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Resumo:

O objetivo principal deste trabalho é a inferência utilizando o método de máxima verossimilhança para o modelo de calibração comparativa estrutural (Barnett, 1969), o qual é frequentemente utilizado nos problemas para avaliar a calibração e a precisão relativa de um conjunto de p instrumentos, que foram projetados para medir a mesma característica, em um grupo comum de n unidades experimentais. Consideramos testes assintóticos para responder as questões de interesse. A metodologia é aplicada a conjunto de dados reais e um pequeno estudo de simulação é apresentada.

HYPOTHESIS TESTING FOR STRUCTURAL CALIBRATION MODEL

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ABSTRACT

The main objective of this paper is to discuss the maximum likelihood inference for the comparative structural calibration model (Barnett,1969), which are frequently used in the problem of assessing the relative calibrations and relative accuracies of a set of p instruments, each designed to measure the same characteristic on a common group of n experimental units. We consider asymptotic tests to answer the outlined questions. The methodology is applied to a real data set and a small simulation study is presented.

1. INTRODUCTION

In this paper we analyze the problem of comparative calibration, where $p > 2$ instruments are used to measure the same unknown quantity x in a common group of n experimental units.

The problem of comparing measurement devices which varies in price, time spent to measure and other features, such as efficiency, has been of growing interest in several areas like engineering, medicine, psychology and agriculture. Grubbs (1948, 1973) considered an experiment designed to compare three types of chronometers and Barnett (1969) considered a comparison of four combinations of two instruments and two operators to measure the vital capacity. Several other examples in the medical area are presented in the literature, specially in Kelly (1984, 1985), Chipkevitch et al. (1996) and Lu et al. (1997). Illustrations in agriculture are considered in Fuller (1987) and for applications in psychology and education, see for example, Dunn (1992).

Suppose that n experimental units (person, physical object, etc.) are randomly selected from a population of such units. Let x_j denote the true value of the quantitative characteristic to be measured in the jth unit, $j = 1, \cdots, n$, which is measured by all the p instruments and y_{ij} the true value of the measurement for the jth unit measured by the ith instrument, $i = 1, \cdots, p$, $j = 1, \cdots, n$. We assume that the y_{ij} satisfies the linear structural relationship with the true (unobserved) x_j and denote by Y_{ij} the observed value (subject to measurement error) of the measurement of the jth unit by the ith instrument. The described model can be represented as

$$y_{ij} = \alpha_i + \beta_i x_j, \tag{1.1}$$

$$Y_{ij} = y_{ij} + e_{ij}, \quad i = 1, \cdots, p \quad \text{and} \quad j = 1, \cdots, n,$$

where $e_{ij} \sim ind.N(0, \phi_i)$, $x_j \sim ind.N(\mu_x, \phi_x)$, x_j independent of e_{ij}, $i = 1, \cdots, p$ and $j = 1, \cdots, n$.

As pointed out by many authors (Bolfarine and Galea -Rojas (1996), Galea-Rojas, Bolfarine and Vilca-Labra (2002) and Barnett (1969), for example) the model (1.1) is not identifiable. The common way to deal with this problem is to impose restrictions on the model parameters. As considered in Barnett (1969), we assume that there is a reference instrument which measures without bias (additive and multiplicative) the quantity of interest. Without loss of generality, we consider that the reference instrument is the first one. Hence, corresponding to instrument 1, we have

$$\alpha_1 = 0 \ \text{ and } \ \beta_1 = 1. \tag{1.2}$$

Another alternative way to identify the model is to consider that the parameters of the characteristic to be measured (x_j) is known. The value of these parameters (μ_x and ϕ_x) are obtained by external information as is pointed out by Lu et al. (1997).

Usually, it is considered $p > 2$ instruments. In the case $p = 2$ instruments the model (1.1) corresponds to the structural measurement error model and additional restrictions are needed to deal with the identifiability of the model (see Fuller (1987)). In the case of $p = 3$ instruments, Barnett (1969) derived the maximum likelihood estimators for the parameters explicitly. However, it is possible that some estimates of the variances are negative. Carter (1981) considered the derivation of nonnegative variance estimates in this case. For $p > 3$ instruments, no explicit form are available for the maximum likelihood estimators of the unknown parameters as was pointed out by Barnett (1969). Bolfarine and Galea-Rojas (1995a) obtained the maximum likelihood estimates using the EM-algorithm for $p > 3$ instruments. Moreover, they considered a study of inference using the Wald statistics. However, as can be seen in the simulation study presented in Section 5, the Wald test statistics may be liberal, i.e., the null rejection rates may be greater than the nominal level of the test.

An important concept in the comparative calibration problem, which is used to assess the relative calibration and relative accuracy of a set of p instruments are the reliability and precision of the instruments (see for example, Shyr and Gleser (1986), Bolfarine and Galea-Rojas (1995a)). These quantities are defined respectively as

$$\rho_i = \frac{\phi_x \beta_i^2}{\phi_x \beta_i^2 + \phi_i} \quad \text{and} \quad \pi_i = \frac{1}{\phi_x \beta_i^2}, \quad i = 1, \cdots, p.$$

Usually in comparative calibration analysis, the problem of comparing different instruments or measurement methods reduce to making inferences about the additive and multiplicative bias, α_i and β_i, and the reliability of the instruments, ρ_i, $i = 1, \cdots, p$. Thus, some hypothesis that may be considered to assess the quality of the instruments are:

i) $H_{01} : \alpha_2 = \cdots = \alpha_p = 0, \quad \beta_2 = \cdots = \beta_p = 1,$

ii) $H_{02} : \alpha_2 = \cdots = \alpha_p = 0, \quad \beta_2 = \cdots = \beta_p = 1, \quad \phi_1 = \cdots = \phi_p,$

iii) $H_{03} : \beta_2 = \cdots = \beta_p = 1, \quad \phi_1 = \cdots = \phi_p,$

iv) $H_{04} : \alpha_2 = \cdots = \alpha_p = 0,$

v) $H_{05} : \beta_2 = \cdots = \beta_p = 1,$

vi) $H_{06} : \phi_1 = \cdots = \phi_p$.

The hypothesi H_{01} means that the measurements of the instruments are without bias (multiplicative and additive), while the hypothesi H_{04} (H_{05}) means that the instruments measure without additive (multiplicative) bias and hypothesi H_{02} means that besides the instruments measurements are without bias, they are equally reliable. Notice that the null hypothesis H_{02} is equivalent to $H'_{02} : \alpha_2 = ... = \alpha_p = 0, \quad \beta_2 = ... = \beta_p = 1, \quad \rho_1 = ... = \rho_p$, which is easier to interpret, but on the other hand, it is more difficult to obtain the likelihood ratio statistics and Score statistics. For that reason, we are going to use the hypothesis H_{02} which is easier to implement. Finally, the Hypothesi H_{03} means that the instruments measure without multiplicative bias and they are equally reliable.

Bolfarine and Galea-Rojas (1995b) considered an equivalent factor analysis version of the Barnett's model presented in Theobald and Mallinson (1978). This Model can be represented as

$$Y_{ij} = \mu_i + \lambda_i F_j + e_{ij}, \tag{1.3}$$

where λ_i denote the unknown calibration factors, F_j and e_{ij} are all mutually independent, with $F_j \sim N(0,1)$ and $e_{ij} \sim N(0, \phi_i)$, $i = 1, ..., p$ and $j = 1, ..., n$. Without loss of generality we can admit that the first instrument is the reference instrument, in this case, the relationship between the models defined in (1.1) and (1.3), can be represented as

$$\mu_x = \mu_1, \quad \phi_x = \lambda_1^2, \quad \beta_i = \frac{\lambda_i}{\lambda_1}, \quad \alpha_i = \mu_i - \beta_i \mu_1,$$

$i = 2, ..., p$. From the last relationship, we have that $\mu_i = \alpha_i + \beta_i \mu_x$.

In the literature, the study of the hypothesis testing for the model (1.1) has been considered based only on the Wald statistics (see Bolfarine and Galea-Rojas (1995a)). In the particular case where $p = 2$, i.e., the measurement error model (Fuller, 1987), Arellano-Valle and Bolfarine (1994) considered the score and likelihood ratio test statistics to test $H_0 : \beta_2 = 1$. Bolfarine and Galea-Rojas (1995b) considered the hypothesis testing using the Wald test statistics and the factor analysis version of the model (1.1) given in (1.3).

In this paper we discuss the problem of hypothesis testing based on these three tests and $p > 2$. Also, we present a small simulation study based on these three tests, as well as different sample sizes, different parameter values and the hypothesis of interest, (i) through (vi), just described. In Section 2 we describe the model and obtain the information matrix in closed form expressions. Also, we present the EM-algorithm to obtain the maximum likelihood estimates under the restrictions imposed by the hypothesis described earlier. In Section 3, we discuss the hypothesis testing and in Section 4 we present the model considering the factor analysis version. Finally, in Section 5 we apply the methodology to the data presented in Chipkevitch et al. (1996) and Barnett (1969) and a small simulation study is presented.

2. THE MODEL

The model defined by (1.1) with the restriction given in (1.2) may be written as

$$\mathbf{Y}_j = \mathbf{a} + \mathbf{b}x_j + \mathbf{e}_j,$$

where $\mathbf{Y}_j = (Y_{1j}, \cdots, Y_{pj})^\top$, $\mathbf{a} = (0, \alpha_2, \cdots, \alpha_p)^\top$, $\mathbf{b} = (1, \beta_2, \cdots, \beta_p)^\top$ and $\mathbf{e}_j = (e_{1j}, \cdots, e_{pj})^\top$, $j = 1, \cdots, n$. Thus, from (2.1), we have that $\boldsymbol{Y}_1, \cdots, \boldsymbol{Y}_n$ are independent and identically distributed with $\boldsymbol{Y}_j \sim N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $\boldsymbol{\mu} = E(\boldsymbol{Y}_j) = \mathbf{a} + \mathbf{b}\mu_x$ and $\boldsymbol{\Sigma} = Var(\boldsymbol{Y}_j) = \phi_x \mathbf{b}\mathbf{b}^\top + D(\boldsymbol{\phi})$, with $D(\boldsymbol{\phi})$ denoting the diagonal matrix with the diagonal elements given by $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_p)^\top$. By letting $\boldsymbol{\alpha} = (\alpha_2, \ldots, \alpha_p)^\top$ and $\boldsymbol{\beta} = (\beta_2, \ldots, \beta_p)^\top$, the model parameters are given by

$$\boldsymbol{\theta} = (\mu_x, \boldsymbol{\alpha}^\top, \boldsymbol{\beta}^\top, \phi_x, \boldsymbol{\phi}^\top)^\top,$$

and the log-likelihood function may be written as

$$l(\boldsymbol{\theta}) = \sum_{j=1}^{n} l_j(\boldsymbol{\theta}) = \sum_{j=1}^{n} \log f(\mathbf{Y}_j, \boldsymbol{\theta}), \qquad (2.1)$$

where $l_j(\boldsymbol{\theta}) = -\frac{p}{2}\log 2\pi - \frac{1}{2}\log|\boldsymbol{\Sigma}| - \frac{1}{2}d_j(\boldsymbol{\theta})$, with $d_j(\boldsymbol{\theta}) = (\mathbf{Y}_j - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\mathbf{Y}_j - \boldsymbol{\mu})$.

2.1 INFORMATION MATRIX

In this section, we present the score function and the information matrix required for the implementation of the Wald and Score statistics.

The score function, which is the derivative of the likelihood function $l(\boldsymbol{\theta})$ with respect to $\boldsymbol{\theta}$ is given by

$$U(\boldsymbol{\theta}) = \frac{\partial l(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \sum_{j=1}^{n} U_j(\boldsymbol{\theta}), \tag{2.1.1}$$

where

$$U_j(\boldsymbol{\theta}) = \frac{\partial l_j(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = -\frac{1}{2}[\frac{\partial \log|\boldsymbol{\Sigma}|}{\partial \boldsymbol{\theta}} + \frac{\partial d_j(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}],$$

with $d_j(\boldsymbol{\theta})$ as given in (2.1). In Appendix A, we present the derivatives of $\log|\boldsymbol{\Sigma}|$ and $d_j(\boldsymbol{\theta})$ with respect to the components of $\boldsymbol{\theta}$ ($\mu_x, \boldsymbol{\alpha}, \boldsymbol{\beta}, \phi_x$ and $\boldsymbol{\phi}$).

From (2.1.1) it follows that the information matrix denoted by $I_F(\boldsymbol{\theta})$ is given by

$$I_F(\boldsymbol{\theta}) = \begin{pmatrix} I_{\mu_x\mu_x} & I_{\mu_x\boldsymbol{\alpha}} & I_{\mu_x\boldsymbol{\beta}} & 0 & \mathbf{0} \\ I_{\boldsymbol{\alpha}\mu_x} & I_{\boldsymbol{\alpha}\boldsymbol{\alpha}} & I_{\boldsymbol{\alpha}\boldsymbol{\beta}} & \mathbf{0} & \mathbf{0} \\ I_{\boldsymbol{\beta}\mu_x} & I_{\boldsymbol{\beta}\boldsymbol{\alpha}} & I_{\boldsymbol{\beta}\boldsymbol{\beta}} & I_{\boldsymbol{\beta}\phi_x} & I_{\boldsymbol{\beta}\boldsymbol{\phi}} \\ 0 & \mathbf{0} & I_{\phi_x\boldsymbol{\beta}} & I_{\phi_x\phi_x} & I_{\phi_x\boldsymbol{\phi}} \\ \mathbf{0} & \mathbf{0} & I_{\boldsymbol{\phi}\boldsymbol{\beta}} & I_{\boldsymbol{\phi}\phi_x} & I_{\boldsymbol{\phi}\boldsymbol{\phi}} \end{pmatrix}, \tag{2.1.2}$$

with the components given by

$$\begin{aligned}
I_{\mu_x\mu_x} &= \frac{(c-1)c^{-1}}{\phi_x}, \quad I_{\mu_x\boldsymbol{\alpha}} = c^{-1}\boldsymbol{\beta}^\top D^{-1}(\boldsymbol{\psi}), \quad I_{\mu_x\boldsymbol{\beta}} = c^{-1}\mu_x\boldsymbol{\beta}^\top D^{-1}(\boldsymbol{\psi}), \\
I_{\boldsymbol{\alpha}\boldsymbol{\alpha}} &= D^{-1}(\boldsymbol{\psi}) - \frac{\phi_x}{c}D^{-1}(\boldsymbol{\psi})\boldsymbol{\beta}\boldsymbol{\beta}^\top D^{-1}(\boldsymbol{\psi}), \quad I_{\boldsymbol{\alpha}\boldsymbol{\beta}} = \mu_x I_{\boldsymbol{\alpha}\boldsymbol{\alpha}}, \\
I_{\boldsymbol{\beta}\boldsymbol{\beta}} &= (\mu_x^2 + \phi_x - \frac{\phi_x}{c})D^{-1}(\boldsymbol{\psi}) + \frac{\phi_x}{c}(\frac{2\phi_x}{c} - \mu_x^2 - \phi_x)D^{-1}(\boldsymbol{\psi})\boldsymbol{\beta}\boldsymbol{\beta}^\top D^{-1}(\boldsymbol{\psi}), \\
I_{\boldsymbol{\beta}\phi_x} &= (c-1)c^{-2}D^{-1}(\boldsymbol{\psi})\boldsymbol{\beta}, \\
I_{\boldsymbol{\beta}\boldsymbol{\phi}} &= \frac{\phi_x}{c}D^{-2}(\boldsymbol{\psi})D(\boldsymbol{\beta})\mathbb{I}_{(p)} - \frac{\phi_x^2}{c^2}D^{-1}(\boldsymbol{\psi})\boldsymbol{\beta}\mathbf{b}^\top D(\mathbf{b})D^{-2}(\boldsymbol{\phi}), \\
I_{\phi_x\phi_x} &= \frac{c^{-2}}{2\phi_x^2}(c-1)^2, \quad I_{\phi_x\boldsymbol{\phi}} = \frac{c^{-2}}{2}\mathbf{b}^\top D(\mathbf{b})D^{-2}(\boldsymbol{\phi}), \\
I_{\boldsymbol{\phi}\boldsymbol{\phi}} &= \frac{1}{2}D^{-2}(\boldsymbol{\phi}) - \frac{\phi_x}{c}D^2(\mathbf{b})D^{-3}(\boldsymbol{\phi}) + \frac{\phi_x^2}{2c^2}D^{-2}(\boldsymbol{\phi})D(\mathbf{b})\mathbf{b}\mathbf{b}^\top D(\mathbf{b})D^{-2}(\boldsymbol{\phi}),
\end{aligned}$$

with $\boldsymbol{\psi} = (\phi_2, \ldots, \phi_p)^\top$, $\mathbb{I}_{(p)} = [\mathbf{0}, \mathbf{I}_{p-1}]$ and $c = 1 + \phi_x\mathbf{b}^\top D^{-1}(\boldsymbol{\phi})\mathbf{b}$.

2.2 MAXIMUM LIKELIHOOD ESTIMATION

In this Section we present the EM algorithm to obtain the maximum likelihood estimates of the parameters under different restrictions. Bolfarine and Galea (1995a) presented the EM algorithm for the obtention of the maximum likelihood estimates of the parameters under the restriction that $\alpha_1 = 0$ and $\beta_1 = 1$. Considering this restriction and the hypothesis described in Section 1, namely H_{0i}, $i = 1, ..., 6$, we obtained the EM algorithm under the restrictions imposed by the hypothesis in question.

The methodology of EM algorithm consists in the augmentation of the observed data $\mathbf{Y} = (\mathbf{Y}_1^\top, \ldots, \mathbf{Y}_n^\top)^\top$ by some unobserved data, $\mathbf{x} = (x_1, \ldots, x_n)^\top$, in a way that the maximum likelihood estimator (MLE) of the parameters based on the augmented data, $\mathbf{Z} = (\mathbf{Y}^\top, x)^\top$ is easy to obtain. Given the estimates of $\boldsymbol{\theta}$ in the mth iteration, $\boldsymbol{\theta}^{(m)}$, the E step consists in the obtention of the expectation of the complete data log-likelihood, $l(\boldsymbol{\theta}/\mathbf{Z})$, with respect to the conditional distribution of x given $\mathbf{Y}$ and $\boldsymbol{\theta}^{(m)}$. The M step consists in the maximization of the function obtained in the E step with respect to $\boldsymbol{\theta}$, which gives the estimates of the parameters at the next iteration, $\boldsymbol{\theta}^{(m+1)}$. Each iteration of the EM algorithm increments the log-likelihood function of the observed data $l(\boldsymbol{\theta}/\mathbf{Y})$, i.e., $l(\boldsymbol{\theta}^{(m)}/\mathbf{Y}) \leq l(\boldsymbol{\theta}^{(m+1)}/\mathbf{Y})$. When the likelihood function of the complete data belongs to the exponential family, the implementation of the EM algorithm is usually simple. In our case, the E step consists in the obtention of $E(x_j/\mathbf{Y})$ and $E(x_j^2/\mathbf{Y})$, $j = 1, \cdots, n$. In the M step, we maximize the log-likelihood function of the complete data where the values of the sufficient statistics were substituted by the expected values obtained in the E step. Next, we are going to present the log-likelihood function of the complete data given by $\boldsymbol{Z}_j = (x_j, \mathbf{Y}_j^\top)^\top$, $j = 1, \cdots, n$, in order to implement the EM algorithm. Considering (1.1), we have $\boldsymbol{Z}_j \sim N_{p+1}(\boldsymbol{\mu}_z, \boldsymbol{\Sigma}_z), j = 1, \ldots, n,$ with

$$\boldsymbol{\mu}_z = \begin{pmatrix} \mu_x \\ \boldsymbol{\mu} \end{pmatrix} \quad \text{and} \quad \boldsymbol{\Sigma}_z = \begin{pmatrix} \phi_x & \mathbf{b}^\top \phi_x \\ \mathbf{b}\phi_x & \phi_x \mathbf{b}\mathbf{b}^\top + D(\boldsymbol{\phi}) \end{pmatrix},$$

which gives the following log-likelihood function

$$l(\boldsymbol{\theta}/\mathbf{Z}) = -\frac{p+1}{2} \log 2\pi - \frac{n}{2} \log |\boldsymbol{\Sigma}_z| - \frac{1}{2} \sum_{j=1}^{n} (\mathbf{Z}_j - \boldsymbol{\mu}_z)^\top \boldsymbol{\Sigma}_z^{-1} (\mathbf{Z}_j - \boldsymbol{\mu}_z), \qquad (2.2.1)$$

where

$$|\Sigma_z| = \phi_x \prod_{i=1}^{p} \phi_i \quad \text{and} \quad \Sigma_z^{-1} = \begin{pmatrix} c/\phi_x & -\mathbf{b}^\top D^{-1}(\phi) \\ -D^{-1}(\phi)\mathbf{b} & D^{-1}(\phi) \end{pmatrix}, \text{ with } c = 1 + \phi_x \mathbf{b}^\top D^{-1}(\phi)\mathbf{b}.$$

Considering the properties of the multivariate normal distribution, the E step consists in the obtention of

$$\begin{aligned} \widehat{x}_j^{(m)} &= E(x_j/\mathbf{Y}_j, \boldsymbol{\theta}^{(m-1)}) \\ &= \mu_x^{(m-1)} + \frac{\phi_x^{(m-1)}}{c^{(m-1)}} \mathbf{b}^{(m-1)^\top} D^{-1}(\phi)^{(m-1)}(\boldsymbol{Y}_j - \boldsymbol{a}^{(m-1)} - \mu_x^{(m-1)}\boldsymbol{b}^{(m-1)}) \end{aligned}$$

and

$$\widehat{x_j^2}^{(m)} = Var(x_j/\mathbf{Y}_j, \boldsymbol{\theta}^{(m-1)}) + (E(x_j/\mathbf{Y}_j, \boldsymbol{\theta}^{(m-1)}))^2 = \frac{\phi_x^{(m-1)}}{c^{(m-1)}} + \left(\hat{x}_j^{(m)}\right)^2.$$

Notice that the complete log-likelihood function and also the estimates of x_j and x_j^2 depend on the values of $\boldsymbol{a}$, $\mathbf{b}$ and c. For the case where there are no additional restrictions, the estimates of the parameters in the M step is given in Bolfarine and Galea-Rojas (1995a). Next, we present the estimates of the parameters in the M step under the null hypothesis H_{0i}, $i = 1, 4, 5$ and 6. Under the null hypothesis given in H_{02} and H_{03}, we do not need to use the EM algorithm, as it is possible to find the restricted estimators of the parameters explicitly. First, we present the values of $\mathbf{a}$, $\mathbf{b}$ and c, for each hypothesis in Table 1.

Table 1: The values of $\boldsymbol{a}$, $\boldsymbol{b}$ and c under different null hypothesis

Null hypothesis	$\boldsymbol{a}$	$\boldsymbol{b}$	c
$H_{01} : \alpha_2 = \cdots = \alpha_p = 0; \beta_2 = \cdots = \beta_p = 1$	$\mathbf{0}_p$	$\mathbf{1}_p$	$1 + \phi_x \mathbf{1}_p^\top D^{-1}(\boldsymbol{\phi})\mathbf{1}_p$
$H_{04} : \alpha_2 = \cdots = \alpha_p = 0$	$\mathbf{0}_p$	$(1, \boldsymbol{\beta}^\top)^\top$	$1 + \phi_x \boldsymbol{b}^\top D^{-1}(\boldsymbol{\phi})\boldsymbol{b}$
$H_{05} : \beta_2 = \cdots = \beta_p = 1$	$(0, \boldsymbol{\alpha}^\top)^\top$	$\mathbf{1}_p^\top$	$1 + \phi_x \mathbf{1}_p^\top D^{-1}(\boldsymbol{\phi})\mathbf{1}_p$
$H_{06} : \phi_1 = \cdots = \phi_p = \phi$	$(0, \boldsymbol{\alpha}^\top)^\top$	$(1, \boldsymbol{\beta}^\top)^\top$	$1 + \frac{\phi_x}{\phi}\boldsymbol{b}^\top\boldsymbol{b}$

Next, we obtain the estimates of the parameters in the M step for each null hypothesis, considering the expressions given in Table 1 and the values of $\hat{x}_j^{(m)}$ and $\widehat{x_j^2}^{(m)}$ with the values of $\boldsymbol{a}$, $\boldsymbol{b}$ and c, substituted by their respective values given in Table 1.

Under H_{0i}, $i=1,4,5,6$ the estimates of the parameters in the M step is given by

1) Estimates under H_{01}

$$\widetilde{\mu}_x^{(m+1)} = \widehat{\overline{x}}^{(m)}, \quad \widetilde{\phi}_x^{(m+1)} = \frac{\sum_{j=1}^n \widehat{x_j^2}^{(m)}}{n} - (\widehat{\overline{x}}^{(m)})^2 \text{ and}$$

$$\widetilde{\phi}_i^{(m+1)} = \frac{1}{n}\left(\sum_{j=1}^n Y_{ij}^2 - 2\sum_{j=1}^n Y_{ij}\hat{x}_j^{(m)} + \sum_{j=1}^n \widehat{x_j^2}^{(m)}\right), \quad i=1,\cdots,p,$$

with $\widehat{\overline{x}}^{(m)} = \frac{\sum_{j=1}^n \hat{x}_j^{(m)}}{n}$.

2) Estimates under H_{04}

$$\widetilde{\mu}_x^{(m+1)} = \widehat{\overline{x}}^{(m)}, \quad \widetilde{\boldsymbol{\beta}}^{(m+1)} = \frac{\sum_{j=1}^n \hat{x}_j^{(m)} \boldsymbol{Y}_{2j}}{\sum_{j=1}^n \widehat{x_j^2}^{(m)}}, \quad \widetilde{\phi}_x^{(m+1)} = \frac{\sum_{j=1}^n \widehat{x_j^2}^{(m)}}{n} - (\widehat{\overline{x}}^{(m)})^2 \text{ and}$$

$$\widetilde{\phi}_i^{(m+1)} = \frac{1}{n}\left(\sum_{j=1}^n Y_{ij}^2 - 2\widetilde{\beta}_i^{(m+1)}\sum_{j=1}^n Y_{ij}\hat{x}_j^{(m)} + (\widetilde{\beta}_i^{(m+1)})^2\sum_{j=1}^n \widehat{x_j^2}^{(m)}\right), \quad i=1,\cdots,p,$$

with $\widetilde{\beta}_1^{(m+1)} = 1$ and $\boldsymbol{Y}_{2j} = (Y_{2j},\cdots,Y_{pj})^\top$.

3) Estimates under H_{05}

$$\widetilde{\mu}_x^{(m+1)} = \widehat{\overline{x}}^{(m)}, \quad \widetilde{\alpha}_i^{(m+1)} = \overline{Y}_{i.} - \widehat{\overline{x}}^{(m)}, \quad \widetilde{\phi}_x^{(m+1)} = \frac{\sum_{j=1}^n \widehat{x_j^2}^{(m)}}{n} - (\widehat{\overline{x}}^{(m)})^2 \text{ and}$$

$$\widetilde{\phi}_i^{(m+1)} = \frac{1}{n}\left(\sum_{j=1}^n (Y_{ij} - \widetilde{\alpha}_i^{(m+1)})^2 - 2\sum_{j=1}^n \hat{x}_j^{(m)}(Y_{ij} - \widetilde{\alpha}_i^{(m+1)}) + \sum_{j=1}^n \widehat{x_j^2}^{(m)}\right),$$

with $\widetilde{\alpha}_1^{(m+1)} = 0$ and $\overline{Y}_{i.} = \frac{1}{n}\sum_{j=1}^n Y_{ij}, \quad i=2,\ldots,p.$

4) Estimates under H_{06}

$$\widetilde{\mu}_x^{(m+1)} = \widehat{\overline{x}}^{(m)}, \quad \widetilde{\boldsymbol{\beta}}^{(m+1)} = \frac{\sum_{j=1}^n (\widehat{x}_j^{(m)} - \widehat{\overline{x}}^{(m)})(\boldsymbol{Y}_{2j} - \overline{\boldsymbol{Y}}_2)}{\sum_{j=1}^n \widehat{x_j^2}^{(m)} - n(\widehat{\overline{x}}^{(m)})^2}, \quad \widetilde{\boldsymbol{\alpha}}^{(m+1)} = \overline{\boldsymbol{Y}}_2 - \widetilde{\boldsymbol{\beta}}^{(m+1)}\widehat{\overline{x}}^{(m)},$$

$$\widetilde{\phi}_x^{(m+1)} = \frac{\sum_{j=1}^n \widehat{x_j^2}^{(m)}}{n} - (\widehat{\overline{x}}^{(m)})^2 \text{ and}$$

$$\widetilde{\phi}^{(m+1)} = \frac{1}{np}\left(\sum_{j=1}^n (\boldsymbol{Y}_j - \widetilde{\boldsymbol{a}}^{(m+1)})^\top(\boldsymbol{Y}_j - \widetilde{\boldsymbol{a}}^{(m+1)}) - 2\sum_{j=1}^n \hat{x}_j^{(m)}(\widetilde{\boldsymbol{b}}^{(m+1)})^\top(\boldsymbol{Y}_j - \widetilde{\boldsymbol{a}}^{(m+1)}) + \right.$$

$$\left.(\widetilde{\boldsymbol{b}}^{(m+1)})^\top \widetilde{\boldsymbol{b}}^{(m+1)} \sum_{j=1}^{n} \widehat{x_j^2}^{(m)}\right),$$

where $\widetilde{\boldsymbol{a}}^{(m+1)}=(0,(\widetilde{\boldsymbol{\alpha}}^{(m+1)})^\top)^\top$, $\widetilde{\boldsymbol{b}}^{(m+1)}=(0,(\widetilde{\boldsymbol{\beta}}^{(m+1)})^\top)^\top$ and $\overline{\mathbf{Y}}_2=\frac{1}{n}\sum_{j=1}^{n}\mathbf{Y}_{2j}$.

Notice that in all cases, the maximization step is given in closed form expressions, with no need to develop numerical iterative process during the M-step, which makes the algorithm extremely simple and computationally inexpensive.

Moreover, as commented earlier, under the hypothesis H_{02} and H_{03} the restricted estimator of the parameters may be obtained explicitly. These estimates under H_{02} is given by

$$\begin{aligned}
\widetilde{\mu}_x &= \overline{Y}_{..}, \quad \widetilde{\phi}=\frac{1}{n(p-1)}\sum_{j=1}^{n}\sum_{i=1}^{p}(Y_{ij}-\overline{Y}_{.j})^2 \text{ and} \\
\widetilde{\phi}_x &= \frac{1}{n}\sum_{j=1}^{n}(\overline{Y}_{.j}-\overline{Y}_{..})^2-\frac{\widetilde{\phi}}{p}=\frac{1}{np}\sum_{j=1}^{n}\sum_{i=1}^{p}(Y_{ij}-\overline{Y}_{..})^2-\widetilde{\phi},
\end{aligned}$$

where $\overline{Y}_{..}=\frac{1}{np}\sum_{j=1}^{n}\sum_{i=1}^{p}Y_{ij}$ and $\overline{Y}_{.j}=\frac{1}{p}\sum_{i=1}^{p}Y_{ij}$. Observe that in this case the estimate of ϕ_x can be negative. If $\widetilde{\phi}_x<0$, then we can proceed considering $\phi_x=0$ and in this case the estimate of ϕ is given by

$$\widetilde{\phi}=\frac{1}{np}\sum_{j=1}^{n}\sum_{i=1}^{p}(Y_{ij}-\overline{Y}_{..})^2.$$

Finally, under the hypothesis H_{03} the restricted estimator of the parameters may be obtained explicitly, which is given by

$$\widetilde{\mu}_x = \overline{Y}_{1.}, \quad \widetilde{\alpha}_i=\overline{Y}_{i.}-\overline{Y}_{1.}, \quad i=2,\ldots,p, \quad \widetilde{\phi}=\frac{1}{n(p-1)}\sum_{j=1}^{n}\sum_{i=1}^{p}[(Y_{ij}-\overline{Y}_{i.})^2-(\overline{Y}_{.j}-\overline{Y}_{..})^2]$$

and

$$\widetilde{\phi}_x = \frac{1}{n}\sum_{j=1}^{n}(\overline{Y}_{.j}-\overline{Y}_{..})^2-\frac{\widetilde{\phi}}{p}.$$

If $\widetilde{\phi}_x<0$, then we can proceed considering $\phi_x=0$ and in this case the estimate of ϕ is given by

$$\widetilde{\phi}=\frac{1}{np}\sum_{j=1}^{n}\sum_{i=1}^{p}(Y_{ij}-\overline{Y}_{i.})^2.$$

3. HYPOTHESIS TESTING

In this Section we present three asymptotic tests to test the hypothesis H_{0i}, $i = 1, \cdots, 6$, which was discussed in the Introduction. Let's denote by $\widehat{\boldsymbol{\theta}}$ the MLE of the parameters $\boldsymbol{\theta} = (\boldsymbol{\alpha}^T, \boldsymbol{\beta}^T, \boldsymbol{\phi}^T, \mu_x, \phi_x)^T$ and $\widetilde{\boldsymbol{\theta}}_i$ the restricted MLE under the ith hypothesi.

3.1 WALD'S STATISTICS

Observe that the 6 hypothesis of interest given in the Introduction may be rewritten as $H_{0i} : \boldsymbol{A_i}\, \boldsymbol{\theta} = \boldsymbol{q}_{0i}$, where $\boldsymbol{A_i}$ is an adequate $r_i \times 3p$ matrix of rank r_i, with $r_i \leq 3p$, $i = 1, \cdots, 6$. Let $q = p - 1$, $\mathbf{I}_q$ an identity matrix of order q, $\mathbf{1}_q$ an $q \times 1$ vector composed by $1's$ and analogously $\mathbf{0}_q$ an $q \times 1$ vector of 0's. Then, the hypothesis H_{0i}, $i = 1, ..., 6$, may be written as,

1) $\mathbf{A}_1 = [\mathbf{I}_{2q} \quad \mathbf{0}_{2q\times(p+2)}]$ and $\mathbf{q}_{01} = (\mathbf{0}_q^\top, \mathbf{1}_q^\top)^\top$,

2) $\mathbf{A}_2 = \begin{pmatrix} \mathbf{I}_{2q} & \mathbf{0} & \mathbf{0}_{2q\times 2} \\ \mathbf{0} & \mathbf{B}_{q\times p} & \mathbf{0}_{q\times 2} \end{pmatrix}$ and $\mathbf{q}_{02} = (\mathbf{0}_q^\top, \mathbf{1}_q^\top, \mathbf{0}_q^\top)^\top$,

3) $\mathbf{A}_3 = \begin{pmatrix} \mathbf{0}_{q\times q} & \mathbf{I}_q & \mathbf{0} & \mathbf{0}_{q\times 2} \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_{q\times p} & \mathbf{0}_{q\times 2} \end{pmatrix}$ and $\mathbf{q}_{03} = (\mathbf{1}_q^\top, \mathbf{0}_q^\top)^\top$,

4) $\mathbf{A}_4 = [\mathbf{I}_q \quad \mathbf{0}_{q\times(q+p+2)}]$ and $\mathbf{q}_{04} = \mathbf{0}_q$,

5) $\mathbf{A}_5 = [\mathbf{0}_{q\times q} \quad \mathbf{I}_q \quad \mathbf{0}_{q\times(p+2)}]$ and $\mathbf{q}_{05} = \mathbf{1}_q$,

6) $\mathbf{A}_6 = [\mathbf{0}_{q\times 2q} \quad \mathbf{B}_{q\times p} \quad \mathbf{0}_{q\times 2}]$ and $\mathbf{q}_{06} = \mathbf{0}_q$,

with $\mathbf{B} = [\mathbf{1}_q \quad -\mathbf{I}_q]$.

Considering A_i, $i = 1, \cdots, 6$, just described, we can test each of the hypothesis H_{0i} considering the Wald test statistics, which is given by

$$W_{0i} = n[\mathbf{A}_i\widehat{\boldsymbol{\theta}} - \mathbf{q}_{0i}]^\top [\mathbf{A}_i I_F^{-1}(\widehat{\boldsymbol{\theta}})\mathbf{A}_i^\top]^{-1} [\mathbf{A}_i\widehat{\boldsymbol{\theta}} - \mathbf{q}_{0i}],$$

where $I_F^{-1}(\boldsymbol{\theta})$ is the inverse of the observed information matrix given in (2.1.2). So considering the Wald test statistics, we only need to obtain the unrestricted MLE to calculate the test statistics.

3.2 SCORE STATISTICS

In this case, the Score statistics for each hypothesis H_{0i}, $i = 1, ..., 6$, is given by

$$S_{0i} = \frac{1}{n}\widetilde{\boldsymbol{U}}_i^{\top} I_F^{-1}(\widetilde{\boldsymbol{\theta}}_i)\widetilde{\boldsymbol{U}}_i,$$

where $\widetilde{\boldsymbol{U}}_i = U(\widetilde{\boldsymbol{\theta}}_i) = \dfrac{\partial l(\widetilde{\boldsymbol{\theta}}_i)}{\partial \boldsymbol{\theta}}$, $i = 1, ..., 6$ and $U(\boldsymbol{\theta})$ as given in (2.1.1). Notice that in this case we only need to obtain the estimated values under H_0. Depending on the situation, it is easier to obtain the parameter estimates under H_0, rather than the unrestricted parameter estimates. So when this is the case, the Score statistics takes advantage over the Wald's statistics.

3.3 LIKELIHOOD RATIO STATISTICS

The likelihood ratio statistics is given by

$$RV_{0i} = -2[l(\widehat{\boldsymbol{\theta}}) - l(\widetilde{\boldsymbol{\theta}}_i)],$$

for each hypothesis H_{0i}, $i = 1, \cdots, 6$.

Under usual regularity conditions, each of the statistics W_{0i}, S_{0i} and RV_{0i}, $i = 1, \cdots, 6$, has an asymptotic $\chi^2_{r_i}$ distribution under the null hypothesis. So we reject H_{0i} with significance level α, if $W_{0i}(S_{0i}, RV_{0i}) > \chi^2_{(1-\alpha, r_i)}$, where $\chi^2_{(1-\alpha, r_i)}$ is the $100(1-\alpha)\%$ percentile of the chi square distribution with r_i degrees of freedom, $i = 1, ..., 6$.

4. THE FACTOR ANALYSIS VERSION

In this Section, we consider the factor analysis version considered by Theobald and Mallison (1978) defined in (1.3). Under the factor analysis version, Bolfarine and Galea-Rojas (1995b) presented the hypothesis testing based on the Wald statistics, where the estimates of the parameters were obtained by the use of the EM algorithm.

4.1 ESTIMATION AND TESTING HYPOTHESIS ABOUT THE CALIBRATION LINES

Under the parametrization of factor analysis, we can reproduce some of the hypothesis defined as H_{0i}, $i = 1, \cdots, 6$ presented in Section 2 as follows:

1) H_{01} is equivalent to $H'_{01} : \mu_1 = ... = \mu_p, \quad \lambda_1 = ... = \lambda_p$.

2) H_{02} is equivalent to $H'_{02} : \mu_1 = ... = \mu_p, \quad \lambda_1 = ... = \lambda_p, \quad \phi_1 = ... = \phi_p$.

3) H_{03} is equivalent to $H'_{03} : \lambda_1 = ... = \lambda_p, \quad \phi_1 = ... = \phi_p$.

4) H_{05} is equivalent to $H'_{05} : \lambda_1 = ... = \lambda_p$.

The hypothesi H'_{06} is the same as the hypothesi H_{06}. The hypothesi H_{04} does not have a simple representation under the factors analysis version as given before for the other hypothesis. It may be written as $H'_{04} : \mu_2 - \frac{\lambda_2}{\lambda_1}\mu_1 = ... = \mu_p - \frac{\lambda_p}{\lambda_1}\mu_1 = 0$. In this case it is more complicated to test the hypothesi (H'_{04}) considering the Score and the likelihood ratio statistics. An alternative is to test the hypothesi H'_{04} based on the asymptotic distribution of the maximum likelihood estimators of the $\gamma = (\mu_2 - \frac{\lambda_2}{\lambda_1}\mu_1, ..., \mu_p - \frac{\lambda_p}{\lambda_1}\mu_1)^\top$ and then consider the Wald statistics.

As presented in Section 2, the maximum likelihood estimator under the hypothesis H'_{02} and H'_{03} may also be obtained explicitly for the factor analysis version. The restricted maximum likelihood estimates under H'_{02} is given by

$$\begin{aligned} \widetilde{\mu} &= \overline{Y}_{..}, \quad \widetilde{\phi} = \frac{1}{n(p-1)} \sum_{j=1}^{n} \sum_{i=1}^{p} (Y_{ij} - \overline{Y}_{.j})^2 \quad \text{and} \\ \widetilde{\lambda} &= \sqrt{\frac{1}{n} \sum_{j=1}^{n} (\overline{Y}_{.j} - \overline{Y}_{..})^2 - \frac{\widetilde{\phi}}{p}} = \sqrt{\frac{1}{np} \sum_{j=1}^{n} \sum_{i=1}^{p} (Y_{ij} - \overline{Y}_{..})^2 - \widetilde{\phi}}. \end{aligned}$$

Observe that λ^2 can be estimated negatively, in this case it may be considered as $\lambda = 0$ and then consider the maximum likelihood estimator of the parameter ϕ as

$$\widetilde{\phi} = \frac{1}{np} \sum_{j=1}^{n} \sum_{i=1}^{p} (Y_{ij} - \overline{Y}_{..})^2.$$

Under the hypothesi H'_{03} the restricted estimator of the parameters may be obtained explicitly, which is given by

$$\begin{aligned} \widetilde{\boldsymbol{\mu}} &= \overline{\boldsymbol{Y}}, \quad \widetilde{\phi} = \frac{1}{n(p-1)} \sum_{j=1}^{n} \sum_{i=1}^{p} [(Y_{ij} - \overline{Y}_{i.})^2 - (\overline{Y}_{.j} - \overline{Y}_{..})^2] \quad \text{and} \\ \widetilde{\lambda} &= \sqrt{\frac{1}{n} \sum_{j=1}^{n} (\overline{Y}_{.j} - \overline{Y}_{..})^2 - \frac{\widetilde{\phi}}{p}}, \end{aligned}$$

with $\boldsymbol{Y}_j = (Y_{1j}, \cdots, Y_{pj})^\top$, $\overline{\boldsymbol{Y}} = \frac{1}{n}\sum_{j=1}^{n} \boldsymbol{Y}_j$ and $\widetilde{\boldsymbol{\mu}} = (\tilde{\mu}_1, \cdots, \tilde{\mu}_p)^\top$.

Observe that $\tilde{\lambda}$ can also be estimated negatively, in this case we can proceed considering $\lambda = 0$ and the maximum likelihood estimator of the parameter ϕ is given by

$$\widetilde{\phi} = \frac{1}{np}\sum_{j=1}^{n}[(\boldsymbol{Y}_j - \overline{\boldsymbol{Y}})^\top \mathbf{1}_p]^2.$$

For the rest of the hypothesis we are going to consider the EM algorithm and the ECM algorithm (Meng and Rubin (1993)) to obtain the restricted estimative of the parameters.

The estimator of F_j and F_j^2 in the E-step is given in Bolfarine and Galea-Rojas (1995b) as

$$\widehat{F}_j^{(m)} = \frac{1}{c^{(m-1)}}(\boldsymbol{\lambda}^{(m-1)})^\top D^{-1}(\boldsymbol{\phi}^{(m-1)})(\mathbf{Y}_j - \boldsymbol{\mu}^{(m-1)}) \quad \text{and}$$

$$\widehat{F^2}_j^{(m)} = \frac{1}{c^{(m-1)}} + (\hat{F}_j^{(m)})^2,$$

where $c^{(m-1)} = 1 + (\boldsymbol{\lambda}^{(m-1)})^\top D^{-1}(\boldsymbol{\phi}^{(m-1)})\boldsymbol{\lambda}^{(m-1)}$ and $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_p)^\top$

Next, we obtained the restricted estimates of the parameters in the CM-step under the hypothesis H'_{01} and H'_{05} and M-step under the hypothesis H'_{06}. The ECM-algorithm replaces each M-step of the EM-algorithm by a sequence of S conditional maximization steps (CM-steps), each of which maximizes the expectation of the complete data log-likelihood, $l(\boldsymbol{\theta}/\mathbf{Z})$, with respect to the conditional distribution of F given $\mathbf{Y}$ and $\boldsymbol{\theta}^{(m)}$, over $\boldsymbol{\theta}$ but with some vector function of $\boldsymbol{\theta}$, $g_s(\boldsymbol{\theta})$ $(s = 1, \cdots, S)$ fixed at its previous value (see Meng and Rubin (1993)).

1) CM-Step

- Estimates under $H'_{01} : \mu_1 = \cdots = \mu_p$ and $\lambda_1 = \cdots = \lambda_p$

$$\widetilde{\lambda}^{(m+1)} = \frac{\mathbf{1}_p^\top D^{-1}(\widetilde{\boldsymbol{\phi}}^{(m)})\sum_{j=1}^{n}\hat{F}_j^{(m)}(\boldsymbol{Y}_j - \overline{\boldsymbol{Y}})}{\mathbf{1}_p^\top D^{-1}(\widetilde{\boldsymbol{\phi}}^{(m)})\mathbf{1}_p\left(\sum_{j=1}^{n}\widehat{F_j^2}^{(m)} - n(\widehat{\overline{F}}^{(m)})^2\right)},$$

$$\widetilde{\mu}^{(m+1)} = \frac{(\overline{\boldsymbol{Y}} - \widetilde{\lambda}^{(m+1)}\mathbf{1}_p\widehat{\overline{F}}^{(m)})^\top D^{-1}(\widetilde{\boldsymbol{\phi}}^{(m)})\mathbf{1}_p}{\mathbf{1}_p^\top D^{-1}(\widetilde{\boldsymbol{\phi}}^{(m)})\mathbf{1}_p},$$

and

$$\widetilde{\boldsymbol{\phi}}^{(m+1)} = \frac{1}{n}\sum_{j=1}^{n}\Big((\widetilde{\lambda}^{(m+1)})^2\widehat{F_j^2}^{(m)}\mathbf{1}_p - 2\widetilde{\lambda}^{(m+1)}\hat{F}_j^{(m)}(\boldsymbol{Y}_j - \mathbf{1}_p\widetilde{\mu}^{(m+1)}) +$$

$$D(\boldsymbol{Y}_j - \mathbf{1}_p\widetilde{\mu}^{(m+1)})(\boldsymbol{Y}_j - \mathbf{1}_p\widetilde{\mu}^{(m+1)})\Big).$$

- Estimates under $H'_{05} : \lambda_1 = \cdots = \lambda_p$

$$\widetilde{\lambda}^{(m+1)} = \frac{\mathbf{1}_p^\top D^{-1}(\widetilde{\boldsymbol{\phi}}^{(m)})\sum_{j=1}^{n}\hat{F}_j^{(m)}(\boldsymbol{Y}_j - \overline{\boldsymbol{Y}})}{\mathbf{1}_p^\top D^{-1}(\widetilde{\boldsymbol{\phi}}^{(m)})\mathbf{1}_p\left(\sum_{j=1}^{n}\widehat{F_j^2}^{(m)} - n(\widehat{\overline{F}}^{(m)})^2\right)}, \quad \widetilde{\boldsymbol{\mu}}^{(m+1)} = \overline{\boldsymbol{Y}} - \widetilde{\lambda}^{(m+1)}\mathbf{1}_p\widehat{\overline{F}}^{(m)},$$

and

$$\widetilde{\boldsymbol{\phi}}^{(m+1)} = \frac{1}{n}\sum_{j=1}^{n}((\widetilde{\lambda}^{(m+1)})^2\widehat{F_j^2}^{(m)}\mathbf{1}_p - 2\widetilde{\lambda}^{(m+1)}\hat{F}_j^{(m)}(\boldsymbol{Y}_j - \widetilde{\boldsymbol{\mu}}^{(m+1)}) +$$

$$D(\boldsymbol{Y}_j - \widetilde{\boldsymbol{\mu}}^{(m+1)})(\boldsymbol{Y}_j - \widetilde{\boldsymbol{\mu}}^{(m+1)})).$$

2) M-Step

- Estimates under $H'_{06} : \phi_1 = \cdots = \phi_p$

$$\widetilde{\boldsymbol{\lambda}}^{(m+1)} = \frac{\sum_{j=1}^{n}\hat{F}_j^{(m)}(\boldsymbol{Y}_j - \overline{\boldsymbol{Y}})}{\sum_{j=1}^{n}\widehat{F_j^2}^{(m)} - n(\widehat{\overline{F}}^{(m)})^2}, \quad \widetilde{\boldsymbol{\mu}}^{(m+1)} = \overline{\boldsymbol{Y}} - \widetilde{\boldsymbol{\lambda}}^{(m+1)}\widehat{\overline{F}}^{(m)},$$

and

$$\widetilde{\phi}^{(m+1)} = \frac{1}{np}\sum_{j=1}^{n}\Big(\widehat{F_j^2}^{(m)}(\widetilde{\boldsymbol{\lambda}}^{(m+1)})^\top\widetilde{\boldsymbol{\lambda}}^{(m+1)} - 2\hat{F}_j^{(m)}(\widetilde{\boldsymbol{\lambda}}^{(m+1)})^\top(\boldsymbol{Y}_j - \widetilde{\boldsymbol{\mu}}^{(m+1)}) +$$

$$(\boldsymbol{Y}_j - \widetilde{\boldsymbol{\mu}}^{(m+1)})^\top(\boldsymbol{Y}_j - \widetilde{\boldsymbol{\mu}}^{(m+1)})\Big).$$

Next, we are going to present the test statistics for the hypothesis H'_{01}, H'_{02}, H'_{03}, H'_{05} and H'_{06}.

4.1.1 WALD STATISTICS

Let $\boldsymbol{\theta} = (\boldsymbol{\mu}^\top, \boldsymbol{\lambda}^\top, \boldsymbol{\phi}^\top)^\top$ be the vector of parameters under the factor analysis version. Then under usual regularity conditions $\sqrt{n}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \rightarrow^d N_{3p}(\mathbf{0}, \mathbf{I}_F^{-1}(\boldsymbol{\theta}))$, where $\widehat{\boldsymbol{\theta}}$ is the maximum likelihood estimator of $\boldsymbol{\theta}$ and $\mathbf{I}_F(\boldsymbol{\theta})$ is the Fisher information matrix presented in Appendix B. Denoting

$$\boldsymbol{\Omega} = \mathbf{I}_F^{-1}(\boldsymbol{\theta}) = \begin{pmatrix} \boldsymbol{\Omega}_{\mu\mu} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Omega}_{\psi} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\Omega}_{\mu\mu} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Omega}_{\lambda\lambda} & \boldsymbol{\Omega}_{\lambda\phi} \\ \mathbf{0} & \boldsymbol{\Omega}_{\phi\lambda} & \boldsymbol{\Omega}_{\phi\phi} \end{pmatrix}$$

and $\boldsymbol{\psi} = (\boldsymbol{\lambda}^\top, \boldsymbol{\phi}^\top)^\top$, with $\boldsymbol{\lambda}^\top = (\lambda_1, \cdots, \lambda_p)$ and $\boldsymbol{\phi}^\top = (\phi_1, \cdots, \phi_p)$, the Wald statistics for the hypothesis H'_{01}, H'_{02}, H'_{03}, H'_{05} and H'_{06} are, respectively, given by

$$\begin{aligned}
W^*_{01} &= n[\widehat{\boldsymbol{\mu}}^\top \boldsymbol{B}^\top (\boldsymbol{B}\widehat{\boldsymbol{\Omega}}_{\mu\mu}\boldsymbol{B}^\top)^{-1}\boldsymbol{B}\widehat{\boldsymbol{\mu}} + \widehat{\boldsymbol{\lambda}}^\top \boldsymbol{B}^\top (\boldsymbol{B}\widehat{\boldsymbol{\Omega}}_{\lambda\lambda}\boldsymbol{B}^\top)^{-1}\boldsymbol{B}\widehat{\boldsymbol{\lambda}}], \\
W^*_{02} &= n[\widehat{\boldsymbol{\mu}}^\top \boldsymbol{B}^\top (\boldsymbol{B}\widehat{\boldsymbol{\Omega}}_{\mu\mu}\boldsymbol{B}^\top)^{-1}\boldsymbol{B}\widehat{\boldsymbol{\mu}} + \widehat{\boldsymbol{\psi}}^\top \mathbb{B}^\top (\mathbb{B}\widehat{\boldsymbol{\Omega}}_{(\lambda,\phi)}\mathbb{B}^\top)^{-1}\mathbb{B}\widehat{\boldsymbol{\psi}}], \\
W^*_{03} &= n\widehat{\boldsymbol{\psi}}^\top \mathbb{B}^\top (\mathbb{B}\widehat{\boldsymbol{\Omega}}_{(\lambda,\phi)}\mathbb{B}^\top)^{-1}\mathbb{B}\widehat{\boldsymbol{\psi}}, \\
W^*_{05} &= n\widehat{\boldsymbol{\lambda}}^\top \boldsymbol{B}^\top (\boldsymbol{B}\widehat{\boldsymbol{\Omega}}_{\lambda\lambda}\boldsymbol{B}^\top)^{-1}\boldsymbol{B}\widehat{\boldsymbol{\lambda}}, \\
W^*_{06} &= n\widehat{\boldsymbol{\phi}}^\top \boldsymbol{B}^\top (\boldsymbol{B}\widehat{\boldsymbol{\Omega}}_{\phi\phi}\boldsymbol{B}^\top)^{-1}\boldsymbol{B}\widehat{\boldsymbol{\phi}},
\end{aligned}$$

where $\mathbb{B} = \begin{pmatrix} \boldsymbol{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix}$, with $\boldsymbol{B}$ as given in Section (3.1) and $\boldsymbol{\mu}^\top = (\mu_1, \cdots, \mu_p)$.

4.1.2 SCORE STATISTICS

The score statistics for each hypothesis H'_{0i}, $i = 1, 2, 3, 5$ and 6, is given by

$$S^*_{0i} = \frac{1}{n}\widetilde{\boldsymbol{U}}_i^\top I_F^{-1}(\widetilde{\boldsymbol{\theta}}_i)\widetilde{\boldsymbol{U}}_i,$$

with $\widetilde{\boldsymbol{U}}_i = U(\widetilde{\boldsymbol{\theta}}_i) = \sum_{j=1}^n U_j(\widetilde{\boldsymbol{\theta}}_i)$, where $U_j(\boldsymbol{\theta}_i)$ was obtained in Appendix B.

4.1.3 LIKELIHOOD RATIO STATISTICS

The likelihood ratio statistics is given by

$$RV^*_{0i} = -2[l(\widehat{\boldsymbol{\theta}}) - l(\widetilde{\boldsymbol{\theta}}_i)],$$

for each of the hypothesis H'_{01}, H'_{02}, H'_{03}, H'_{05} and H'_{06}, where $l(\boldsymbol{\theta}) = \sum_{j=1}^n log f(\mathbf{Y}_j, \boldsymbol{\theta})$, with $\mathbf{Y}_j = (Y_{1j}, \cdots, Y_{pj})^\top \sim N_p(\boldsymbol{\mu}, \Sigma)$, $\boldsymbol{\mu} = (\mu_1, \cdots, \mu_p)^\top$ and $\boldsymbol{\Sigma} = \boldsymbol{\lambda}\boldsymbol{\lambda}^\top + D(\boldsymbol{\phi})$.

Under usual regularity conditions, each of the statistics W^*_{0i}, S^*_{0i} and RV^*_{0i}, presented above has an asymptotic $\chi^2_{r_i}$ distribution under the null hypothesis. So we reject H_{0i} with significance level α, if $W^*_{0i}(S^*_{0i}, RV^*_{0i}) > \chi^2_{(1-\alpha,r_i)}$, where $\chi^2_{(1-\alpha,r_i)}$ is the $100(1-\alpha)\%$ percentile of the chi square distribution with r_i degrees of freedom. Considering the hypothesis under question, we have that $r_1 = 2(p-1)$, $r_2 = 3(p-1)$, $r_3 = 2(p-1)$, $r_5 = p-1$ e $r_6 = p-1$.

5. APPLICATION AND SIMULATION

In this section we apply the results obtained in the previous sections to the data set presented in Barnett (1969) and Chipkevitch et al. (1996). Barnett (1969) considered a comparison of four combinations of two instruments and two operators to measure the lung capacity in 72 patients. In Table 2 we present the maximum likelihood estimates of the parameters and in Table 3 we present the restricted maximum likelihood estimates of the parameters under the null hypothesis.

Table 2: The estimated values of the parameters with the standard deviations between parenthesis

μ_x	α_2	α_3	α_4	β_2	β_3	β_4
2246.11	-204.46	-528.58	-437.25	1.060	1.192	1.131
(90.08)	(105.66)	(121.88)	(123.27)	(0.045)	(0.052)	(0.052)

ϕ_x	ϕ_1	ϕ_2	ϕ_3	ϕ_4
534042.4	50248.08	19150.75	29235.73	38843.20
(9.71×10^4)	(9607.77)	(5255.33)	(7260.35)	(8272.19)

In Table 4 we show the Wald, Likelihood Ratio and Score test statistics together with the p_values.

Looking at Table 4, we reject all the hypothesis at $\alpha = 5\%$, which means that considering these four combinations of instruments and operators, none o then are measuring without bias (additive, multiplicative or both) and they are not equally reliable. If we consider $\alpha = 1\%$, we do not reject the last hypothesi if we use the Wald or Likelihood Ratio test statistics, which means that these combinations are equally reliable. However, if we consider the Score test statistics we reject this hypothesi.

Table 3: The estimated values of the parameters under the null hypothesis

θ	H_{01}	H_{02}	H_{03}	H_{04}	H_{05}	H_{06}
μ_x	2170.33	2168.16	2246.11	2218.29	2246.11	2246.11
α_2	-	-	-70.42	-	-70.42	-183.58
α_3	-	-	-97.50	-	-97.50	-513.75
α_4	-	-	-143.89	-	-143.89	-430.75
β_2	-	-	-	0.978	-	1.05
β_3	-	-	-	0.980	-	1.19
β_4	-	-	-	0.956	-	1.13
ϕ_x	627813.30	638788	645769.70	655311.80	629064.90	538315.4
ϕ_1	56755.62	42067.94	38446.91	53193.62	49979.14	34849.27
ϕ_2	12765.54	42067.94	38446.91	13163.44	14128.62	34849.27
ϕ_3	45622.65	42067.94	38446.91	45425.38	43830.90	34849.27
ϕ_4	53250.35	42067.94	38446.91	49810.37	46330.41	34849.27

Table 4: The Wald, Likelihood Ratio and Score test statistics.

Hypothesis	Wald	p_value	Likelihood Ratio	p_value	Score	p_value
H_{01}	33.45	0	34.59	0	29.8	0
H_{02}	43.29	0	51.56	0	55.23	0
H_{03}	28.29	0.0001	32.12	0	33.27	0
H_{04}	24.78	0	27.07	0	23.40	0
H_{05}	18.45	0.0004	19.21	0.0002	16.67	0.0008
H_{06}	8.98	0.03	11.19	0.01	12.73	0.005

Chipkevitch (1996) considered the measurements of the testicular size of 42 adolescents obtained based on five methods to calculate the volume of the testicle. The objective was to test the lack of bias and the accuracy of the instruments. In Chipkevitch et al. (1996), the transformed data was considered by taking the cube root of the volume to satisfy the normal assumption of the data. We will also consider the transformed data set.

Table 5 shows the estimated values of the parameters, while Table 6 presents the restricted maximum likelihood estimates of the parameters under the null hypothesis.

Table 5: The estimated values of the parameters with the standard deviations between parenthesis

μ_x	α_2	α_3	α_4	α_5	β_2	β_3	β_4	β_5	ϕ_x
2.10	0.07	0.03	0.03	0.38	0.93	0.97	1.03	0.90	0.12
(0.06)	(0.11)	(0.11)	(0.11)	(0.10)	(0.05)	(0.05)	(0.05)	(0.05)	(0.03)

ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5
0.007	0.008	0.007	0.007	0.006
(1.88×10^{-3})	(2.07×10^{-3})	(1.83×10^{-3})	(1.90×10^{-3})	(1.61×10^{-3})

Table 6: The estimated values of the parameters under the null hypothesis

θ	H_{01}	H_{02}	H_{03}	H_{04}	H_{05}	H_{06}
α_2	-	-	-0.074	-	-0.075	0.068
α_3	-	-	-0.036	-	-0.036	0.030
α_4	-	-	0.100	-	0.100	0.033
α_5	-	-	0.172	-	0.172	0.387
β_2	-	-	-	0.964	-	0.932
β_3	-	-	-	0.982	-	0.968
β_4	-	-	-	1.047	-	1.032
β_5	-	-	-	1.077	-	0.897

θ	H_{01}	H_{02}	H_{03}	H_{04}	H_{05}	H_{06}
μ_x	2.100	2.133	2.100	2.103	2.100	2.100
ϕ_x	0.118	0.113	0.116	0.112	0.115	0.123
ϕ_1	0.005	0.017	0.007	0.007	0.007	0.007
ϕ_2	0.150	0.017	0.007	0.008	0.008	0.007
ϕ_3	0.008	0.017	0.007	0.007	0.006	0.007
ϕ_4	0.020	0.017	0.007	0.007	0.008	0.007
ϕ_5	0.040	0.017	0.007	0.009	0.007	0.007

Table 7 shows the test statistic values considering the Wald, Likelihood Ratio and Score test statistics together with the p_values.

Table 7: The Wald, Likelihood Ratio and Score test statistics.

Hypothesis	Wald	p_value	Likelihood Ratio	p_value	Score	p_value
H_{01}	273.297	0	133.439	0	77.324	0
H_{02}	274.067	0	159.040	0	127.978	0
H_{03}	10.343	0.242	9.011	0.341	8.597	0.377
H_{04}	19.681	0.001	15.776	0.003	14.045	0.007
H_{05}	9.574	0.048	8.528	0.074	8.173	0.085
H_{06}	0.614	0.961	0.591	0.964	0.557	0.967

Analyzing Table 7, we notice that the hypothesis 1,2 and 4 are rejected, while the hypothesis 3 and 6 are not rejected. Considering the hypothesi 5, we reject at $\alpha = 5\%$ if we use the Wald test statistic, meaning that the instruments measure with multiplicative bias. However, if we consider the Likelihood Ratio or Score test statistics we do not reject this hypothesi. Motivated by this result we conducted a small simulation study where we observed that for the sample size in consideration, the hypothesi in question and the estimated parameter values obtained for the Chipkevitch data set, the Wald test statistic is liberal under the nominal level 5%. So the result obtained by the Likelihood Ratio and Score test statistics may be more accurate, i.e., the instruments are measuring without multiplicative bias.

We have conducted a small simulation study considering all the 6 hypothesis to compare the three asymptotic test statistics for different sample sizes and parameter values. Let $\phi_x = 0.123$, $\phi_1 = \cdots, \phi_p = \phi$, with $\phi = 0.007$ and $\phi = 0.05$, μ_x=0.1, μ_x=2 and μ_x=5. For the intercept and slope parameters, we have considered $\alpha_2 = 0.07$, $\alpha_3 = 0.03$, $\alpha_4 = 0.03$, $\alpha_5 = 0.38$, $\beta_2 = 0.93$, $\beta_3 = 0.97$, $\beta_4 = 1.03$ and $\beta_5 = 0.90$. So we have considered $p = 5$ and the parameter values near the estimated values of the parameters of the Chipkevitch data set and some other values. To study the behavior of W_{0i}, S_{0i} and RV_{0i}, $i = 1, \cdots, 6$ for moderate and large sample sizes we generated a thousand samples of sizes 40, 50 and 100 considering the model defined in (2.1). Considering the nominal significance level $\alpha = 5\%$ we obtained the corresponding empirical significance levels.

When we changed the value of the parameter μ_x, keeping all the other parameters fixed, as well as the sample size and the different hypothesis, the conclusion of the simulation study were the same, implying that the value of the parameter μ_x seems not to change the behavior of the test statistics in the cases considered here. The same happened when we considered different values for the parameter ϕ. So we are going to present here only the results of the simulation study for $\mu_x = 2$ and $\phi = 0.007$, which are the values close to the estimated values of Chipkevitch data set. Table 8 shows the empirical significance levels obtained in this simulation study.

Table 8: Empirical significance levels considering the Wald, Likelihood Ratio and Score test statistics.

	$H_{01}: \alpha_2 = \cdots = \alpha_5 = 0;$ $\beta_2 = \cdots = \beta_5 = 1.$			$H_{02}: \alpha_2 = \cdots = \alpha_5 = 0;$ $\beta_2 = \cdots = \beta_5 = 1;$ $\phi_1 = \cdots = \phi_5 = \phi.$			$H_{03}: \beta_2 = \cdots = \beta_5 = 1;$ $\phi_1 = \cdots = \phi_5 = \phi.$		
n	W_{01}	RV_{01}	S_{01}	W_{02}	RV_{02}	S_{02}	W_{03}	RV_{03}	S_{03}
40	11	7.3	4.9	12.1	7.6	5.2	11.1	7.8	6.3
50	11	6.1	4.8	8.8	5.4	4.8	7.1	5.8	5.6
100	5.6	4.5	4.8	5.8	5.3	4.9	6.5	5.8	5.5
	$H_{04}: \alpha_2 = \cdots = \alpha_5 = 0.$			$H_{05}: \beta_2 = \cdots = \beta_5 = 1.$			$H_{06}: \phi_2 = \cdots = \phi_5 = \phi.$		
n	W_{01}	RV_{01}	S_{01}	W_{02}	RV_{02}	S_{02}	W_{03}	RV_{03}	S_{03}
40	9.1	7.0	5.5	9.9	7.5	5.9	9.6	8.2	5.3
50	8.7	5.9	4.2	8.2	5.9	4.0	6.0	6.0	5.4
100	5.7	4.7	4.2	5.1	4.3	4.1	6.0	5.9	5.2

Analyzing the Table 8, we observe that regardless of the hypothesis in question, we conclude that the Wald test statistics is liberal, i.e., it is displaying null rejection rates that are greater then the nominal level of the test even for n=100. For moderate sample sizes the Score test statistics seems better than the other two test statistics. The same conclusion were obtained for the other parameter values that are not shown here.

Considering the factor analysis version, the estimated values and the values of the test statistics are very close to the ones obtained earlier. Table 9 and Table 10 (Table 11 and Table 12), shows respectively, the estimated values of the parameters and the Wald, Likelihood Ratio and Score test statistics for the Barnett (Chipkevitch) data set.

Table 9: The estimated values of the parameters with the standard deviations between parenthesis

μ_1	μ_2	μ_3	μ_4	λ_1	λ_2	λ_3	λ_4
2246.11	2175.69	2148.61	2102.22	730.78	774.39	871.03	826.23
(90.08)	(92.71)	(104.61)	(100.10)	(66.46)	(66.62)	(75.40)	(72.73)

ϕ_1	ϕ_2	ϕ_3	ϕ_4
50248.08	19150.75	29235.73	38843.20
(9607.40)	(5258.81)	(7259.32)	(8269.97)

Table 10: The Wald, Likelihood Ratio and Score test statistics.

Hypothesis	Wald	p_value	Likelihood Ratio	p_value	Score	p_value
H_{01}	34.81	0	34.59	0	29.8	0
H_{02}	44.58	0	51.56	0	55.23	0
H_{03}	29.12	0.0001	32.11	0	33.45	0
H_{05}	19.35	0.0002	19.21	0.0002	16.67	0.0008
H_{06}	8.98	0.03	11.19	0.01	12.73	0.005

So, considering the Barnett data set and the hypothesis of interest we obtained the same result as the one obtained earlier.

Table 11: The estimated values of the parameters with the standard deviations between parenthesis

μ_1	μ_2	μ_3	μ_4	μ_5	λ_1	λ_2	λ_3	λ_4	λ_5
2.10	2.03	2.06	2.20	2.27	0.35	0.33	0.34	0.36	0.32
(0.06)	(0.05)	(0.05)	(0.06)	(0.05)	(0.04)	(0.04)	(0.04)	(0.04)	(0.04)

ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5
0.007	0.008	0.007	0.007	0.006
(1.88×10^{-3})	(2.07×10^{-3})	(1.83×10^{-3})	(1.90×10^{-3})	(1.61×10^{-3})

Table 12: The Wald, Likelihood Ratio and Score test statistics.

Hypothesis	Wald	p_value	Likelihood Ratio	p_value	Score	p_value
H_{01}	268.021	0	133.438	0	77.323	0
H_{02}	268.763	0	159.040	0	127.978	0
H_{03}	8.825	0.357	9.009	0.341	8.669	0.371
H_{05}	8.084	0.088	8.528	0.074	8.173	0.085
H_{06}	0.614	0.961	0.591	0.964	0.554	0.968

Considering the Chipkevitch data set, we obtained the same results as the one obtained earlier considering the model defined in (1.1) except for the hypothesi H'_{05}. In this case, if we use the Wald test statistics we had rejected the hypothesi H'_{05}. On the other hand, if we consider the Likelihood Ratio or the Score test statistics we had not rejected this hypothesi. Based on the simulation study, we have concluded that the result obtained under the Likelihood Ratio or the Score test statistics were more reliable. Considering the factor analysis version, we reached the same conclusion meaning that the instruments measure without multiplicative bias at the significance level $\alpha = 5\%$.

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APPENDIX A: Computing the first derivatives under the Barnett's model

In this section we present the score function which is given by $U(\boldsymbol{\theta}) = \sum_{j=1}^{n} U_j(\boldsymbol{\theta})$, where

$$U_j(\boldsymbol{\theta}) = \frac{\partial \ell_j(\boldsymbol{\theta})}{\partial \gamma} = -\frac{1}{2}\frac{\partial \log|\boldsymbol{\Sigma}|}{\partial \gamma} - \frac{1}{2} d_{j\gamma}, \tag{A.1}$$

where $d_{j\gamma} = \dfrac{\partial d_j(\boldsymbol{\theta})}{\partial \gamma}$, $\gamma = \mu_x, \boldsymbol{\alpha}, \boldsymbol{\beta}, \phi_x, \boldsymbol{\phi}$ and $d_j(\boldsymbol{\theta})$ as in (2.1). Further, using results in Nel (1980) related to vector derivatives it follows that,

$$\begin{aligned}
\frac{\partial \log|\boldsymbol{\Sigma}|}{\partial \gamma} &= 0, \quad \text{for } \gamma = \mu_x, \boldsymbol{\alpha}, \quad \frac{\partial \log|\boldsymbol{\Sigma}|}{\partial \boldsymbol{\beta}} = 2\frac{\phi_x}{c} D^{-1}(\boldsymbol{\psi})\boldsymbol{\beta}, \\
\frac{\partial \log|\boldsymbol{\Sigma}|}{\partial \phi_x} &= \frac{1}{c}\frac{c-1}{\phi_x}, \quad \frac{\partial \log|\boldsymbol{\Sigma}|}{\partial \boldsymbol{\phi}} = -\frac{\phi_x}{c} D(\mathbf{b}) D^{-2}(\boldsymbol{\phi})\mathbf{b} + D^{-1}(\boldsymbol{\phi})\mathbf{1}_p,
\end{aligned}$$

$$\begin{aligned}
d_{j\mu_x} &= -2\mathbf{b}^\top \boldsymbol{\Sigma}^{-1}\boldsymbol{W}_j, \quad d_{j\boldsymbol{\alpha}} = -2\mathbf{I}_{(p)}\boldsymbol{\Sigma}^{-1}\boldsymbol{W}_j, \\
d_{j\boldsymbol{\beta}} &= -2q_j D^{-1}(\boldsymbol{\psi})\boldsymbol{W}_{2j} + 2\frac{\phi_x}{c} a_j q_j D^{-1}(\boldsymbol{\psi})\boldsymbol{\beta}, \quad d_{j\phi_x} = -c^{-2}a_j^2, \\
d_{j\boldsymbol{\phi}} &= -D^{-2}(\boldsymbol{\phi})D(\mathbf{W}_j)\mathbf{W}_j - \frac{\phi_x^2}{c^2}a_j^2 D(\mathbf{b})D^{-2}(\boldsymbol{\phi})\mathbf{b} + 2\frac{\phi_x}{c}a_j D(\mathbf{b})D^{-2}(\boldsymbol{\phi})\mathbf{W}_j,
\end{aligned}$$

where $\boldsymbol{\psi} = (\phi_2, \ldots, \phi_p)^\top$, $q_j = \mu_x + c^{-1}\phi_x a_j$, with $a_j = \boldsymbol{W}_j^\top D^{-1}(\boldsymbol{\phi})\boldsymbol{b}$, $\boldsymbol{W}_j = \boldsymbol{Y}_j - \boldsymbol{\mu}$, $\boldsymbol{W}_{2j} = \boldsymbol{Y}_{2j} - \boldsymbol{\alpha} - \boldsymbol{\beta}\mu_x$, $\boldsymbol{Y}_{2j}=(y_{2j}, \ldots, y_{pj})^\top$ and $\mathbf{b}$, c as defined in Section 2.

APPENDIX B: Computing the first derivatives under factor analysis

In this appendix we obtained the score function under the parametrization of factor analysis. In This case

$$U_j(\boldsymbol{\theta}) = \frac{\partial l_j(\boldsymbol{\theta})}{\partial \gamma} = -\frac{1}{2}\frac{\partial \log|\boldsymbol{\Sigma}|}{\partial \gamma} - \frac{1}{2} d_{j\gamma}, \tag{B.1}$$

where $d_{j\gamma} = \dfrac{\partial d_j(\boldsymbol{\theta})}{\partial \gamma}, \gamma = \boldsymbol{\mu}, \boldsymbol{\lambda}, \boldsymbol{\phi}$ and $d_j(\boldsymbol{\theta}) = \boldsymbol{W}_j^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{W}_j$, with $\boldsymbol{W}_j = \boldsymbol{Y}_j - \boldsymbol{\mu}$ and $\boldsymbol{\Sigma} = \boldsymbol{\lambda}\boldsymbol{\lambda}^\top + D(\boldsymbol{\phi})$. Thus, it follows that

$$\frac{\partial \log|\boldsymbol{\Sigma}|}{\partial \boldsymbol{\mu}} = 0, \frac{\partial \log|\boldsymbol{\Sigma}|}{\partial \boldsymbol{\lambda}} = 2c^{-1}D^{-1}(\boldsymbol{\phi})\boldsymbol{\lambda}, \quad \frac{\partial \log|\boldsymbol{\Sigma}|}{\partial \boldsymbol{\phi}} = D^{-1}(\boldsymbol{\phi})\mathbf{1}_p - c^{-1}D^{-2}(\boldsymbol{\phi})D(\boldsymbol{\lambda})\boldsymbol{\lambda},$$

$$\begin{aligned} d_{j\mu} &= -2\boldsymbol{\Sigma}^{-1}\boldsymbol{W}_j, \\ d_{j\lambda} &= -2c^{-1}D^{-1}(\boldsymbol{\phi})\boldsymbol{W}_j\boldsymbol{W}_j^\top D^{-1}(\boldsymbol{\phi})\boldsymbol{\lambda} + 2c^{-2}c_{j1}D^{-1}(\boldsymbol{\phi})\boldsymbol{\lambda}, \\ d_{j\phi} &= -D^{-2}(\boldsymbol{\phi})D(\boldsymbol{W}_j)\boldsymbol{W}_j + 2c^{-1}c_{j2}D^{-2}(\boldsymbol{\phi})D(\boldsymbol{\lambda})\boldsymbol{W}_j - c^{-2}c_{j1}D^{-2}(\boldsymbol{\phi})D(\boldsymbol{\lambda})\boldsymbol{\lambda}. \end{aligned}$$

where c as given in Section 4, $c_{j1} = \boldsymbol{W}_j^\top \boldsymbol{M}\boldsymbol{W}_j$ and $c_{j2} = \boldsymbol{W}_j^\top D^{-1}(\boldsymbol{\phi})\boldsymbol{\lambda}$, with $\boldsymbol{M} = D^{-1}(\boldsymbol{\phi})\boldsymbol{\lambda}\boldsymbol{\lambda}^\top D^{-1}(\boldsymbol{\phi})$.

The Information Matrix is given by $I_F(\boldsymbol{\theta}) = \begin{pmatrix} I_{\boldsymbol{\mu}\boldsymbol{\mu}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{\boldsymbol{\lambda}\boldsymbol{\lambda}} & I_{\boldsymbol{\lambda}\boldsymbol{\phi}} \\ \mathbf{0} & I_{\boldsymbol{\phi}\boldsymbol{\lambda}} & I_{\boldsymbol{\phi}\boldsymbol{\phi}} \end{pmatrix}$, with $I_{\boldsymbol{\mu}\boldsymbol{\mu}} = \boldsymbol{\Sigma}^{-1}$; $I_{\boldsymbol{\lambda}\boldsymbol{\lambda}} = \frac{1}{c}\left[(c-1)D^{-1}(\boldsymbol{\phi}) + (\frac{2}{c} - 1)M\right]$; $I_{\boldsymbol{\lambda}\boldsymbol{\phi}} = I_{\boldsymbol{\phi}\boldsymbol{\lambda}}^\top = \frac{1}{c}\left[D^{-2}(\boldsymbol{\phi})D(\boldsymbol{\lambda}) - \frac{1}{c}\boldsymbol{M}D^{-1}(\boldsymbol{\phi})D(\boldsymbol{\lambda})\right]$ and $I_{\boldsymbol{\phi}\boldsymbol{\phi}} = \frac{1}{2}\left[D^{-2}(\boldsymbol{\phi}) - \frac{2}{c}D^{-3}(\boldsymbol{\phi})D(\boldsymbol{\lambda})D(\boldsymbol{\lambda}) + \frac{1}{c^2}D^{-1}(\boldsymbol{\phi})D(\boldsymbol{\lambda})\boldsymbol{M}D(\boldsymbol{\lambda})D^{-1}(\boldsymbol{\phi})\right]$.

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