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Bayesian Analysis for the Accelerated Life Tests with informative prior distributions obtained from fixed stresses

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Abstract

Accelerated life testing of an item, under more severe than normal conditions, is commonly used by industrial statisticians to reduce test time and costs. In this paper we apply the Bedrick, Christensen, and Johnson's (1996) ideas to formulate an informative prior for the parameters involved in a power rule model with an exponential distribution under the type II censoring mechanism. An illustrative example showing how to specify an informative prior is considered. Also, the sensitivity of the posterior quantiles and reliability function of the mean time to failure under the usual stress are analysed via the Gibbs-Metropolis algorithm.

Key Words: Data augmentation prior; exponential distributions; Gibbs-Metropolis algorithms; power rule model.

1 Introduction

In engineering applications, accelerated conditions are produced by testing items at higher than normal temperature, voltage, pressure, load, etc.(see Mann, Shaffer & Singpurwalla (1974), Nelson (1990), Rodrigues, (1993)). To apply Bayesian methods to estimate the reliability function of an item under the usual stress level and the exponential censored data, the specification of a prior distribution for the parameters involved in the model is an important issue. In this paper, we apply Bedrick, Christensen and Johnson's

(1996) (hereafter referred to as DCJ) ideas on specifying priors to provide an appropriated Bayesian method to obtain the marginal posterior distributions of the parameters and an estimate of the reliability function under the usual stress level. The BCJ's approach consists in specifying independent prior distributions on the conditional mean time to failure under the two fixed stress levels where the joint prior distribution on regression coefficients induced from these priors is of the same form as the likelihood. This specific prior on the regression coefficients is called ' Data Augmentation Prior" (DAP). The motivation to use this method to specifying an informative prior follows from the idea that it is inherently easier to think about mean time to failure given the stresses than it is to think about thecoefficient regressions. O'Haggan, Woodward, and Moodaley (1990), p. 1092 state that, in general, is extremely difficult to directly specify a prior distribution on regression coefficients. With an illustrative numerical example, it is showed in this paper that making use of Gibbs-Metropolis the priors obtained via " prior observations" result in tractable Bayesian inference.

2 Formulation of the model

We will consider the following model: Suppose that we perform life tests on a set of n_i devices under the stress V_i and the life time T has an exponential density

$$\begin{aligned} f(t \mid \theta_i) &= \frac{1}{\theta_i}\exp\{-\frac{t}{\theta_i}\}, \quad \theta_i > 0, t > 0, \\ i &= 1, \ldots, k. \end{aligned} \tag{1}$$

The unknown parameter θ_i, $i = 1, \ldots, k$,is the mean time to failure under the stress V_i. The k tests yield, as data, the set $\{V_i, n_i, \hat{\theta}_i, n_i\}, i = 1, \ldots, k$ where $\hat{\theta}_i = \frac{A_i}{r_i}$, $A_i = \sum_{j=1}^{r_i} t_{ij} + (n_i - r_i)t_{i,r_i}$ and t_{ij}, $i = 1, \ldots, r_i$ are observables failures under the stress V_i for n_i items on test ($r_i < n_i$). Given the data set and the usual stress V_1 our purpose is to get information of the mean time θ_1 and the reliability function of an item under the usual stress. To make the Bayesian inference, we adopt the power rule model (see for example,Mann, Schaffer & Singpurwalla, 1974)

$$\theta_i = \frac{\alpha_1}{V_i^{\alpha_2}}, \quad \alpha_1 > 0, \quad -\infty < \alpha_2 < \infty. \tag{2}$$

From (2). the link function that specifies the relationships between the mean time to failure and the stress V_i is given by

$$ln(\theta_i) = X_i'\beta, \tag{3}$$

where $X_i' = (1, ln(V_i^{-1}))$ and the vector of coefficient regressions is $\beta' = (\beta_1, \beta_2)$ is given by the following parametrization:

$$\beta_1 = ln(\alpha_1), \quad \beta_2 = \alpha_2. \tag{4}$$

The mean time to failure under the stress V_i can be written as

$$\theta_i = \exp\{X_i'\beta\}. \tag{5}$$

The likelihood function for β_1 and β_2 is given by

$$L(\beta) = \Pi_{i=1}^{k} e^{-r_i X_i'\beta} \exp\{-r_i \hat{\theta}_i e^{-X'\beta}\}. \tag{6}$$

For a prior $\pi(\beta)$, the posterior of β is

$$\pi(\beta \mid Data) = \frac{L(\beta)\pi(\beta)}{\int L(\beta)\pi(\beta)d\beta}$$

The two interesting aspects of a Bayesian analysis are the specification of the informative prior $\pi(\beta)$ and the various integrals involving $\pi(\beta \mid data)$. Since the integrals are intratable we make use of the Gibbs-Metropolis algorithm. To specify an informative prior for the vetor of coefficient regressions, we apply the DAP procedure discussed by BCJ.

3 Specifying the prior $\pi(\beta)$

An important issue in any Bayesian analysis is the specification of the informative prior distribution $\pi(\beta)$. With this purpose in mind, this article follows the BCJ's approach (1996) to formulate a prior on β called " Data Augmentation Priors" (DAP). This DAP's procedure consists in obtaining a prior on β which has the functional form of the likelihood function. The main idea behind this procedure is to find " prior observations" and "weights" for the mean time to failure at two suitably selected covariates or stresses.

Data Augmentation Prior:

Let:

(i)- $\tilde{X}_i' = (1, \tilde{X}_i)$:selected covariates or stresses such that the matrix $\tilde{X} = \tilde{X}(2 \times 2)$ is nonsingular ($\tilde{X}_1 \neq \tilde{X}_2$).

(ii)-

$$\tilde{\theta}_i = \exp\{\tilde{X}_i'\beta\}, \quad i = 1, 2.$$
$$\Longrightarrow \beta = \tilde{X}^{-1} ln(\tilde{\theta}).$$

Following BCJ's procedure we have that

$$\pi(\beta) = \Pi_{i=1}^{2} e^{-w_i \tilde{X}_i'\beta} \exp\{-w_i \tilde{\hat{\theta}}_i e^{-\tilde{X}'\beta}\}, \tag{7}$$

where $\tilde{\hat{\theta}}_i$ is the prior sufficient information about $\tilde{\theta}_i$ and w_i is the weight of $\tilde{\hat{\theta}}_i$.

Remarks:

1- Eliciting information about $\tilde{\theta}_i$ is much easier than eliciting information about β.

2-The conditional mean priors on $\tilde{\theta}_i$ (BCJ,1996) induced by (1), via the change-of-variables technique, are independent Invgamma$(w_i, w_i\tilde{\hat{\theta}}_i)$ priors given by

$$\pi(\tilde{\theta}) = \Pi_{i=1}^{2} \tilde{\theta}_i^{-(w_i+1)} \exp\{-\frac{w_i \tilde{\hat{\theta}}_i}{\tilde{\theta}_i}\} \tag{8}$$

The procedure of assessing the prior data $(w_i, \tilde{\hat{\theta}}_i)$ is as follows: For appropriate choice of $\tilde{A}_i$ and the fixed stress $\tilde{X}_i$ define:

$$\tilde{p}_i = Pr[A_i > \tilde{A}_i \mid \tilde{X}_i, \tilde{\theta}_i] = 1 - F(\frac{2\tilde{A}_i}{\tilde{\theta}_i}),$$

where $F(.)$ is the $\chi^{(2)}$-distribution with $2r_i$ degrees of freedom.

It is reasonable to think $\tilde{\hat{\theta}}_i$ as a prior information about $\tilde{\theta}$ having the same weights as $\hat{\theta}$, that is, $w_i = r_i$. For specified values τ_i, we elicit γ_i's such that

$$Pr[\tilde{p}_i > \gamma_i] = \tau_i, \quad i = 1, 2.$$

$$\Longrightarrow \tilde{\hat{\theta}}_i = \frac{\tilde{A}_i}{r_i} U_i,$$

where

$$U_i = \frac{2G^{-1}(\tau_i)}{F^{-1}(1-\gamma_i)},$$

and $G(.)$ is the $\Gamma(w_i, 1)$-distribution, for $i = 1, 2$.
Assuming that $V_1 < V_2 < \ldots, < V_k$, to garantee independence between the parameters $\tilde{\theta}_1$ and $\tilde{\theta}_2$, we choose the following fixed stresses (see BCJ (1996), Section 5, for more details about the justification of these choices):

$$\tilde{X}_1' = (1, ln(V_1^{-1})) \Longrightarrow w_1 = r_1$$
$$\tilde{X}_2' = (1, ln(V_K^{-1})) \Longrightarrow w_2 = r_k$$

Using the change-in-variables technique in (4), we finally obtain the following posterior distribution:

$$\pi(\alpha_1, \alpha_2 \mid Data) \propto \alpha_1^{-(r+r_1+r_k+1)} \exp\{-\frac{1}{\alpha_1}\sum_{i=1}^{k} r_i \hat{\eta}_i V_i^{\alpha_2}\}[(\Pi_{i=1}^{k}(V_i^{r_i})V_1^{r_1}V_k^{r_k}]^{\alpha_2}, \quad (9)$$

where,

$$\begin{aligned} \hat{\eta}_i &= \hat{\theta}_i, \quad i = 2, \ldots, k-1, \\ &= \hat{\theta}_1 + \hat{\tilde{\theta}}_1, \quad i = 1, \\ &= \hat{\theta}_k + \hat{\tilde{\theta}}_2, \quad i = k. \end{aligned}$$

The joint distribution (9) immediately lead to the following full conditional densities for α_1 and α_2:

$$(i) \quad - \quad \pi(\alpha_1 \mid \alpha_2, data) \sim \text{Invgamma}(r + r_1 + r_k, \sum_{i=1}^{k} r_i \hat{\eta}_i V_i^{\alpha_2})$$

$$(ii) \quad - \quad \pi(\alpha_2 \mid \alpha_1, data) \sim \exp\{-\frac{1}{\alpha_1}\sum_{i=1}^{k} r_i \hat{\eta}_i V_i^{\alpha_2}\}[(\Pi_{i=1}^{k} V_i^{r_i})V_1^{r_1}V_k^{r_k}]^{\alpha_2}.$$

Since, (i) is a known distribution and (ii) is an intractable distribution we make use of the Gibbs-Metropolis algorithm (see, Chib and Greenberg (1995)) to obtain the posterior marginal distributions and the reliability function under the usual stress V_1.

4 An illustrative example

As an illustrative example, consider the data of table 1 which was generated according to (1) and the power rule (2) for 5 stress with $\alpha_1 = 400$ and $\alpha_2 = 0.7$.

TABLE 1- Generated data with $\alpha_1 = 400$ and $\alpha_2 = 0.70$

i	V_i	n_i	r_i	θ_i	Uncensored observations
1	5	10	4	129,65	16,30,41,63
2	10	10	6	79,81	15,21,36,39,54,59
3	15	10	7	60,09	18,29,39,42,46,57,60
4	20	10	8	49,13	19,20,36,37,41,45,45,57
5	25	10	9	42,02	8,12,13,14,23,33,42,51,67

We may be willing to assert independently that we are 90% certain that the proportion of items with total time on test A_1 past 528 is at least 20% (usual stress) and that we have probability 0.25 that the proportion of items with total time on test A_5 past 330 exceeds 70% (high stress). Thus we have $\tilde{A}_1 = 528$, $\tilde{A}_2 = 330$, $\tau_1 = 0.90$, $\tau_2 = 0.20$, $\gamma_1 = 0.20$, $\gamma_2 = 0.70$. Given the inverse gamma prior on $\tilde{\theta}_i$, if $w_1 = r_1 = 4$ and $w_2 = r_5 = 9$, these assessments imply

$$\begin{aligned} \tilde{\hat{\theta}}_1 &= 159.90 \quad \text{and} \quad \tilde{\hat{\theta}}_2 = 34.72 \\ U_1 &= 1.21 \quad \text{and} \quad U_2 = 0.94, \end{aligned} \tag{10}$$

for our choice of $\tilde{A}_i, \tau_i$ and γ_i, $i = 1, 2$. Figure 1,2,3 and 4 present the marginal posterior and predictive reliability functions. We performed a posterior analysis using the Gibbs- Metropolis algorithm. We run a long sequence of size 10.000 and from the second half of this sequence we choose final sample of size 500 after each 10 observations. For this short sequence the potencial scale reduction factors for the parameters α_1 and α_2 (Gelman and Rubin, (1992)),$\sqrt{R}$, are estimtated by 1.05.

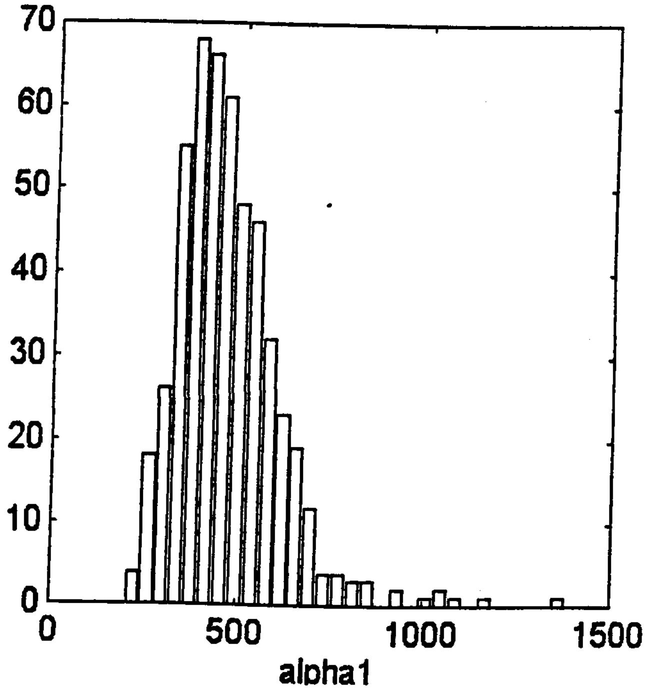

Figure 1: Histogram of 500 draws from the marginal posterior of α_1.

Figure 4 presents estimates of predictive reliability functions,

$$Pr(T_f > t \mid V_i, Data) = \int \exp\{-\frac{tV_i^{\alpha_2}}{\alpha_1}\}\pi(\alpha_1, \alpha_2 \mid Data)d\alpha_1 d\alpha_2.$$

The two curves use the minimum and maximum stress values, $V_1 = 5$ and $V_5 = 25$. From figure 4 , having the usual stress $V_1 = 5$ is much better than having $V_5 = 25$. Also, the predictive reliability function estimated via Gibbs-Metropolis algorithm is very close to the real reliability function given by $\theta_1 = 129.65$.

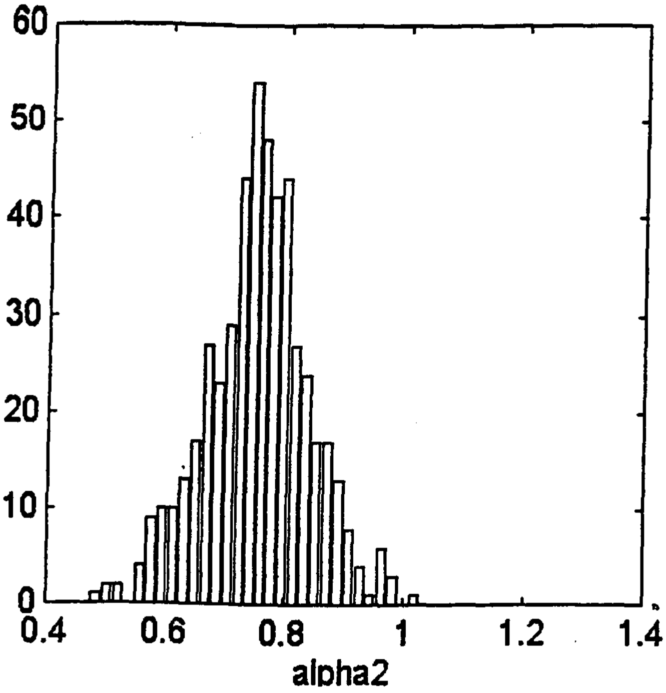

Figure 2: Histogram of 500 draws from the marginal posterior of α_2

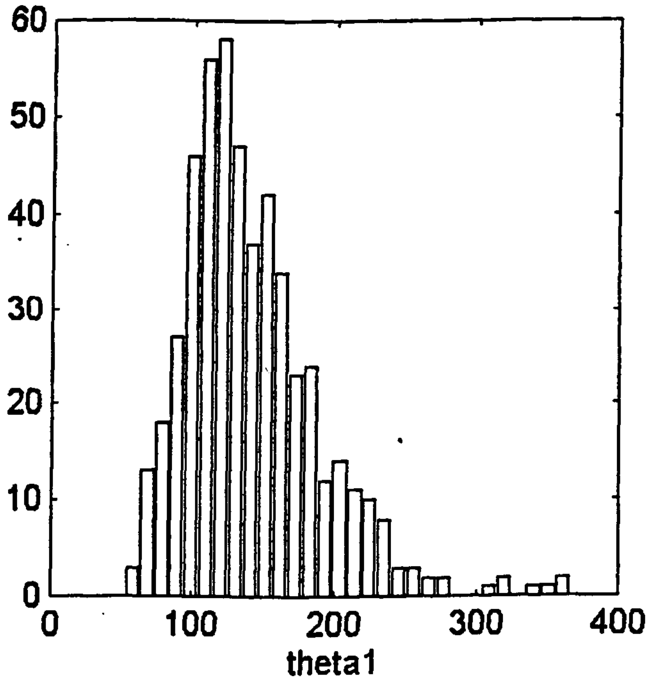

Figure 3: Histogram of 500 draws of posterior distribution of θ_1

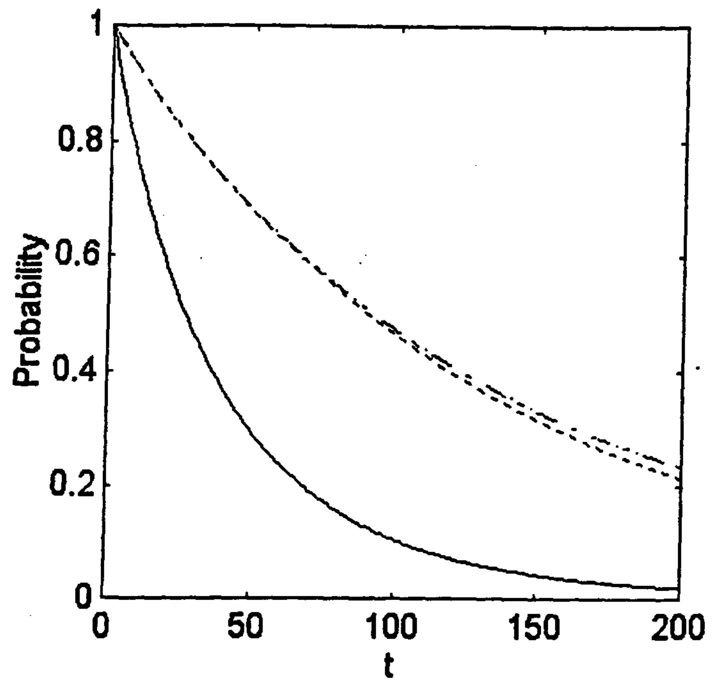

Figure 4: Predictive Reliability functions at two stresses: The solid line represents $V_5 = 25$; the dashdot line , $V_1 = 5$; the dashed line the real reliability function for $V_1 = 5$ and $\theta_1 = 129.65$.

TABLE 2- Summary of posterior inference for θ_1 based on 500 draws of the posterior marginal of θ_1 for different values of the vector $U = (U_1, U_2)$.

$U = (U_1, U_2)$	2.5%	25%	Median	75%	97.%
(0.5.0.5)	63.11	94.25	116.25	142.51	223.49
(1.0,0.5)	67.44	98.04	124.53	151.88	224.88
(0.5,1.0)	76.00	106.20	128.38	159.41	240.21
(1.0,1.0)	79.67	110.48	134.14	170.50	253.24
(1.5,1.0)	80.23	114.25	143.85	174.11	241.51
(1.0,1.5)	80.92	117.67	144.82	180.86	271.62

Table 2 suggests no much influence on the posterior quantiles of θ_1 for different "data prior observations".

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