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BAYESIAN AND MAXIMUM LIKELIHOOD APPROACHES FOR ARCH MODELS CONSIDERING BRAZILIAN FINANCIAL SERIES

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Abstract

The purpose of this work is to approach the inference problem on the parameters for autoregressive conditional heteroscedasticity (ARCH) models. The estimates are obtained by using both Maximum Likelihood (ML) estimation and Bayesian estimation. In the ML estimation we get confidence intervals by using the Bootstrap simulation technique. In the Bayesian estimation we present a reparametrization of model which allows us to use prior Normal densities to transformed parameters. The posterior estimates are obtained using Monte Carlo Markov Chain (MCMC) methods. The methodology is exemplified considering two Brazilian financial time series: Bovespa Stock Index - IBovespa and Telebras series. The order of each ARCH model is selected by using Bayesian Information Criterion (BIC).

1 Introduction

A simple graphical analysis of financial series reveals that these series present a high volatility for some periods of time. It is fundamental to describe how this volatility changes with time in order to evaluate the risk of investments and the price of options.

The most utilized models to describe the volatility of financial series are the ARCH models proposed by Engle[1]. An ample review of the properties of these models may be found in Bollerslev et al[2] and Degiannakis & Xekalaki[3]. In the ARCH models the volatility in a given instant of time is considered to depend on the past values of the series. The determination of Maximum Likelihood (ML) estimators for the ARCH model parameters demands the maximization of a non-linear function. Engle[1] suggests the use of Newton method as an iterative method for the determination of the ML estimates. This procedure slackens the restrictions imposed on the parameters that ensure stationarity in the covariance. In addition, procedures to identify, adjust and diagnose models, as well as to predict future values of econometric series, use properties from the asymptotic theory. Since the models are very distant from linearity it becomes analytically untreatable. An alternative to deal with these difficulties is to consider the Bayesian approach to these models.

One of the first authors to propose the Bayesian approach to ARCH models was Geweke[4], who considered a particular case of parameterization, enabling the use of non-informative prior densities, and the parameter estimates are obtained by means of the Monte Carlo simulation. A Bayesian approach for GARCH models was proposed by Migon & Mazucheli[5] within the class of dynamic models. Nakatsuma[6] developed a MCMC method for linear regression models with ARMA-GARCH errors, utilizing Normal prior distributions for their parameters. More recently Polasek & Kozumi[7] and Polasek[8] developed a Bayesian approach with hierarchical structure to VAR-VARCH and PAR-ARCH models using MCMC, but they suggest that the computational effort must be studied.

This work presents the comparison between the Bayesian estimates and the ML estimates in the inference of the ARCH process parameters. To compare them, we have used Bovespa Stock Index – IBovespa and Telebras financial series. We have obtained the ML estimated intervals of the model parameters by using the asymptotic properties and by using the Bootstrap simulation. The ML estimates and the confidence intervals, given by the percentiles 2.5% and 97.5% of the estimates sample, are compared to the Bayesian estimates and to the 95% credibility intervals of the posterior distributions generated by the MCMC simulation techniques. The results obtained herein show that the Bayesian approach, besides allowing us to incorporate the prior information about the model parameters to the modeling, it also demonstrates to be robust than the ML approach to estimate the credibility intervals of these parameters. This result is particularly noticed when the order of these models is increased.

2 Bootstrapped-Likelihood Approach for ARCH(q) Models

Since our goal is to model the volatility, we consider a version of original ARCH(q) linear model, as proposed by Engle[1], without additional regressors and the observed process z_t can be described by the model

$$z_t|\Omega_{t-1} \backsim P\left(\mathbf{0}, h_t\right) \tag{1}$$

$$h_t = \alpha_0 + \sum_{j=1}^{q} \alpha_j z_{t-j}^2 \tag{2}$$

where $P(.)$ is a parametric distribution and Ω_{t-1} represents the available information set up to the period t. Let be z_t be an error that satisfies the model

$$z_t = h_t^{\frac{1}{2}} \varepsilon_t \tag{3}$$

with $\{\varepsilon_t, t \geq 0\}$ being independent and identically distributed (i.i.d.) process following a standardized Normal distribution.

The ML estimates for the parameters $\alpha_j, j = 0, 1, \ldots, q$ are are derived from the hypothesis imposed to the distribution of z_t, or of ε_t. In order to make the model given by equations (2.1)-(2.3) reasonable ($h_t > 0$ for all t) we must have $\alpha_0 > 0$ and $\alpha_j \geq 0$ for $j = 1, \ldots, q$. Besides, for the process z_t to have finite variance, the necessary and sufficient condition is that all roots of polynomial

$$1-\sum_{j=1}^{q}\alpha_j l^j \tag{4}$$

are outside the unit circle. When this condition is accomplished, the unconditional variance of z_t is given by

$$Var(z_t)=\alpha_0\left(1-\sum_{j=1}^{q}\alpha_j\right)^{-1} \tag{5}$$

Then the necessary condition for the stationary covariance is that $\sum_{j=1}^{q}\alpha_j < 1$.

These constraints will not be considered in this work to avoid the difficulties in maximizing the likelihood function subjected to these constraints (see, e.g., Geweke[9]). The main disadvantage of doing unconstrained estimates of the model parameters is that we might have negative estimates for the parameters, which makes the volatility representing inconsistent. The negative estimates, when they occurred, weren't considered. It is certain that when the negative estimates to a series frequently occur, that is a problem, which clearly reflects one of the weaknesses of the ML method. However, for some series, especially those generated by low order ARCH models, this disadvantage in not a serious problem.

Given a trajectory observed $\mathbf{Z}=\{z_t,\ t=1,2,\ldots,T\}$ of the return process z_t, the likelihood function for $t=q+1,\ldots,T$ is given by

$$L\left(\alpha|\mathbf{Z}\right)\propto\prod_{t=q+1}^{T}\left(\frac{1}{h_t}\right)^{\frac{1}{2}}\exp\left\{-\frac{z_t^2}{2h_t}\right\} \tag{6}$$

where $\alpha=\left(\alpha_0,\alpha_1,...,\alpha_q\right)'$. Then, the likelihood function may be maximized with respect to unknown parameters α.Denoting the log-likelihood function as

$$l\left(\alpha|\mathbf{Z}\right)=\frac{1}{T}\sum_{t=q+1}^{T}l_t\left(\alpha|\mathbf{Z}\right) \tag{7}$$

and using the expressions (2) and (3), we get

$$l_t\left(\alpha|\mathbf{Z}\right)=-\frac{1}{2}\log h_t-\frac{z_t^2}{2h_t} \tag{8}$$

The gradient of the function (7) with respect to α is given by

$$\frac{\partial l(\alpha|\mathbf{Z})}{\partial \alpha} = \frac{1}{T}\sum_{t=q+1}^{T} \frac{\partial l_t(\alpha|\mathbf{Z})}{\partial \alpha} \tag{9}$$

where

$$\frac{\partial l_t(\alpha|\mathbf{Z})}{\partial \alpha} = \frac{1}{2h_t} v_t \left(\frac{z_t^2}{2h_t} - 1\right) \tag{10}$$

and

$$v_t = \frac{(1, z_{t-1}^2, \ldots, z_{t-q}^2)}{h_t}, \quad t = q+1, \ldots, T. \tag{11}$$

Using the notation of $\mathbf{v}' = (v'_{q+1}, \ldots, v'_T)$ and taking conditional expectations of the Hessian, we may write the observed (empirical) information matrix as

$$\hat{I} = -E\left[\frac{1}{T}\sum_{t=q+1}^{T} \frac{\partial l_t(\alpha|\mathbf{Z})}{\partial \alpha_i \partial \alpha_j}\right] = \frac{1}{2T}\sum_{t=q+1}^{T} v'_t v_t = \frac{1}{2T}\mathbf{v}'\mathbf{v} \tag{12}$$

with $i = 0, 1, \ldots, q$ and $j = 0, 1, \ldots, q$.

Some works have considered quasi-ML information matrix (see, e.g., Davidson[10]). In this work we opt to use Newton method for the determination of the ML estimates (see, e.g., Engle[1]). Then, an iterative method to calculate the ML estimators for the parameters $\alpha_j, j = 0, 1, \ldots, q$ consists of iterating the following recursive equation:

$$\hat{\alpha}^{(k+1)} = \hat{\alpha}^{(k)} + \left[\hat{I}^{(k)}\right]^{-1} \left[\frac{\partial l(\alpha|\mathbf{Z})}{\partial \alpha}\right]_{\hat{\alpha}=\hat{\alpha}^{(k)}} \tag{13}$$

where $\left[\hat{I}^{(k)}\right]$ is the empirical information matrix in (2.12). Under some regularity conditions, Engle[1] showed the ML estimators $\hat{\alpha}$ are asymptotically Normal with limit distribution given by $\sqrt{T}(\hat{\alpha} - \alpha) \xrightarrow{D} N\left(\mathbf{0}, \hat{I}^{-1}\right)$. An review of the properties of those models may be found in Bollerslev et al[2] and Degiannakis & Xekalaki[3].

By Bootstrap method it is more usual to use the empirical distribution of residuals to generate bootstrapped parameters sample (see Pascual et al[11]), but in this work in order to correct the large interval obtained by this procedure, i.e., due the use of the ML approach, we use a parametric Bootstrap procedure to accomplish better confidence intervals for the estimated parameters. The procedure can be summarized as follow:

Algorithm 1 ***Step 1:*** *Get $\hat{\alpha} = (\hat{\alpha}_0, \hat{\alpha}_1, ..., \hat{\alpha}_q)'$, the α Maximum Likelihood estimates.*

Step 2: *Get a burned period $t_0 >> q$, a sample size T and a counter $j = 1$.*

Step 3: *Execute $z_t = 0$ for $t = 1, ..., q$ and $t = q + 1$.*

Step 4: *Compute $h_t = \hat{\alpha}_0 + \hat{\alpha}_1 z_{t-1}^2 + ... + \hat{\alpha}_q z_{t-q}^2$.*

Step 5: *Generate $\varepsilon_t = N(0,1)$ and compute $z_t = \sqrt{h_t}\varepsilon_t$.*

Step 6: *Execute $t = t + 1$ and return to Step 4 until $t = T$.*

Step 7: *Use the remaining sample $(z_{t_0}, ..., z_T)$ to compute the Bootstrap ML estimates, $\hat{\alpha}^{*(j)} = (\alpha_0^*(j), \alpha_1^*(j), \ldots, \alpha_q^*(j))$.*

Step 8: *Execute $j = j + 1$ and return to Step 3 until $j = B$.*

After $B = 1000$ iterations, we have computed the confidence intervals through percentile Bootstrap methodology. Supposing that there is a transformation $u^* = f(\hat{\alpha}^* - \hat{\alpha})$ wich makes the symmetric Bootstrap distribution around 0, the symmetric relation $u^*_{(B+1),\frac{p}{2}} = -u^*_{(B+1),1-\frac{p}{2}}$ for the quantiles of u^* may be used with the back-transformation. That procedure leads to the confidence interval, $\hat{\alpha} - u^*_{(B+1),\frac{p}{2}} S_{(B+1)} \leq \hat{\alpha} \leq \hat{\alpha} + u^*_{(B+1),1-\frac{p}{2}} S_{(B+1)}$ where $S_{(B+1)}$ is the estimate of the standard deviation of the $\hat{\alpha}^*$. Denoting as $r^*_{(B+1),\frac{p}{2}} = \hat{\alpha} - u^*_{(B+1),\frac{p}{2}} S_{(B+1)}$ and as $r^*_{(B+1),1-\frac{p}{2}} = \hat{\alpha} + u^*_{(B+1),1-\frac{p}{2}} S_{(B+1)}$, the confidence interval is written as:

$$r^*_{\left((B+1),\frac{\alpha}{2}\right)} \leq \alpha \leq r^*_{\left((B+1),1-\frac{\alpha}{2}\right)} \tag{14}$$

These intervals are called the percentile Bootstrap intervals by Efron[12]. In this work, we compute intervals given by the percentiles 2.5% and 97.5% of those estimates samples.

3 Bayesian Approach for ARCH(q) Models

Consider an observed trajectory of the return $\mathbf{Z} = \{z_t,\ t = 1, 2, \ldots, T\}$. The Bayesian approach to the inference problem of ARCH(q) model parameters can be done by combining the likelihood function $L(\alpha|\mathbf{Z})$ of this trajectory and the prior density $\pi(\alpha)$, resulting in a posterior density given as

$$\pi(\alpha|\mathbf{Z}) \propto L(\alpha|\mathbf{Z})\pi(\alpha) \tag{15}$$

The necessary condition for the stationary covariance, allows us to ensure variation ranges for the parameters for ARCH(q) models. Then we may be guaranteed that there are intervals $[a_j, b_j]$, $j = 1, 2, \ldots, q$ with $a_j > 0$ and $b_j < 1$, so that $a_j \leq \alpha_j \leq b_j$. We may also consider $a_0 \leq \alpha_0 \leq b_0$ with $a_0 \geq 0$ and $b_0 \leq E\left(z_t^2\right)$. This analysis leads us to choose a transformation that maps the intervals $[a_j, b_j]$ in a domain $(-\infty, +\infty)$. This reparameterization may present advantages when it is intended to use MCMC algorithms. Thus, we opt to use for a reparameterization given by Oliveira[15]:

$$\phi_j = \log\left(\frac{\alpha_j - a_j}{b_j - \alpha_j}\right),\ j = 0, 1, \ldots, q \tag{16}$$

where a_j and b_j are chosen by prior information. With this reparameterization the likelihood function is expressed with the volatility h_t rewritten with the parameters α_j, adequately transformed in ϕ_j by (16). Denoting $\Sigma(\phi)$ as diagonal matrix given by $\Sigma(\phi) = diag\left(h_{q+1}(\phi), \ldots, h_T(\phi)\right)$, we have

$$L(\phi|\mathbf{Z}) \propto \left[\det\left(\Sigma(\phi)\right)\right]^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\mathbf{Z}'\left[\Sigma^{-1}(\phi)\right]\mathbf{Z}\right\} \tag{17}$$

where $\phi = (\phi_0, \phi_1, \ldots, \phi_q)$. Assume that ϕ_j are independent, with Normal prior distributions $\pi(\phi_j) \sim N(0, \sigma_j^2)$, $j = 0, 1, \ldots, q$.Then the joint prior distribution for $\phi = (\phi_0, \phi_1, \ldots, \phi_q)$ is given by

$$\pi(\phi) \propto \prod_{j=0}^{q}\left(\frac{1}{\sigma_j}\right)^{\frac{1}{2}} \exp\left\{-\frac{\phi_j^2}{2\sigma_j^2}\right\} \tag{18}$$

With (17) and (18) we may write the joint posterior distribution for ϕ as

$$\pi(\phi|\mathbf{Z}) \propto \prod_{t=q+1}^{T}\left(\frac{1}{h_t(\phi)}\right)^{\frac{1}{2}} \exp\left\{-\frac{z_t^2}{2h_t(\phi)}\right\} \times \prod_{j=0}^{q}\left(\frac{1}{\sigma_j}\right)^{\frac{1}{2}} \exp\left\{-\frac{\phi_j^2}{2\sigma_j^2}\right\} \tag{19}$$

The conditional posterior distributions for $\phi_j, j = 0, 1, \ldots, q$ used in the Metropolis-Hastings MCMC simulation algorithm (see, e.g., Chib & Greenberg[13]), to obtain a sample of the joint posterior distribution (19), are given by

$$\pi(\phi_j|\phi_{-j}, \mathbf{Z}) \propto L(\phi|\mathbf{Z}) \times \pi(\phi_j) \tag{20}$$

where ϕ_{-j} is a vector with the model parameters except for the parameter ϕ_j.

The Metropolis-Hastings, a MCMC method, is one of the most popular techniques used by statisticians. It is currently one of used as a way to simulate observations from unwieldy distributions. This algorithm can draw samples from any probability distribution $p(x)$, requiring only that the density can be calculated at x. The algorithm generates a set of states x^t which is a Markov chain because each state x^t depends only on the previous state x^{t-1}. The algorithm depends on the creation of a proposal density $Q(x'; x^t)$, which depends on the current state x^t and which can generate a new proposed sample x'.

Then, in order to get samples from the conditional distribution (20), we apply the Metropolis-Hastings algorithm, which is summarized as follow:

Algorithm 2 ***Step 1:*** *Provide the arbitrary initial values* $\phi^{(0)} = \left(\phi_j^{(0)}\right), j = 0, 1, \ldots, q$ *and compute* $\alpha^{(0)} = \left(\alpha_j^{(0)}\right), j = 0, 1, \ldots, q$. *Execute* $i = 1$.

Step 2: *Generate a new value* γ *of the conditional density* $\pi(\phi_j|\phi_{-j}, \mathbf{Z})$, *where* ϕ_{-j} *is given by:*

$$\phi_{-j} = \left(\phi_0^{(i-1)}, \phi_1^{(i-1)}, \ldots, \phi_{j-1}^{(i-1)}, \phi_{j+1}^{(i-1)}, \ldots, \phi_q^{(i-1)}\right)$$

Then we have a new vector given by:

$$\phi^c = \left(\phi_0^{(i-1)}, \phi_1^{(i-1)}, \ldots, \phi_{j-1}^{(i-1)}, \gamma_1, \phi_{j+1}^{(i-1)}, \ldots, \phi_q^{(i-1)}\right)$$

Step 3: *Calculate the probability of accepting of the new value* γ *in the* $\phi_j^{(i-1)}$ *place:*

$$\lambda\left(\phi_j^{(i-1)}, \gamma\right) = \begin{cases} \min\left\{1, \frac{\pi(\phi^c|\mathbf{Z})}{\pi\left(\phi^{(i-1)}|\mathbf{Z}\right)}\right\}, & \text{if } \pi\left(\phi^{(i-1)}|\mathbf{Z}\right) \neq 0 \\ 1, & \text{otherwise} \end{cases}$$

Step 4: *Generate a random variable* u *from an standard Uniform distribuiton* $U(0,1)$ *and execute:*

$$\phi_j^{(i)} = \begin{cases} \gamma, & \text{if } u \leq \lambda\left(\phi_j^{(i-1)}, \gamma\right) \\ \phi_j^{(i-1)}, & \text{otherwise} \end{cases}$$

Step 5: *Repeat the Step 2 to Step 4 for* $j = 0, 1, \ldots, q$.

Step 6: *Execute $i = i + 1$ and return to Step 2 until until the convergence to be verified with some criterion (in this work we used the criterion of Geweke)*

After the MCMC convergence, the original parameters α_j, $j = 0, 1, \ldots, q$, may be recuperated by means of the inverse transformation

$$\alpha_j = \frac{b_j e^{\phi_j} + a_j}{1 + e^{\phi_j}} \tag{21}$$

Bayesian inference may be expressed as the expectation evaluation of a function of interest $g(\alpha)$ with respect to the posterior density $\pi(\alpha|\mathbf{Z})$:

$$E[g(\alpha)] = \int_{\alpha} g(\alpha)\, \pi(\alpha|\mathbf{Z})\, \mathrm{d}\alpha \tag{22}$$

The exact solution of the multiple integral in (22) may not be analytically available in this case. However, whenever $E[g(\alpha)]$ exists, an approximation to (22) may be obtained through Monte Carlo simulation (see, e.g., Geweke[14]).

4 Empirical Applications

We have considered two financial series for accomplishing this work. The first one is the Bovespa Index (Ibovespa), which is the most important Brazilian stock market index. It shows the present value, in our currency, of a theoretical stock exchange rating average performance. The second series is the daily ratings of Telebras shares, (Telebrás used to be a Brazilian holding controlled by the Federal Union). The stock market closing time values have been considered in both series. In studying those series, it is usual to consider the instantaneous rate of return, calculated by using a log-difference transformation, i.e., for each price series z_t, $z_t^* = 100 \times \Delta \ln z_t$ was computed. The charts of prices and their return are presented in Figures 1 and 2 below. As is typical of financial series, they show sign of heteroscedasticity and volatility clustering. The results for IBovespa and Telebras series were obtained from the ARCH models selected by BIC criterion (see, e.g., Schwarz[16]), in order to compare the Maximum Likelihood and Bayesian approaches.

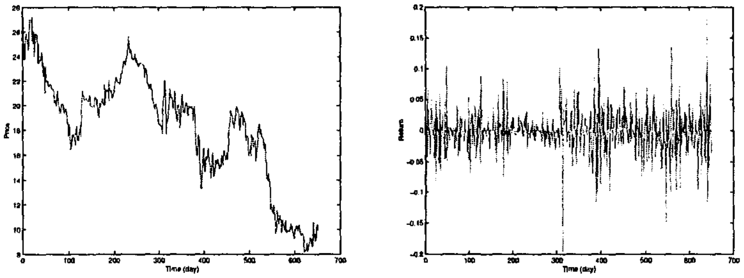


Fig. 1: IBovespa daily rating and return time series.

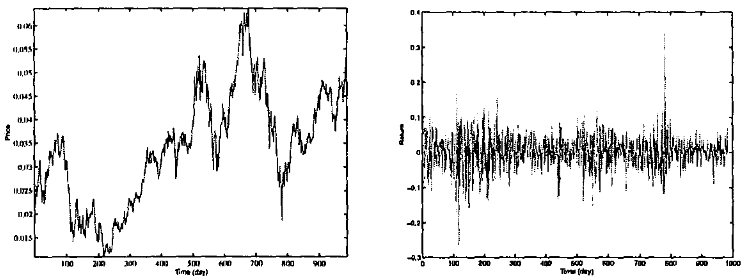


Fig. 2: Telebras daily rating and return time series.

The adjusted model for IBovespa is an ARCH(3) model, while the Telebras series is better fitted by an ARCH(7) model. Those results are obtained through the algorithms described above. In the Bayesian approach, the prior parameters were chosen considering past studies of those series by Oliveira[15]). When implementing the Metropolis-Hastings algorithm, a 5000-iteration chain was simulated for each parameter. By discarding 50% and taking 5 by 5 values, we'll have a sample with a size of 500. The convergence of the parameters was verified according to the criterion of Geweke[17]. This criterion is a test that consists of the division of the generated chain into two sequences. If the chain were stationary, then the means of those sequences will be close and the convergence will be reached. The size of the test is set at 5%.

Table 1 and Table 2 show the 95% confidence intervals of the ML and Bootstrapped ML estimates and, the 95% credibility intervals of the Bayesian estimators, respectively, for the IBovespa series. The GC column represents the values of the criterion of Geweke and AR column represents the acceptance rates of the Metropolis-Hastings algorithm.

Table 1. Estimation Results and Confidence Intervals for ML and Bootstrapped ML estimators Techniques - IBovespa Series; BIC=-4.00671.

	MLE	C.I.(95%)	Bootstrapped C.I.(95%)
α_0	0.00062	[0.00051; 0.00073]	[0.00054; 0.00078]
α_1	0.15578	[0.05738; 0.25418]	[0.04532; 0.27163]
α_2	0.08240	[−0.00414; 0.16066]	[0.00885; 0.17609]
α_3	0.30628	[0.17219; 0.44035]	[0.14654; 0.44029]

Table 2. Estimation Results and Credibility Intervals - IBovespa Series.

	Mean	Std.Dev.	Median	C.I.(95%)	GC	AR
α_0	0.00064	0.00002	0.00057	[0.00059; 0.00068]	0.03220	73.658
α_1	0.15857	0.01608	0.16862	[0.12752; 0.18999]	0.03714	93.104
α_2	0.08976	0.01279	0.07421	[0.06480; 0.11431]	0.02015	80.140
α_3	0.31966	0.02518	0.31329	[0.27042; 0.36933]	0.00930	89.138

Table 3 and Table 4 show the 95% confidence intervals of the ML and Bootstrapped ML estimates and, the 95% credibility intervals of the Bayesian estimators, respectively, for the Telebras series

Table 3. Estimation Results and Confidence Intervals for ML and Bootstrapped ML estimators Techniques - Telebras Series; BIC=-3.35832

	MLE	C.I.(95%)	Bootstrapped C.I.(95%)
α_0	0.00066	[0.00046; 0.00085]	[0.00042; 0.00102]
α_1	0.10622	[0.02666; 0.18578]	[0.01178; 0.22299]
α_2	0.14749	[0.06009; 0.23488]	[0.03109; 0.27018]
α_3	0.10513	[0.02448; 0.18578]	[0.01157; 0.21718]
α_4	0.09410	[0.01533; 0.17287]	[0.01155; 0.21235]
α_5	0.05602	[−0.01471; 0.12675]	[0.00357; 0.17233]
α_6	0.10602	[0.02657; 0.18546]	[0.01369; 0.22360]
α_7	0.11114	[0.03073; 0.19156]	[0.01372; 0.23595]

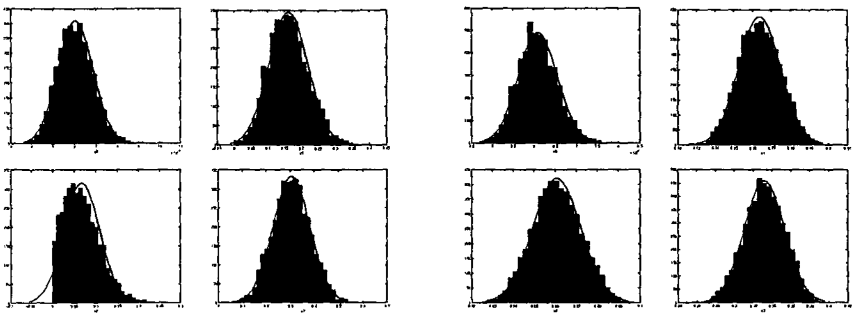
Table 4. Estimation Results and Credibility Intervals - Telebras Series.

	Mean	Std. Dev.	Median	C.I.(95%)	GC	AR
α_0	0.00069	0.00008	0.00069	[0.00054; 0.00085]	−1.35154	52.332
α_1	0.10234	0.01922	0.10271	[0.06595; 0.13691]	0.32766	84.444
α_2	0.15044	0.01952	0.15058	[0.11407; 0.18682]	0.92066	86.990
α_3	0.10155	0.01918	0.10199	[0.06566; 0.13677]	0.74452	83.814
α_4	0.09868	0.01946	0.09818	[0.06385; 0.13534]	1.01998	84.010
α_5	0.05267	0.01903	0.05285	[0.01703; 0.08759]	−0.20901	81.420
α_6	0.10252	0.01914	0.10263	[0.06675; 0.13739]	0.65443	83.976
α_7	0.10327	0.01893	0.10326	[0.06735; 0.13693]	0.40127	82.452

We have observed in the Tables 1, 2, 3, and 4 that the ML and the Bayesian estimates are close. We have also observed that the intervals obtained from the Bootstrap simulation are more accurate than those obtained asymptotically from the information matrix. This is due to the fact that the Bootstrapped confidence intervals take into consider the asymmetric property of the empirical distribution, such as the parameters α_2 for IBovespa as well as α_5 for Telebras time series.

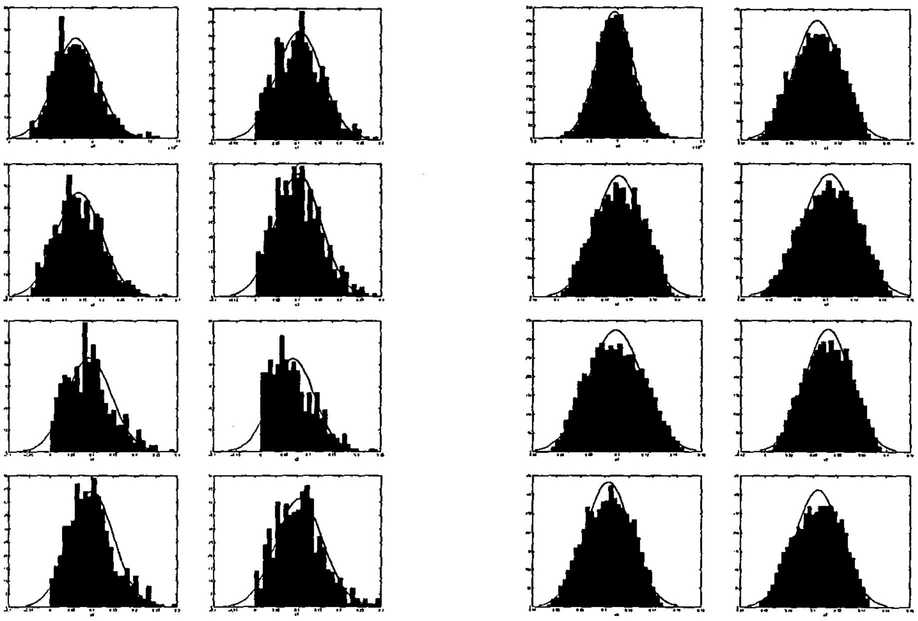
The histograms representing the empirical distributions of the ML estimators and the posterior distributions obtained through the Metropolis-Hastings MCMC simulation algorithm of IBovespa time series are shown in Figure 3. The same is shown in Figure 4 for the Telebras time series.

In both figures we have noticed that the histograms which correspond to the empirical distribution of the ML estimators, always present a larger amplitude than the histograms that represent the posterior distributions, which means that the ML estimators have a larger variance rate than the Bayesian estimators. We have also noticed in these figures that some of the ML estimator histograms present a truncated shape close to zero because we have not been constraining the parameters in the ML method, which is mentioned in section 2.



(a) (b)

Fig. 3: Empirical Distribution of the ML Estimators (a) and Posterior Distribution of the Bayesian Estimators (b) - IBovespa Series.



(c) (d)

Fig. 4: Empirical Distribution of the ML Estimators (c) and Posterior Distribution of the Bayesian Estimators (d) - Telebras Series.

5 Conclusion

The asymptotic confidence interval analysis of the ML method has been based on Normal approximation; therefore, mistaken results might occur when applying this methodology. On the other hand, considering the Bayesian estimate approach, the 95% credibility intervals demonstrate to be more accurate than the 95% confidence intervals that have been estimated through the ML approach and the Bootstrap technique. This fact indicates that the Bayesian approach shows a better performance level concerning the interval estimation of non-linear model parameters of the time series. Consequently, we have concluded that even though the computational effort required by the MCMC algorithms is higher when compared to the Newton method used to calculate ML estimates, the estimate accuracy obtained herein, especially to high order models, is recommended because the Bayesian approach is more accurate and robust when inferring the ARCH(q) model parameters.

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