, both over the ring $\mathcal{P}_V(\Gamma)$. The modules $\mathcal{Q}_V(\Gamma)$, $\vec{\mathcal{P}}_V(\Gamma)$ and $\vec{\mathcal{Q}}_V(\Gamma)$ are finitely generated and graded over the ring $\mathcal{P}_V(\Gamma)$, which is also finitely generated and graded (see [1]). When σ is trivial, then $\mathcal{P}_V(\Gamma)$ and $\mathcal{Q}_V(\Gamma)$, as well as $\vec{\mathcal{P}}_V(\Gamma)$ and $\vec{\mathcal{Q}}_(\Gamma)$ coincide.

Consider now the space $\mathcal{P}_V$ of polynomial functions $V \to \mathbb{R}$ and the space $\vec{\mathcal{P}}_V$ of polynomial mappings $V \to V$. Consider also the actions of Γ on these spaces induced by the action of Γ on V:

$$\begin{array}{ccc} \Gamma \times \mathcal{P}_V \to \mathcal{P}_V & \text{and} & \Gamma \times \vec{\mathcal{P}}_V \to \vec{\mathcal{P}}_V, \\ (\gamma, f) \mapsto \gamma \odot f & & (\gamma, g) \mapsto \gamma \star g \end{array} \tag{3}$$

where $\gamma \odot f(x) = f(\gamma x)$ and $\gamma \star g(x) = \gamma^{-1} g(\gamma x)$, $\forall x \in V$, $\forall \gamma \in \Gamma$. We then define the Reynolds operators on $\mathcal{P}_V(\Gamma_+)$, namely $R, S : \mathcal{P}(\Gamma_+) \to \mathcal{P}(\Gamma_+)$, by

$$R(f) = \frac{1}{2} \sum_{\gamma\Gamma_+} \gamma \odot f = \frac{1}{2} \left(f + \delta \odot f \right),$$

$$S(f) = \frac{1}{2} \sum_{\gamma\Gamma_+} \sigma(\gamma) \gamma \odot f = \frac{1}{2} \left(f - \delta \odot f \right).$$

We have the equivariant versions of the operators above: consider $\vec{R}, \vec{S} : \vec{\mathcal{P}}_V(\Gamma_+) \to \vec{\mathcal{P}}_V(\Gamma_+)$ defined by

$$\vec{R}(g) = \frac{1}{2} \sum_{\gamma\Gamma_+} \gamma \star g = \frac{1}{2} \left(g + \delta \star g \right),$$

$$\vec{S}(g) = \frac{1}{2} \sum_{\gamma\Gamma_+} \sigma(\gamma) \gamma \star g = \frac{1}{2} \left(g - \delta \star g \right).$$

In [1], the Reynolds operators has been used to prove the decompositions of $\mathcal{P}_V(\Gamma)-$modules

$$\mathcal{P}_V(\Gamma_+) = \mathcal{P}_V(\Gamma) \oplus \mathcal{Q}_V(\Gamma) \quad \text{and} \quad \vec{\mathcal{P}}_V(\Gamma_+) = \vec{\mathcal{P}}_V(\Gamma) \oplus \vec{\mathcal{Q}}_V(\Gamma),$$

which are used to give algorithms to compute generators of $\mathcal{Q}_V(\Gamma)$ and $\vec{\mathcal{Q}}_V(\Gamma)$ from the knowledge of generators of $\mathcal{P}_V(\Gamma_+)$ and $\vec{\mathcal{P}}_V(\Gamma_+)$. These algorithms are applied here, in Section 5.

A result in [3] provides a simple way to compute a set of generators of $\mathcal{P}_V(\Gamma)$ from generators of $\mathcal{P}_V(\Gamma_+)$. The result is stated below:

Theorem 2.1 *[3, Theorem 3.1] Let* Γ *be a compact Lie group acting linearly on* V *and let* $\sigma : \Gamma \to \mathbf{Z}_2$ *be a homomorphism as in (2). Let* $u_1, \ldots, u_s$ *be a Hilbert basis for the ring* $\mathcal{P}_V(\Gamma_+)$*. Then the set*

$$\{R(u_i), S(u_i)S(u_j), \ \forall\ 1 \leq i, j \leq s\}$$

is a Hilbert basis for the ring $\mathcal{P}_V(\Gamma)$*.*

3 Invariant Theory for the semidirect product

Given two groups Γ_1 and Γ_2, recall that a semidirect product $\Gamma_1 \rtimes \Gamma_2$ is the direct product $\Gamma_1 \times \Gamma_2$ as a set with a group operation induced by a homomorphism $\mu : \Gamma_2 \to \mathrm{Aut}(\Gamma_1)$. We consider now Γ_1 and Γ_2 acting on V and let (ρ, V) and (η, V) denote their representations, respectively. Now, define the operation $(\Gamma_1 \rtimes \Gamma_2) \times V \to V$,

$$(\gamma_1, \gamma_2)v = \gamma_1(\gamma_2 v). \tag{4}$$

We then have:

Proposition 3.1 *The operation (4) defines an action of the semidirect product* $\Gamma_1 \rtimes \Gamma_2$ *on* V *if, and only if, the representation of* $\mu(\gamma_2)(\gamma_1)$ *is a conjugation, that is,* $\rho(\mu(\gamma_2)(\gamma_1)) = \eta(\gamma_2)\rho(\gamma_1)\eta(\gamma_2)^{-1}$*.*

The proof of the proposition above is direct from the definition of an action. It is worthwhile mentioning that

$$\rho(\gamma_1)\eta(\gamma_2) = \eta(\gamma_2)\rho(\mu(\gamma_2^{-1})(\gamma_1)) \tag{5}$$

highlights the non-commutativity of the actions of Γ_1 and Γ_2.

We observe that when μ is the trivial homomorphism, sending every element of Γ_2 to the identity automorphism of Γ_1, then $\Gamma_1 \rtimes \Gamma_2$ is the direct product $\Gamma_1 \times \Gamma_2$. In this case, the operation in (4) is an action if, and only if, the actions of Γ_1 and Γ_2 commute. Moreover, when Γ_1 have symmetries and reversing symmetries simultaneously, for each $\gamma_2 \in \Gamma_2$, the automorphism $\mu(\gamma_2)$ preserves the symmetries and reversing symmetries of Γ_1.

In this work, we assume that Γ_1 and Γ_2 admit a semidirect product with a representation in the conditions of Proposition 3.1. Notice that this assumption may fail: consider the groups $\Gamma_1 = \langle \kappa_1 \rangle$ and $\Gamma_2 = \langle \kappa_2 \rangle$ acting on $\mathbb{R}^2$ as $\kappa_1(x, y) = (y, x)$ and $\kappa_2(x, y) = (x, -y)$. In this case, it is not possible even to give the set $\Gamma_1 \times \Gamma_2$ a group structure. Nevertheless, there may be applications where we wish to work with $\Gamma = \langle \kappa_1, \kappa_2 \rangle$. Obviously, in this case, we have $\Gamma = \tilde{\Gamma}_1 \rtimes \tilde{\Gamma}_2$, with $\tilde{\Gamma}_1 = \langle \kappa_1\kappa_2 \rangle$ and $\tilde{\Gamma}_2 = \langle \kappa_2 \rangle$, and apply our theory to these two new groups.

We consider now the symmetry structure of Γ_2 endowed with a homomorphism as in (2) and introduce a homomorphism on $\Gamma_1 \rtimes \Gamma_2$ in order to preserve this structure: we define

$$\begin{array}{rccc} \tilde{\sigma}: & \Gamma_1 \rtimes \Gamma_2 & \rightarrow & \mathbf{Z}_2. \\ & (\gamma_1, \gamma_2) & \mapsto & \sigma(\gamma_2) \end{array} \tag{6}$$

The next result relates the invariant theory of $\Gamma_1 \rtimes \Gamma_2$ to that of each group, Γ_1 and Γ_2, that compose the semidirect product.

Proposition 3.2 *Let Γ_1 and Γ_2 be compact Lie groups acting linearly on V and consider the homomorphism $\tilde{\sigma}$ defined by (6). Then:*

(i) $\mathcal{P}_V(\Gamma_1 \rtimes \Gamma_2) = \mathcal{P}_V(\Gamma_1) \cap \mathcal{P}_V(\Gamma_2)$;

(ii) $\vec{\mathcal{P}}_V(\Gamma_1 \rtimes \Gamma_2) = \vec{\mathcal{P}}_V(\Gamma_1) \cap \vec{\mathcal{P}}_V(\Gamma_2)$;

(iii) $\mathcal{Q}_V(\Gamma_1 \rtimes \Gamma_2) = \mathcal{P}_V(\Gamma_1) \cap \mathcal{Q}_V(\Gamma_2)$;

(iv) $\vec{\mathcal{Q}}_V(\Gamma_1 \rtimes \Gamma_2) = \vec{\mathcal{P}}_V(\Gamma_1) \cap \vec{\mathcal{Q}}_V(\Gamma_2)$.

Proof:
If $f \in \mathcal{P}_V(\Gamma_1 \rtimes \Gamma_2)$, then $f((\gamma_1, \gamma_2)v) = f(v)$. If $\gamma_1 = Id$, we have $f(\gamma_2 v) = f(v)$, that is, $f \in \mathcal{P}_V(\Gamma_2)$. Also, if $\gamma_2 = Id$, then $f(\gamma_1 v) = f(v)$, implying that $f \in \mathcal{P}_V(\Gamma_1)$. On the other hand, if $f \in \mathcal{P}_V(\Gamma_1) \cap \mathcal{P}_V(\Gamma_2)$, then $f((\gamma_1, \gamma_2)v) = f(\gamma_1(\gamma_2 v)) = f(\gamma_2 v) = f(v)$.

The proofs of the other three equalities are similar to this. ▯

The proposition above is used in Section 4 for the deduction of normal forms of $\Gamma-$reversible-equivariant systems, where the algebraic approach is applied to $\mathbf{S} \rtimes \Gamma$ for a convenient group $\mathbf{S}$ of symmetries. Parallel results to Proposition 3.2 for Γ_1 and Γ_2 having both symmetries and reversibilities can be found in [13].

We end this section recalling the diagonal representation of $\Gamma_1 \times \Gamma_2$ on $V \times W$:

$$(\gamma_1, \gamma_2)(v, w) = (\gamma_1 v, \gamma_2 w).$$

The equalities of Proposition 3.2 applied to $V \times W$ are immediate for the direct product and they are directly related to the invariant theory of each group Γ_1 and Γ_2. The result is as follows:

Lemma 3.1 *Let Γ_1 and Γ_2 be groups acting linearly on V and W, respectively. Let $\{u_1, \ldots, u_r\}$ and $\{\alpha_1, \ldots, \alpha_s\}$ be Hilbert bases for $\mathcal{P}_V(\Gamma_1)$ and $\mathcal{P}_W(\Gamma_2)$, respectively. Let $\{f_1, \ldots f_m\}$ be a* *generating set of* $\vec{\mathcal{P}}_V(\Gamma_1)$ *over the ring* $\mathcal{P}_V(\Gamma_1)$ *and let* $\{g_1,\dots g_n\}$ *be a generating set of* $\vec{\mathcal{P}}_W(\Gamma_2)$ *over the ring* $\mathcal{P}_W(\Gamma_2)$. *Then,* $\{u_1,\dots u_r,\alpha_1,\dots,\alpha_s\}$ *is a Hilbert basis for* $\mathcal{P}_{V\times W}(\Gamma_1\times\Gamma_2)$ *and*

$$\left\{\begin{pmatrix} f_i \\ 0_W\end{pmatrix}, \begin{pmatrix} 0_V \\ g_j\end{pmatrix},\ 1\le i\le m,\ \ 1\le j\le n\right\}$$

is a generating set of the module $\vec{\mathcal{P}}_{V\times W}(\Gamma_1\times\Gamma_2)$ *over the ring* $\mathcal{P}_{V\times W}(\Gamma_1\times\Gamma_2)$, *where* 0_V *and* 0_W *denote the zero vectors of* V *and* W, *respectively.*

We register below one case that is used in Section 5 in the construction of nonresonant $\mathbf{R}\times\mathbf{T}^n-$reversible-equivariant normal forms on $\mathbb{C}^{n+1}$.

Example 3.1 *Consider the action of the group torus* $\mathbf{T}$ *on* $\mathbb{C}$ *given by* $\theta z = e^{i\theta}$. *Then* $\mathcal{P}_{\mathbb{C}}(\mathbf{T}) = \langle z\bar{z}\rangle$ *and* $\vec{\mathcal{P}}_{\mathbb{C}}(\mathbf{T}) = \mathcal{P}_{\mathbb{C}}(\mathbf{T})\{z, iz\}$. *Consider the action of* $\mathbf{T}^n$ *on* $\mathbb{C}^n$ *given by* $(\theta_1,\dots,\theta_n).(z_1,\dots z_n) = (e^{i\theta_1}z_1,\dots,e^{i\theta_n}z_n)$. *From Lemma 3.1 we get* $\mathcal{P}_{\mathbb{C}^n}(\mathbf{T}^n) = \langle z_1\bar{z_1},\dots,z_n\bar{z_n}\rangle$ *and*

$$\vec{\mathcal{P}}_{\mathbb{C}^n}(\mathbf{T}^n) = \mathcal{P}_{\mathbb{C}^n}(\mathbf{T}^n)\left\{\begin{pmatrix} z_1\\0\\0\\ \vdots\\0\end{pmatrix}, \begin{pmatrix} iz_1\\0\\0\\ \vdots\\0\end{pmatrix}, \begin{pmatrix} 0\\z_2\\0\\ \vdots\\0\end{pmatrix}, \begin{pmatrix} 0\\iz_2\\0\\ \vdots\\0\end{pmatrix},\dots \begin{pmatrix} 0\\ \vdots\\0\\0\\z_n\end{pmatrix}, \begin{pmatrix} 0\\ \vdots\\0\\0\\iz_n\end{pmatrix}\right\}.$$

4 Computing normal form by algebraic tools

In this section we provide an algebraic method to obtain the normal form of a $\Gamma-$reversible equivariant vector field on a finite dimensional vector space V. As we shall see, the method takes the action of Γ into account to simplify the deduction of the normal form significantly. We now follow [10] to briefly recall the normal form theory without symmetries. Our approach adapts that method.

Consider a system of ODEs

$$\dot{x} = h(x),\ x\in V, \tag{7}$$

where $h\in C^\infty$. The interest is local around a singular point that we assume to be the origin. So $h(0) = 0$. Although the results hold in a neighborhood of the origin, we shall use the notation of the whole vector space V. Let L denote the linear part of h at the origin and consider the Taylor expansion of h about the origin

$$L(x) + h_2(x) + h_3(x) + \dots \tag{8}$$

with h_k homogeneous of degree k, $k \geq 2$. The method to obtain the normal form for (7) consists of successive changes of coordinates on the source of the form $I + \xi_k$ about the origin, for $k \geq 2$, where I is the identity and ξ_k is a homogeneous polynomial of degree k. As a result, the new system has a "simpler" form at its degree-k level, no effect on its lower-order terms and with any form of its terms of degree greater than k. Here we do not deal necessarily with analytic mappings, so the systems are formally conjugate in the sense that their corresponding formal Taylor series are conjugate as formal vector fields.

After applying changes of coordinates up to degree k and rewriting in the x variable again, one obtains the new system with an intermediate form

$$\dot{x} = Lx + \sum_{n=2}^{k-1} g_n(x) + \tilde{h}_k(x) - (D\xi_k)_{(x)}Lx - L\xi_k(x)) + O|x|^{k+1}. \tag{9}$$

Now consider the homological operator $Ad_L : \vec{\mathcal{P}}_V \to \vec{\mathcal{P}}_V$ given by

$$Ad_L(p)(x) = (Dp)_{(x)}Lx - Lp(x), \tag{10}$$

where $\vec{\mathcal{P}}_V$ is the vector space of polynomial mappings $V \to V$. Since $\vec{\mathcal{P}}_V$ is a graded algebra, that is $\vec{\mathcal{P}}_V = \bigoplus_{k=0}^{\infty} \vec{\mathcal{P}}^k_V$, we can consider for each $k \geq 0$ the operator Ad^k_L, the restriction of Ad_L to $\vec{\mathcal{P}}^k_V$. From (9) it follows that the new system can be simpler if its term of degree k is of the form $g_k = \tilde{h}_k - Ad^k_L(\xi_k)$, for some ξ_k. This is clearly not unique, and a choice for g_k was proposed by Belitskii [5] through an appropriate complement, $(\mathrm{Im}\, Ad^k_L)^c = \ker Ad^k_{L^t}$, where L^t denotes the adjoint operator of L. So, the method consists of determining the polynomial solutions of the PDE $Ad^k_{L^t} = 0$, and (7) turns out to be formally conjugate to

$$\dot{x} = Lx + g_2(x) + g_3(x) + \ldots$$

with $g_k \in \ker Ad^k_{L^t},\ k \geq 2$.

In general, it may not be easy to find $\ker Ad^k_{L^t}$ since it involves solving a PDE. An alternative method to find the complement has been proposed by Elphick *et al.* in [7] recognizing a group of symmetries on this space. More precisely, from the linear part L, consider the group $\mathbf{S}$ defined as in (1). So, $\mathbf{S}$ is a one-parameter closed subgroup of $GL(n)$ acting on V by matrix product. The authors prove that $\ker Ad^k_{L^t} = \vec{\mathcal{P}}^k_V(\mathbf{S})$ (see [7, Theorem 2]) and, therefore,

$$\vec{\mathcal{P}}^k_V = \vec{\mathcal{P}}^k_V(\mathbf{S}) \oplus Ad_L(\vec{\mathcal{P}}^k_V). \tag{11}$$

Now, we are interested in obtaining the normal form for (7) when h is $\Gamma-$reversible-equivariant. Changes of coordinates in this setting are assumed to be $\Gamma-$equivariant, so that all the symmetries and reversing symmetries of the original system are preserved in the normal form. As mentioned in the introduction, some authors have obtained the normal form by first solving a partial differential equation and after that imposing the symmetries and reversing symmetries constraints. The result that we show here provides a different method for this process, based on the knowledge of the invariant theory for Γ and $\mathbf{S}$. Although there are applications for which the group $\mathbf{S}$ is not compact, the theory developed in [1] can be applied here, as long as there is a finite set of generators for the ring $\mathcal{P}_V(\mathbf{S})$ and for the module $\vec{\mathcal{P}}_V(\mathbf{S})$ over $\mathcal{P}_V(\mathbf{S})$.

The next two lemmas are useful in the remainder of this section.

Lemma 4.1 *The homological operator Ad_L interchanges the modules in the sum decomposition $\vec{\mathcal{P}}_V(\Gamma_+) = \vec{\mathcal{P}}_V(\Gamma) \oplus \vec{\mathcal{Q}}_V(\Gamma)$.*

Proof: If $p \in \vec{\mathcal{P}}_V(\Gamma)$, then

$$\begin{aligned} Ad_L(p)(\gamma x) &= (Dp)_{(\gamma x)} L(\gamma x) - Lp(\gamma x) = \gamma (Dp)_{(x)} \gamma^{-1} \sigma(\gamma)\gamma L(x) - \sigma(\gamma)\gamma Lp(x) \\ &= \sigma(\gamma)\gamma \left((Dp)_{(x)} L - Lp(x) \right) = \sigma(\gamma)\gamma Ad_L(p)(x), \end{aligned}$$

so $Ad_L(p) \in \vec{\mathcal{Q}}_V(\Gamma)$. The other permutation is analogous, just use $\sigma^2(\gamma) = 1 \ \forall \gamma \in \Gamma$. ▯

Lemma 4.2 *Para todo $p \in \vec{\mathcal{P}}_V(\Gamma_+)$, we have $\vec{S}(Ad_L(p)) = Ad_L(\vec{R}(p))$.*

Proof: We start by noting that

$$Ad_L(\gamma \star p) = \sigma(\gamma)\gamma \star (Ad_L(p)), \ \forall p \in \vec{\mathcal{P}}_V. \tag{12}$$

From this, we have

$$\vec{S}(Ad_L(p)) = \frac{1}{2} \left(Ad_L(p) + Ad_L(\delta \star p) \right). \tag{13}$$

Now, just use the definitions of Ad_L and $\vec{R}$ in (13). ▯

We now present our main theorem. We have to find a complement to the homological operator Ad_L^k to the $\Gamma-$equivariants inside the vector space of $\Gamma-$reversible-equivariants. Our result is the recognition of $\vec{\mathcal{Q}}_V^k(\mathbf{S} \rtimes \Gamma)$ as this complement. Consider the semidirect product $\mathbf{S} \rtimes \Gamma$ for which the homomorphism $\mu : \Gamma \to \mathrm{Aut}(\mathbf{S})$ is defined by

$$\mu(\gamma)(e^{sL^t}) = e^{\sigma(\gamma) s L^t}.$$

By Proposition 3.1, μ defines the action of $\mathbf{S} \rtimes \Gamma$ on V as

$$(e^{sL^t}, \gamma) \cdot v = e^{sL^t}(\gamma v).$$

In this case, the equality (5) is

$$e^{sL^t}(\gamma v) = \gamma(e^{\sigma(\gamma)sL^t} v). \tag{14}$$

Theorem 4.1 *For $k \geq 2$, we have*

$$\vec{\mathcal{Q}}_V^k(\Gamma) = \vec{\mathcal{Q}}_V^k(\mathbf{S} \rtimes \Gamma) \oplus Ad_L^k(\vec{\mathcal{P}}_V^k(\Gamma)). \tag{15}$$

To prove Theorem 4.1, we present three lemmas. First, we use the action of Γ on $\vec{\mathcal{P}}_V$ given in (3) to define the mapping $\pi : \vec{\mathcal{P}}_V \to \vec{\mathcal{P}}_V$:

$$\pi(p) = \frac{1}{2}\left(\int_{\Gamma_+} \tau \star p \; d\tau - \int_{\Gamma_+} (\delta\tau) \star p \; d\tau\right), \tag{16}$$

where $\displaystyle\int_{\Gamma_+}$ is the normalized Haar integral over Γ_+. Notice that π is an extension of the operator $\vec{S}$. More then that, they have the same target space. In fact, we have:

Lemma 4.3 *The mapping $\pi : \vec{\mathcal{P}}_V \to \vec{\mathcal{Q}}_V(\Gamma)$ is a linear projection which preserves the graduation of the algebra $\vec{\mathcal{P}}_V$.*

Proof: By the linearity of the Haar integral it follows that π is linear and and if p has degree k, so does $\pi(p)$. To prove that $\pi(p) \in \vec{\mathcal{Q}}_V^k(\Gamma)$, use

$$\sigma(\gamma)\gamma \star \pi(p) = \frac{1}{2}\left(\int_{\Gamma_+} \sigma(\gamma)(\tau\gamma) \star p \; d\tau - \int_{\Gamma_+} \sigma(\gamma)(\delta\tau\gamma) \star p \; d\tau\right),$$

and check that $\pi(p) = \sigma(\gamma)\gamma \star \pi(p)$ for elements γ in Γ_+ and $\delta\Gamma_+$ separately, using the left and right invariance of Haar integral and also the normality of the subgroup Γ_+. That $\pi^2 = \pi$ follows from the fact that $\pi(p) = p$, $\forall p \in \vec{\mathcal{Q}}_V(\Gamma)$. □

Lemma 4.4 *The projection π defined in (16) satisfies $\pi(\vec{\mathcal{P}}_V(\mathbf{S})) = \vec{\mathcal{Q}}_V(\mathbf{S} \rtimes \Gamma)$.*

Proof: For $p \in \vec{\mathcal{P}}_V^k(\mathbf{S})$, we want $\pi(p) \in \vec{\mathcal{Q}}_V^k(\mathbf{S} \rtimes \Gamma)$, that is, $\pi(p) = \sigma(\gamma)(\gamma e^{\sigma(\gamma)sL^t}) \star \pi(p)$ for $\gamma \in \Gamma$. To prove this, we use (14): if $\gamma \in \Gamma_+$, we have

$$\begin{aligned}
\sigma(\gamma)(\gamma e^{\sigma(\gamma)sL^t}) \star \pi(p) &= \frac{1}{2}\left(\int_{\Gamma_+} (\tau\gamma e^{sL^t}) \star p \; d\tau - \int_{\Gamma_+} (\delta\tau\gamma e^{sL^t}) \star p \; d\tau\right) \\
&= \frac{1}{2}\left(\int_{\Gamma_+} (e^{sL^t}\tau) \star p \; d\tau - \int_{\Gamma_+} (e^{\sigma(\delta\tau)sL^t}\delta\tau) \star p \; d\tau\right) \\
&= \frac{1}{2}\left(\int_{\Gamma_+} \tau \star p \; d\tau - \int_{\Gamma_+} (\delta\tau) \star p \; d\tau\right) = \pi(p).
\end{aligned}$$

If $\gamma \in \delta\Gamma_+$, we have

$$\begin{aligned}
\sigma(\gamma)(\gamma e^{\sigma(\gamma)sL^t}) \star \pi(p) &= \frac{1}{2}\left(\int_{\Gamma_+} (\delta\tau\gamma e^{-sL^t}) \star p \; d\tau - \int_{\Gamma_+} (\tau\gamma e^{-sL^t}) \star p \; d\tau\right) \\
&= \frac{1}{2}\left(\int_{\Gamma_+} (\delta\tau\gamma) \star p \; d\tau - \int_{\Gamma_+} (\tau\gamma) \star p \; d\tau\right) \\
&= \frac{1}{2}\left(\int_{\Gamma_+} (\delta\tau\delta\lambda) \star p \; d\tau - \int_{\Gamma_+} (\tau\delta\lambda) \star p \; d\tau\right) \\
&= \frac{1}{2}\left(\int_{\Gamma_+} (\tilde{\tau}\delta^2\lambda) \star p \; d\tilde{\tau} - \int_{\Gamma_+} (\delta\tilde{\tau}\lambda) \star p \; d\tilde{\tau}\right) \\
&= \frac{1}{2}\left(\int_{\Gamma_+} \tilde{\tau} \star p \; d\tilde{\tau} - \int_{\Gamma_+} (\delta\tilde{\tau}) \star p \; d\tilde{\tau}\right) = \pi(p).
\end{aligned}$$

To prove the other inclusion, set $g \in \vec{\mathcal{Q}}_V^k(\mathbf{S} \rtimes \Gamma)$. So $g = \sigma(\gamma)\gamma \star (g)$, for all $\gamma \in \Gamma$. Then,

$$\pi(g) = \frac{1}{2}\left(\int_{\Gamma_+} \tau \star (g) d\tau - \int_{\Gamma_+} (\delta\tau) \star (g) d\tau\right) = \frac{1}{2}\left(\int_{\Gamma_+} g d\tau - \int_{\Gamma_+} -g d\tau\right) = \int_{\Gamma_+} g d\tau = g.$$

□

For the next lemma, observe that

$$\int_{\Gamma} Ad_L(\gamma \star p) \; d\gamma = Ad_L \int_{\Gamma} \gamma \star p \; d\gamma,$$

which follows directly from the linearity of Ad_L.

Lemma 4.5 *The projection π given by (16) satisfies $\pi(Ad_L(\vec{\mathcal{P}}_V)) = Ad_L(\vec{\mathcal{P}}_V(\Gamma))$.*

Proof: Let $p \in \vec{\mathcal{P}}_V$ and consider the equality (12). Then,

$$\begin{aligned}\pi(Ad_L(p)) &= \frac{1}{2}\left(\int_{\Gamma_+} \gamma \star (Ad_L(p))d\gamma - \int_{\Gamma_+} (\delta\gamma) \star (Ad_L(p))d\gamma\right) \\ &= \frac{1}{2}\left(\int_{\Gamma_+} \sigma(\gamma)Ad_L(\gamma \star p)d\gamma - \int_{\Gamma_+} \sigma(\delta\gamma)Ad_L((\delta\gamma) \star p)d\gamma\right) \\ &= \frac{1}{2}\left(\int_{\Gamma_+} Ad_L(\gamma \star p)d\gamma + \int_{\Gamma_+} Ad_L((\delta\gamma) \star p)d\gamma\right) \\ &= \frac{1}{2}\left(Ad_L \int_{\Gamma_+} \gamma \star p \ d\gamma + Ad_L \int_{\Gamma_+} (\delta\gamma) \star p \ d\gamma\right) \\ &= Ad_L \left[\frac{1}{2}\left(\int_{\Gamma_+} \gamma \star p \ d\gamma + \int_{\Gamma_+} (\delta\gamma) \star p \ d\gamma\right)\right] \\ &= Ad_L \left(\int_{\Gamma} \gamma \star p \ d\gamma\right).\end{aligned}$$

The last equality uses Fubini theorem (see Bröcker and Dieck [8, Proposition I 5.16]). Now, $\int_{\Gamma} \gamma \star p \ d\gamma \in \vec{\mathcal{P}}_V(\Gamma)$ and any element in $\vec{\mathcal{P}}_V(\Gamma)$ is of the form $\int_{\Gamma} \gamma \star p \ d\gamma$, for some $p \in \vec{\mathcal{P}}_V(\Gamma)$. ▯

Proof of the Theorem 4.1: We apply the projection π given in (16) on equality (11) and now we use Lemmas 4.3, 4.4 and 4.5 to obtain

$$\vec{\mathcal{Q}}_V^k(\Gamma) = \vec{\mathcal{Q}}_V^k(\mathbf{S} \rtimes \Gamma) + Ad_L^k(\vec{\mathcal{P}}_V^k(\Gamma)). \tag{17}$$

By Lemma 3.2, $\vec{\mathcal{Q}}_V^k(\mathbf{S} \rtimes \Gamma) = \vec{\mathcal{P}}_V^k(\mathbf{S}) \cap \vec{\mathcal{Q}}_V^k(\Gamma)$, which together with (11) gives

$$\vec{\mathcal{Q}}_V^k(\mathbf{S} \rtimes \Gamma) \cap Ad_L^k(\vec{\mathcal{P}}_V^k(\Gamma)) \subset \vec{\mathcal{P}}_V^k(\mathbf{S}) \cap Ad_L(\vec{\mathcal{P}}_V^k) = \{0\}.$$

▯

From all the discussion of this section, the following is now a direct consequence of Theorem 4.1.

Theorem 4.2 *Let Γ be a compact Lie group acting linearly on V and consider $h : V \to V$ a $\Gamma-$reversible-equivariant vector field, with $h \in C^\infty$, $h(0) = 0$ and $L = (dh)_0$. Then (7) is formally conjugate to*

$$\dot{x} = Lx + g_2(x) + g_3(x) + \dots$$

where, for each $k \geq 2$, $g_k \in \vec{\mathcal{Q}}_V^k(\mathbf{S} \rtimes \Gamma)$.

In practice we may be able to find the general form of elements in $\vec{\mathcal{Q}}_V(\mathbf{S} \rtimes \Gamma)$. The main tool for that is given in [1, Algorithm 3.7]. From that, one has just to select the general polynomial mapping in this module of the degree one wishes to truncate the normal form.

5 Normal forms under $\mathbf{Z}_2-$actions

In this section we use Theorem 4.1 and 4.2 to calculate the normal forms for two types of $\mathbf{Z}_2-$reversible-equivariant vector fields for the system

$$\dot{x} = h(x) \tag{18}$$

with $h : \mathbb{R}^{2n+2} \to \mathbb{R}^{2n+2}$, $h \in C^\infty$, $h(0) = 0$ and whose linearization about the origin has matrix of type

$$L = \begin{pmatrix} 0 & 1 & & & & & & \\ 0 & 0 & & & & & & \\ & & 0 & \omega_1 & & & & \\ & & -\omega_1 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & & 0 & \omega_n \\ & & & & & & -\omega_n & 0 \end{pmatrix}, \tag{19}$$

for certain $\omega_i \in \mathbb{R}$, $i = 1, \dots, n$ (as in Subsections 5.1 and 5.2 below). We consider vector fields that are reversible equivariant under the action of $\mathbf{Z}_2$ generated by

$$\phi(x_1, x_2, \dots, x_{2n+1}, x_{2n+2}) = (x_1, -x_2, \dots, x_{2n+1}, -x_{2n+2}). \tag{20}$$

One case, presented in Subsection 5.1, is a nonresonant system on $\mathbb{R}^{2n+2}$ when the linearization has a 2-dimensional nilpotent part and n nonresonant purely imaginary eigenvalues. The other case, in Subsection 5.2, is resonant on $\mathbb{R}^6$ and its linearization has a nilpotent part and two resonant purely imaginary eigenvalues.

Let $L = D + N$ be the Jordan-Chevalley decomposition of L, where D is the semisimple part of L, N is the nilpotent part of L and $DN = ND$. Suppose that the nonzero eigenvalues of L are purely imaginary. Then, the group $\mathbf{S}$ defined in (1) has a particular structure:

Theorem 5.1 [11, Proposition XVI 5.7] *Let $L = D + N$ be the Jordan-Chevalley decomposition of L. If $N = 0$, then $\mathbf{S} = \mathbf{T}^k$, and if $N \neq 0$, then $\mathbf{S} = \mathbf{R} \times \mathbf{T}^k$*, where k is the number of algebraically independent eigenvalues of L, $\mathbf{T}^k = \overline{\{e^{sD}, s \in \mathbb{R}\}}$ and $\mathbf{R} = \left\{e^{sN^t}, s \in \mathbb{R}\right\}$.

5.1 Nonresonant case

In this subsection we obtain the normal form of (18) when L is nonresonant, namely, $\omega_1, \ldots, \omega_n$ are algebraically independent. Then $\mathbf{S} = \mathbf{R} \times \mathbf{T}^n$, where $\mathbf{R} \cong \left\{ \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}, s \in \mathbb{R} \right\}$. Here we use complex coordinates and the diagonal action of $\mathbf{S} = \mathbf{R} \times \mathbf{T}^n$ on $\mathbb{R}^2 \times \mathbb{C}^n$,

$$s(x_1, x_2) = (x_1, sx_1 + x_2) \quad \text{and} \quad (\theta_1, \ldots, \theta_n).(z_1, \ldots, z_n) = (e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n). \tag{21}$$

In this coordinates, the action of $\phi \in \mathbf{Z}_2$ is

$$\phi(x_1, x_2, z_1, \ldots, z_n) = (x_1, -x_2, \bar{z}_1, \ldots, \bar{z}_n). \tag{22}$$

We use Theorems 4.1 and 4.2, so we start by determining a set of generators of $\vec{\mathcal{Q}}_{\mathbb{R}^2 \times \mathbb{C}^n}(\tilde{\Gamma})$ over the ring $\mathcal{P}_{\mathbb{R}^2 \times \mathbb{C}^n}(\tilde{\Gamma})$ for $\tilde{\Gamma} = (\mathbf{R} \times \mathbf{T}^n) \rtimes \mathbf{Z}_2$. We have that $\tilde{\Gamma}_+ = \mathbf{S} = \mathbf{R} \times \mathbf{T}^n$. For that, we use [1, Algorithm 3.7], which requires a set of generators of $\vec{\mathcal{P}}_{\mathbb{R}^2 \times \mathbb{C}^n}(\tilde{\Gamma}_+)$ over $\mathcal{P}_{\mathbb{R}^2 \times \mathbb{C}^n}(\tilde{\Gamma}_+)$.

For the action of $\mathbf{R}$ on $\mathbb{R}^2$, $\vec{\mathcal{P}}_{\mathbb{R}^2}(\mathbf{R})$ is generated over $\mathcal{P}_{\mathbb{R}^2}(\mathbf{R}) = \langle x_1 \rangle$ by the set

$$\{(x_1, x_2),\ (0, 1)\}. \tag{23}$$

From Lemma 3.1 and Example 3.1, $\{x_1, |z_1|^2, \ldots, |z_n|^2\}$ is a Hilbert basis for the ring $\mathcal{P}_{\mathbb{R}^2 \times \mathbb{C}^n}(\tilde{\Gamma}_+)$.

Again, we apply the Lemma 3.1 together with (23) and Example 3.1 in order to obtain generators for $\vec{\mathcal{P}}_{\mathbb{R}^2 \times \mathbb{C}^n}(\tilde{\Gamma}_+)$:

$$\begin{pmatrix} x_1 \\ x_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ z_1 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ iz_1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ z_n \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ iz_n \end{pmatrix}.$$

Now, apply again the algorithm to get generators for $\vec{\mathcal{Q}}_{\mathbb{R}^2 \times \mathbb{C}^n}(\tilde{\Gamma})$ over $\mathcal{P}_{\mathbb{R}^2 \times \mathbb{C}^n}(\tilde{\Gamma})$:

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ iz_1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ iz_n \end{pmatrix}.$$

Using the Reynolds operators R and S, it follows from Theorem 2.1 that $\mathcal{P}_{\mathbb{R}^2\times\mathbb{C}^n}(\tilde{\Gamma})$ has the same set of generators as $\mathcal{P}_{\mathbb{R}^2\times\mathbb{C}^n}(\tilde{\Gamma}_+)$, since that $R(x_1) = x_1$, $R(|z_i|^2) = |z_i|^2$ and $S(x_1) = S(|z_i|^2) = 0$, for $i = 1, \ldots, n$.

Therefore, the $\mathbf{Z}_2-$reversible-equivariant normal form of the system (18), in complex coordinates, is:

$$\begin{aligned}
\dot{x}_1 &= x_2;\\
\dot{x}_2 &= f_0(x_1, |z_1|^2, \ldots, |z_n|^n)\\
\dot{z}_1 &= -i\omega_1 z_1 + iz_1 f_1(x_1, |z_1|^2, \ldots, |z_n|^n)\\
\dot{z}_2 &= -i\omega_2 z_2 + iz_2 f_2(x_1, |z_1|^2, \ldots, |z_n|^n)\\
&\vdots\\
\dot{z}_n &= -i\omega_2 z_n + iz_n f_n(x_1, |z_1|^2, \ldots, |z_n|^n)
\end{aligned} \tag{24}$$

for some $f_i : \mathbb{R}^{n+1} \to \mathbb{R}$, $i = 0, \ldots, n$.

Remark 5.1 *Normal form (24) has been obtained by Lima and Teixeira in [12]. The authors use the classical methods by Belitskii without symmetries to get a pre-normal form on which they impose the reversibility of $\mathbf{Z}_2$ as a final step. The alternative method given by Theorem 4.1 for a compact Lie group was in fact motivated by this example.*

5.2 Resonant case

In this subsection we find the $\mathbf{Z}_2-$reversible-equivariant normal form of (18) for $n = 2$, when ω_1 and ω_2 in (19) satisfy

$$n_1\omega_2 - n_2\omega_1 = 0,$$

for some nonzero $n_1, n_2 \in \mathbb{N}$. Under this condition, system (18) is called $(n_1 : n_2)-$resonant. The deduction of a normal form via $\ker Ad^k_{L^t}$ becomes harder in computation as n_1 and n_2 get larger. We refer to [12] and [15] for an illustration, where the authors deal with some particular choices of n_1 and n_2. We emphasize that the usage of Theorem 4.2 skeeps from this problem, and works equality well for any values of n_1 and n_2. By Theorem 5.1, $\mathbf{S} = \mathbf{R} \times \mathbf{S}^1$. In complex coordinates, the action of $\mathbf{R}$ on $\mathbb{R}^2$ is given in (21) and the action of $\mathbf{S}^1$ on $\mathbb{C}^2$ is

$$\psi(z_1, z_2) = (e^{in_1\psi} z_1, e^{in_2\psi} z_2). \tag{25}$$

Invariant and equivariant generators on $\mathbb{C}^2$ under $\mathbf{S}^1$ can be found in [11, Theorem 4.2, Chapter XIX]:

$$|z_1|^2, |z_2|^2, \mathrm{Re}(z_1^{n_2}\bar{z}_2^{n_1}), \mathrm{Im}(z_1^{n_2}\bar{z}_2^{n_1})$$

and

$$\begin{pmatrix} z_1 \\ 0 \end{pmatrix}, \begin{pmatrix} z_1 i \\ 0 \end{pmatrix}, \begin{pmatrix} \bar{z}_1^{n_2-1} z_2^{n_1} \\ 0 \end{pmatrix}, \begin{pmatrix} \bar{z}_1^{n_2-1} z_2^{n_1} i \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ z_2 \end{pmatrix}, \begin{pmatrix} 0 \\ z_2 i \end{pmatrix}, \begin{pmatrix} 0 \\ z_1^{n_2} \bar{z}_2^{n_1-1} \end{pmatrix}, \begin{pmatrix} 0 \\ z_1^{n_2} \bar{z}_2^{n_1-1} i \end{pmatrix},$$

respectively.

The result is:

Theorem 5.2 *Let $\dot{x} = Lx + h(x)$ be a $\mathbf{Z}_2-$reversible-equivariant system, with L defined in (19) for $n = 2$. Then, this system is formally conjugate to*

$$\begin{aligned}
\dot{x}_1 &= x_2 + x_1 \operatorname{Im}(z_1^{n_2} \bar{z}_2^{n_1}) f_0(X) \\
\dot{x}_2 &= g_1(X) + x_2 \operatorname{Im}(z_1^{n_2} \bar{z}_2^{n_1}) f_0(X) \\
\dot{z}_1 &= -i\omega_1 z_1 + i z_1 f_2(X) + i \bar{z}_1^{n_2-1} z_2^{n_2} f_3(X) + z_1 \operatorname{Im}(z_1^{n_2} \bar{z}_2^{n_1}) f_4(X) + \bar{z}_1^{n_2-1} z_2^{n_2} \operatorname{Im}(z_1^{n_2} \bar{z}_2^{n_1}) f_5(X) \\
\dot{z}_2 &= -i\omega_2 z_2 + i z_2 f_6(X) + i z_1^{n_2} \bar{z}_2^{n_1-1} f_7(X) + z_2 \operatorname{Im}(z_1^{n_2} \bar{z}_2^{n_1}) f_8(X) + z_1^{n_2} \bar{z}_2^{n_1-1} \operatorname{Im}(z_1^{n_2} \bar{z}_2^{n_1}) f_9(X),
\end{aligned}$$

for $f_i : \mathbb{R}^4 \to \mathbb{R}$, $i = 0, \ldots, n$, and X being the vector $(x_1, |z_1|^2, |z_2|^2, \operatorname{Re}(z_1^{n_2} \bar{z}_2^{n_1}))$.

Proof: Here the whole group is $\tilde{\Gamma} = (\mathbf{R} \times \mathbf{S}^1) \rtimes \mathbf{Z}_2$. From Theorem 4.1, we need to compute the general form of elements in $\vec{\mathcal{Q}}_{\mathbb{R}^2 \times \mathbb{C}^2}(\tilde{\Gamma})$. We have that $\tilde{\Gamma}_+ = \mathbf{R} \times \mathbf{S}^1$. From Lemma 3.1,

$$\{x_1, |z_1|^2, |z_2|^2, \operatorname{Re}(z_1^{n_2} \bar{z}_2^{n_1}), \operatorname{Im}(z_1^{n_2} \bar{z}_2^{n_1})\}$$

is a Hilbert basis for $\mathcal{P}_{\mathbb{R}^2 \times \mathbb{C}^2}(\tilde{\Gamma}_+)$, and the generators for $\vec{\mathcal{P}}_{\mathbb{R}^2 \times \mathbb{C}^2}(\tilde{\Gamma}_+)$ over $\mathcal{P}_{\mathbb{R}^2 \times \mathbb{C}^2}(\tilde{\Gamma}_+)$ are given by

$$\begin{pmatrix} x_1 \\ x_2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ z_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ i z_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \bar{z}_1^{n_2-1} z_2^{n_1} \\ 0 \end{pmatrix}, \begin{pmatrix} ,0 \\ 0 \\ i \bar{z}_1^{n_2-1} z_2^{n_1} \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ z_2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ i z_2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ z_1^{n_2} \bar{z}_2^{n_1-1} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ i z_1^{n_2} \bar{z}_2^{n_1-1} \end{pmatrix}.$$

Then, we apply [1, Algorithm 3.7] to obtain generators for $\vec{\mathcal{Q}}_{\mathbb{R}^2 \times \mathbb{C}^2}(\tilde{\Gamma})$ over $\mathcal{P}_{\mathbb{R}^2 \times \mathbb{C}^2}(\tilde{\Gamma})$:

$$\begin{pmatrix} x_1 \operatorname{Im}(z_1^{n_2} \bar{z}_2^{n_1}) \\ x_2 \operatorname{Im}(z_1^{n_2} \bar{z}_2^{n_1}) \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ i z_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ i \bar{z}_1^{n_2-1} z_2^{n_1} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ z_1 \operatorname{Im}(z_1^{n_2} \bar{z}_2^{n_1}) \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \bar{z}_1^{n_2-1} z_2^{n_1} \operatorname{Im}(z_1^{n_2} \bar{z}_2^{n_1}) \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ iz_2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ iz_1^{n_2}\bar{z}_2^{n_1-1} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ z_2 \operatorname{Im}(z_1^{n_2}\bar{z}_2^{n_1}) \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ z_1^{n_2}\bar{z}_2^{n_1-1} \operatorname{Im}(z_1^{n_2}\bar{z}_2^{n_1}) \end{pmatrix}.$$

Using the Reynolds operators R and S and Theorem 2.1, we obtain

$$\{x_1, |z_1|^2, |z_2|^2, \operatorname{Re}(z_1^{n_2}\bar{z}_2^{n_1})\} \tag{26}$$

as a Hilbert basis for $\mathcal{P}_{\mathbb{R}^2\times\mathbb{C}^2}(\tilde{\Gamma})$.

In fact, we have $R(x_1) = x_1$, $R(|z_1|^2) = |z_1|^2$, $R(|z_2|^2) = |z_2|^2$, $R(\operatorname{Re}(z_1^{n_2}\bar{z}_2^{n_1})) = \operatorname{Re}(z_1^{n_2}\bar{z}_2^{n_1})$, $R(\operatorname{Im}(z_1^{n_2}\bar{z}_2^{n_1})) = 0$, and $S(x_1) = S(|z_1|^2) = S(|z_2|^2) = S(\operatorname{Re}(z_1^{n_2}\bar{z}_2^{n_1})) = 0$, $S(\operatorname{Im}(z_1^{n_2}\bar{z}_2^{n_1})) = \operatorname{Im}(z_1^{n_2}\bar{z}_2^{n_1})$. But $(\operatorname{Im}(z_1^{n_2}\bar{z}_2^{n_1}))^2$ is obtained from (26).▯

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