UNIVERSIDADE DE SÃO PAULO

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Software Reliability Considering the Superposition of Non-homogeneous Poisson Processes in the Presence of a Covariate.

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ABSTRACT "In this paper, we introduce a Bayesian analysis for the superposition of nonhomogeneous Poisson processes in the presence of a covariate x. Posterior summaries of interest are obtained using Markov Chain Monte Carlo Methods. A numerical example is given."

Key Words: Superposition of Poisson processes, Bayesian Analysis, Covariate.

1 Introduction

Consider the modelling of the number of failures of a software by a point process to count failures (see for example, Musa, Iannino and Okumoto, 1987). Let M(t) be the cumulative number of failures of the software that are observed during time (0,t] and assume that M(t) is modeled by a nonhomogeneous Poisson process (NHPP). The intensity function, $\lambda(t) = dE[M(t)]/dt$ is often assumed to be a monotonic function of t (see for example, Musa and Okumoto, 1984 ; or Cox and Lewis, 1966).

An alternative to monotonic intensity functions, is to consider the superposition of several independent NHPP with simple intensity functions. Kuo and Yang (1996) develop Bayesian inferences and model selection methodologies for the superposition model.

The points of failure of a superposition process are defined to be the union of the points of failure from several component point processes.

Let $M_j(t)$ denote the NHPP for the failures from the j-th component in (0,t], with intensity function $\lambda_j(t|\underline{\beta_j})$, where the function form of $\lambda_j(t|\beta_j)$ is assumed to be known with unknown parameter $\underline{\beta_j}$ that may be a vector. We also assume that the number of failures from the j-th component $M_j(t), j = 1, \cdots, J$ is independent. A process $M(t) = \sum_{j=1}^{J} M_j(t)$ which counts the number of failures in the interval (0,t] for the superposition model, is also a nonhomogeneous Poisson process with intensity function $\lambda(t|\underline{\beta}) = \sum_{j=1}^{J} \lambda_j(t|\underline{\beta_j})$, where $\underline{\beta} = (\underline{\beta_1}, \cdots, \underline{\beta_J})$.

With superposition of NHPP, we can have many different forms for the intensity function: bathtub-shaped functions, polynomials with peaks and valleys, and simple monotonic functions. Some special cases are given by,

$$\begin{aligned}
\text{(i)} \quad \lambda(t) &= \alpha_1\beta_1 t^{\alpha_1 - 1} + \alpha_2\beta_2 t^{\alpha_2 - 1} \quad \text{(two Weibull intensity functions).} \\
\text{(ii)} \quad \lambda(t) &= \frac{\alpha_1}{t+\beta_1} + \alpha_2 t + \alpha_3 \quad \text{(Gaver and Acar, 1979).} \qquad (1) \\
\text{(iii)} \quad \lambda(t) &= \beta_0 + \beta_1 t + \ldots + \beta_m t^m
\end{aligned}$$

Kuo and Yang (1996) develop a unified theory that relates the NHPP to either record value statistics or to general order statistics. They show that both record value statistics and general order statistics give rise to NHPP process.

Kuo and Yang (1996) develop Gibbs sampling for Bayesian inference in NHPP models.

In this paper, we consider the use of Gibbs sampling algorithms (see for example, Gelfand and Smith, 1990) with Metropolis-Hastings algorithms (see for example, Chib and Greenberg, 1995; or Smith and Roberts, 1993) for some special cases of superposition of NHPP. We also introduce a Bayesian approach for the superposition of NHPP in the presence of covariates.

2 Bayesian Inference for the Superposition Model

Let $\mathcal{D}_t$ denote the data set observed until time t. It consists of n observed ordered epochs $(t_1, \cdots, t_n)$, where $0 < t_1 < \cdots < t_n < t$. Assuming that the point process of the ordered epochs follow the superposition model, the likelihood function of the NHPP given the data for the time truncated model is,

$$L(\beta|\mathcal{D}_t) = \Big(\prod_{i=1}^{n} \lambda(t_i)\Big) exp(-m(t)) \qquad (2)$$

where $m(t) = \int_0^t \lambda(u)du$ (see for example, Cox and Lewis, 1966; or B.P.Rao, 1980). For the failure truncated model, a similar expression can be applied with t replaced by t_n.

Assume that the prior distributions for the $\underline{\beta_j}$, $j = 1, \cdots, J$ are independent.

The posterior density of $\underline{\beta}$ is given by

$$\pi(\underline{\beta}|\mathcal{D}_t) \propto (\prod_{i=1}^{n}\sum_{j=1}^{J}\lambda_j(t_i))exp(-\sum_{j=1}^{J}m_j(t))\prod_{j=1}^{J}\pi_j(\beta_j) \tag{3}$$

To simplify the conditional distributions need for the Gibbs sampling algorithm, we consider the introduction of latent variables (see Tanner and Wong, 1987), $\underline{I_i} = (I_{i1}, \cdots, I_{iJ})$, where $I_{ij} = 1$ if the i-th failure is caused by the j-th type and $I_{ij} = 0$ otherwise, $i = 1, ..., n$. Observe that $\sum_{j=1}^{J} I_{ij} = 1$ for $i = 1, \cdots, n$. Assume that the conditional distribution for $\underline{I_i}$ given $\underline{\beta}$ and $\mathcal{D}_t$ is a multinomial distribution MN with parameters 1 and cell probabilities $(p_{i1}, ..., p_{iJ})$, where

$$p_{ij} = \frac{\lambda_j(t_i)}{\sum_{j=1}^{J}\lambda_j(t_i)}$$

Therefore, we have,

$$\pi(\underline{I_i}|\underline{\beta}, \mathcal{D}_t) \propto \prod_{j=1}^{J} p_{ij}^{I_{ij}} \propto \prod_{j=1}^{J}\left[\frac{\lambda_j(t_i)}{\sum_{j=1}^{J}\lambda_j(t_i)}\right]^{I_{ij}} \tag{4}$$

In this way, the posterior density of $\underline{\beta}$ given $I = (\underline{I_1}, \cdots, \underline{I_n})^T$, (an nxJ matrix), and $\mathcal{D}_t$ is,

$$\pi(\underline{\beta}|I, \mathcal{D}_t) \quad \propto \quad L(\underline{\beta}|\mathcal{D}_t)(\prod_{i=1}^{n}\pi(\underline{I_i}|\underline{\beta}, \mathcal{D}_t))(\prod_{j=1}^{J}\pi_j(\underline{\beta_j})) \tag{5}$$

That is,

$$\pi(\underline{\beta}|I, \mathcal{D}_t) \propto \prod_{j=1}^{J}\prod_{i:I_{ij}=1}\lambda_j(t_i)\prod_{j=1}^{J}exp(-m_j(t))\prod_{j=1}^{J}\pi_j(\underline{\beta_j}) \tag{6}$$

As a special case, consider the superposition of a Musa and Okumoto (1984) process with intensity function $\lambda_1(t) = \frac{\alpha}{\beta_1+t}$ and an exponential process with intensity function $\lambda_2(t) = \beta_2 t$. Therefore, $\lambda(t) = \frac{\alpha}{\beta_1+t} + \beta_2 t$ and mean value function $m(t) = \alpha\log(1 + \frac{t}{\beta_1}) + \frac{\beta_2 t^2}{2}$.

Assume the following prior distributions for α, β_1 and β_2:

$$\begin{aligned} \alpha &\sim \Gamma(a_1, b_1); \quad a_1, b_1 \text{ known} \\ \beta_1 &\sim \Gamma(a_2, b_2); \quad a_2, b_2 \text{ known} \\ \beta_2 &\sim \Gamma(a_3, b_3); \quad a_3, b_3 \text{ known} \end{aligned} \tag{7}$$

where $\Gamma(a, b)$ denotes a gamma distribution with mean a/b and variance a/b^2. We further assume independence among the parameters.

From (6), we have the joint posterior distribution for α, β_1 and β_2,

$$\pi(\alpha, \beta_1, \beta_2 | I, \mathcal{D}_t) \propto \frac{\alpha^{a_1 - 1 + \sum_{i=1}^n I_{i1}} \beta_1^{a_2 - 1} \beta_2^{n + a_3 - 1 - \sum_{i=1}^n I_{i1}}}{\prod_{i: I_{i1}=1}^n (\beta_1 + t_i)} exp(-b_2\beta_1 - (b_1 + \log(1 + \frac{t_n}{\beta_1}))\alpha - (b_3 + \frac{t_n^2}{2})\beta_2) \quad (8)$$

The conditional distributions for the Gibbs algorithm are given by

(i) Construct I given β and $\mathcal{D}_t$ by generating independent variables I_{i1} from the Bernoulli distribution with parameter

$$p_{i1} = \frac{\alpha}{\alpha + \beta_2(\beta_1 + t_i)t_i}$$

(ii)

$$\pi(\alpha | \beta_1, \beta_2, I, \mathcal{D}_t) \sim \Gamma(a_1 + \sum_{i=1}^n I_{i1}; b_1 + \log(1 + \frac{t_n}{\beta_1})$$

(iii)

$$\pi(\beta_2 | \alpha, \beta_1, I, \mathcal{D}_t) \sim \Gamma(a_3 + n - \sum_{i=1}^n I_{i1}; b_3 + \frac{t_n^2}{2})$$

(iv)

$$\pi(\beta_1 | \alpha, \beta_2, I, \mathcal{D}_t) \propto \beta_1^{a_2 - 1} e^{-b_2\beta_1} \psi(\alpha, \beta_1, \beta_2), \quad (9)$$

where $\psi(\alpha, \beta_1, \beta_2) = exp(-\alpha \log(1 + \frac{t_n}{\beta_1}) - \sum_{i=1}^n I_{i1} \log(\beta_1 + t_i))$

Observe that, we need to use the Metropolis-Hastings algorithm to generate the variable β_1.

3 A Superposition Model in the Presence of a Covariate

Assume now that we have a covariate x which could be related to type of the input or to different programmers. In the presence of a covariate x, we assume the superposition model with intensity function

$$\lambda(t | \underline{\beta}, \gamma, x) = \sum_{j=1}^J \theta_j \lambda_j(t | \underline{\beta_j}) \quad (10)$$

where $\underline{\beta} = (\beta_1, \cdots, \beta_J)$, $\underline{\gamma} = (\gamma_1, \cdots, \gamma_K)$ are parameters related to the incidence probabilities θ_j with $\sum_{j=1}^J \theta_j = 1$

Logistic regression links could be used for the incidence probabilities θ_j, $j = 1, \cdots, J$

For the special case J=2, we have,

$$\lambda(t_i) = \theta_{1i}\lambda_1(t_i) + \theta_{2i}\lambda_2(t_i) \quad (11)$$

where

$$\begin{aligned}\theta_{1i} &= \frac{e^{\gamma+\tau x_i}}{1+e^{\gamma+\tau x_i}} \\ \theta_{2i} &= 1-\theta_{1i}.\end{aligned}$$

Observe that in this case, $\underline{\gamma} = (\gamma, \tau)$

It is interesting to observe that Yamada and Osaki (1984,1985) considered the superposition of two Goel-Okumoto (1979) models, each with intensity function
$\lambda_j(t) = \xi\theta_j\beta_j e^{-\beta_j t}$, j=1,2, where $\xi > 0, \beta_j > 0, 0 < \theta_j < 1$ and $\theta_1 + \theta_2 = 1$, but not including covariates.

Different choices for $\lambda_j(t), j = 1,2$ in (11) could be considered. As a special case, consider a Musa and Okumoto (1984) process with intensity function $\lambda_1(t) = \frac{\alpha}{\beta_1+t}$ and an exponential process with intensity function $\lambda_2(t) = \beta_2 t$

Assuming a failure truncated model, the likelihood function for $\alpha, \beta_1, \beta_2, \gamma$ and τ is (from (2)) given by,

$$L(\alpha, \beta_1, \beta_2, \gamma, \tau|\mathcal{D}_t) = \Big(\prod_{i=1}^{n}((\frac{\alpha}{\beta_1+t_i})\theta_{1i} + \beta_2 t_i\theta_{2i})\Big) exp\Big(-\theta_{1n}\alpha\log(1+\frac{t_n}{\beta_1}) - \frac{\theta_{2n}\beta_2 t_n^2}{2}\Big) \quad (12)$$

where

$$\begin{aligned}\theta_{1n} &= \frac{e^{\gamma+\tau x_n}}{1+e^{\gamma+\tau x_n}} \\ \text{and} \quad \theta_{2n} &= 1-\theta_{1n}\end{aligned}$$

Assume the following prior distributions for $\alpha, \beta_1, \beta_2, \gamma, and, \tau$

$$\begin{aligned}\alpha &\sim \Gamma(a_1, b_1) \\ \beta_1 &\sim \Gamma(a_2, b_2) \\ \beta_2 &\sim \Gamma(a_3, b_3) \\ \gamma &\sim N(d_1, c_1) \\ \tau &\sim N(d_2, c_2)\end{aligned} \quad (13)$$

where $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2$ are known and $N(\mu, \sigma^2)$ denotes a normal distribution with mean μ and variance σ^2. Also assume prior independence among the parameters.

Also assuming the introduction of latent variables $\underline{I_i} = (I_{i1}, I_{i2})$, where $I_{ij} = 1$ if the i-th failure is caused by the j-th type and $I_{ij} = 0$ otherwise, $j = 1,2, i = 1,\cdots,n$, we obtain the joint posterior distribution for $\alpha, \beta_1, \beta_2, \gamma$ and τ (see (6)), given by

$$\pi(\alpha,\beta_1,\beta_2,\gamma,\tau|I,\mathcal{D}_t) \propto \frac{(\prod_{i:I_{i1}=1}^n \theta_{1i})(\prod_{i:I_{i2}=1}^n t_i\theta_{2i})}{\prod_{i:I_{i1}=1}^n(\beta_1+t_i)}$$
$$\alpha^{a_1-1+\sum_{i=1}^n I_{i1}}\beta_1^{a_2-1}\beta_2^{a_3-1+n-\sum_{i=1}^n I_{i1}}e^{-b_2\beta_1}$$
$$exp(-(b_1+\theta_{1n}\log(1+\frac{t_n}{\beta_1}))\alpha-(b_3+\theta_{2n}\frac{t_n^2}{2})\beta_2)$$
$$exp(-\frac{(\gamma-d_1)^2}{2c_1}-\frac{(\tau-d_2)^2}{2c_2}) \quad (14)$$

where

$$\theta_{1i} = \frac{e^{\gamma+\tau x_i}}{1+e^{\gamma+\tau x_i}}$$
$$\theta_{2i} = 1-\theta_{1i}$$

The conditional distributions for the Gibbs algorithm are given by

(i) Construct I given $\alpha,\beta_1,\beta_2,\gamma,\tau$ and $\mathcal{D}_t$ by generating independent variables I_{i1} from the Bernoulli distribution with parameter

$$p_{i1} = \frac{\alpha\theta_{1i}}{\alpha\theta_{1i}+\beta_2\theta_{2i}t_i(\beta_1+t_i)}$$

where $I_{i2}=1-I_{i1}$

(ii)

$$\pi(\alpha|\beta_1,\beta_2,\gamma,\tau,I,\mathcal{D}_t) \sim \Gamma\Big(a_1+\sum_{i=1}^n I_{i1}; b_1+\theta_{1n}\log(1+\frac{t_n}{\beta_1})\Big)$$

(iii)

$$\pi(\beta_2|\alpha,\beta_1,\gamma,\tau,I,\mathcal{D}_t) \sim \Gamma\Big(a_3+n-\sum_{i=1}^n I_{i1}; b_3+\frac{\theta_{2n}t_n^2}{2}\Big) \quad (15)$$

(iv)

$$\pi(\beta_1|\alpha,\beta_2,\gamma,\tau,I,\mathcal{D}_t) \propto \beta_1^{a_2-1}e^{-b_2\beta_1}\psi_1(\alpha,\beta_1,\beta_2\gamma,\tau)$$

where

$$\psi_1(\alpha,\beta_1,\beta_2,\gamma,\tau) = exp\Big(-\alpha\theta_{1n}\log(1+\frac{t_n}{\beta_1})-\sum_{i=1}^n I_{i1}\log(\beta_1+t_i)\Big)$$

(v)

$$\pi(\gamma|\alpha,\beta_1,\beta_2,\tau,I,\mathcal{D}_t) \propto exp\Big(-\frac{(\gamma-d_1)^2}{2c_1}-\frac{(\tau-d_2)^2}{2c_2}\Big)\psi_2(\alpha,\beta_1,\beta_2,\gamma,\tau)$$

where

$$\psi_2(\alpha,\beta_1,\beta_2,\gamma,\tau) = exp\Big(\sum_{i=1}^{n} I_{i1}\log\theta_{1i} + \sum_{i=1}^{n} I_{i2}\log\theta_{2i} - \theta_{1n}\alpha\log(1+\frac{t_n}{\beta_1}) - \frac{\theta_2\beta_2 t_n^2}{2}\Big)$$

Observe that we need to use the Metropolis-Hastings algorithm to generate the variables β_1, γ and τ Similar results could be obtained considering more than one covariate x.

4 Some Considerations on Bayesian Estimators and Model Selection

Suppose that we are interested to get inferences on the mean value function $m(t) = \sum_{j=1}^{J} m_j(t)$ in a specified time t. A Bayes estimator for m(t) considering the squared error loss function is given by the posterior mean $E(m(t)|\mathcal{D}_t)$, which can be approximated by its Monte Carlo estimate using the S generated Gibbs samples. As a special case, consider the superposition of the Musa and Okumoto process and an exponential process

$$\widehat{E}(m(t)|\mathcal{D}_t) = \frac{1}{S}\sum_{s=1}^{S}\Big(\alpha^{(s)}\log(1+\frac{t}{\beta_1^{(s)}}) + \frac{\beta_2^{(s)}t^2}{2}\Big) \tag{16}$$

where $\alpha^{(s)}, \beta_1^{(s)}$ and $\beta_2^{(s)}$ denote the variates for α, β_1 and β_2 drawn in the s-th Gibbs sample.

Considering the superposition of a Musa and Okumoto process and an exponential process, in the presence of a covariate x_i (see section 3), we have,

$$\widehat{E}(m(t_i|\mathcal{D}_t) = \frac{1}{S}\sum_{s=1}^{S}(-\theta_{1i}^{(s)}\alpha^{(s)}\log(1+\frac{t_i}{\beta_1^{(s)}}) - \frac{\theta_{2i}^{(s)}\beta_2^{(s)}t_i^2}{2}) \tag{17}$$

where

$$\theta_{1i}^{(s)} = \frac{e^{\gamma^{(s)}+\tau^{(s)}x_i}}{1+e^{\gamma^{(s)}+\tau^{(s)}x_i}}$$
$$\theta_{2i}^{(s)} = 1-\theta_{1i}^{(s)}$$

$\alpha_1^{(s)}, \beta_1^{(s)}, \beta_2^{(s)}, \gamma^{(s)}), and, \tau^{(s)}$ denote the variates for $\alpha_1, \beta_1, \beta_2, \gamma, and, \tau$ drawn in the s-th Gibbs sample

Similar Monte Carlo estimates are given for other functions of the parameters like θ_{1i}, θ_{2i} or $\lambda(t_i)$, i : $1, \cdots, n$

The predictive survival function evaluated at t distance away from t_n for the superposition of nonhomogeneous Poisson process is given (see Kuo and Yang ,1996) by

$$E(S(t)|\mathcal{D}_t) = \int exp(-m(t_n+t)+m(t_n))\pi(\underline{\beta}|\mathcal{D}_t)d\underline{\beta} \tag{18}$$

For the special case of the superposition of a Musa Okumoto process, a Monte Carlo estimator of (18) is given by

$$\widehat{E}(S(t)|\mathcal{D}_t) = \frac{1}{S}\sum_{s=1}^{S} exp(-\alpha^{(s)}\log(1+\frac{t_n+t}{\beta_1^{(s)}}) - \frac{\beta_2^{(s)}(t_n+t)^2}{2}) + \alpha^{(s)}\log(1+\frac{t_n}{\beta_1^{(s)}}) + \frac{\beta_2^{(s)}t^2}{2}) \quad (19)$$

For the model selection, we could consider (see Kuo and Yang ,1996) the marginal likelihood of the whole data set $\mathcal{D}_t$ for a model M, given by

$$V_M = \int L(\underline{\beta_M}|\mathcal{D}_t)\pi_M(\underline{\beta_M}|\mathcal{D}_t)d\underline{\beta_M} \quad (20)$$

where β_M denotes the set of all unknown parameters in the model M. The posterior Bayes factor criterion prefers model M_1 to M_2 if $V_{M_2}/V_{M_1} < 1$

5 A numerical Ilustration

The data of table 1 was simulated from a NHPP, with intensity function

$$\lambda(t_i) = \left(\frac{e^{\gamma+\tau x_i}}{1+e^{\gamma+\tau x_i}}\right)\left(\frac{\alpha}{\beta_1+t_i}\right) + \left(\frac{1}{1+e^{\gamma+\tau x_i}}\right)\beta_2 t_i \quad (21)$$

considering $\alpha = 20, \beta_1 = 100, \beta_2 = 0.0005, \gamma = -1.0$ *and* $\tau = 2$; 60 observations were generated considering the covariate x with values 2, 3 and 4 respectively (20 observations for each level of the covariate x).

t_i	x_i	t_i	x_i	t_i	x_i	t_i	x_i
1.6590	3	88.3958	2	165.0632	4	252.9478	4
3.4440	2	89.2106	2	172.1504	3	253.5369	2
4.3809	2	92.9258	3	175.2387	3	256.0304	2
22.1900	4	95.6049	4	182.6241	3	272.1693	4
25.2149	4	97.4439	3	184.8264	4	283.5702	4
37.6644	3	105.7311	3	186.5892	4	288.3712	4
39.9029	3	106.8248	3	190.4671	3	291.4028	3
43.4150	3	122.2828	2	193.0431	2	293.3793	2
43.5898	2	131.7938	2	195.1684	2	297.1773	4
59.3859	3	135.8979	3	198.6102	3	298.3395	4
61.9592	2	140.5997	2	224.8489	4	306.5081	3
64.1077	3	145.1028	2	227.2276	4	314.0550	2
70.5231	4	145.4650	4	230.9596	4	329.2328	4
73.6860	2	151.4792	3	235.2451	2	345.7896	4
82.1966	3	160.3269	2	247.9304	4	355.1561	2

Table 1. Generated data from the superposition of a Musa-Okumoto process and an exponencial process with $\alpha = 20, \beta_1 = 100, \beta_2 = 0.0005, \gamma = -1.0, \tau = 2.0$

For a Bayesian analysis of the software reliability data of Table 1, first we assume the superposition of a Musa-Okumoto process and an exponential process without including the covariate x_i with the prior distributions for α, β_1 *and* β_2 given in (7), where $a_1 = 400, b_1 = 20, a_2 = 2500, b_2 = 25, a_3 = 0.25$, *and* $b_3 = 500$.

From the conditional distributions for I, α, β_1 *and* β_2 given in (9),we generate 10 separate Gibbs chains each of which ran for 1200 iterations. In order to diminish the effect of the starting distribution, we discarded the first 200 elements of each chain. We monitored the convergence of the Gibbs samples using the Gelman and Rubin (1992) method that uses the analysis of variance technique to determine if further iterations are needed. For each parameter, we considered every 10th. draw, and so we finally got a sample of size 1000.

In table 2, we have the obtained posterior summaries of the parameters α, β_1 *and* β_2 and in figure 1, we have plots of the approximate marginal densities considering S=1000 Gibbs samples. We also have in table 2, the estimated potential scale reductions $\widehat{R}$ (see Gelman and Rubin, 1992) for all parameters. In this case, the considered number of iterations were sufficient for approximate convergence ($\sqrt{\widehat{R}} < 1.1$ for all parameters).

Parameter	Mean	S.D.	95% *Credible* Interval	$\hat{R}$
α	20.285905	1.0103	(18.3325;22.1793)	1.0021
β_1	99.948741	1.9856	(96.0156;103.7820)	1.0002
β_2	0.000395	0.0001	(0.000198;0.000625)	0.9983

Table 2.Posterior summaries (Superposition of Musa-Okumoto and an exponencial process)

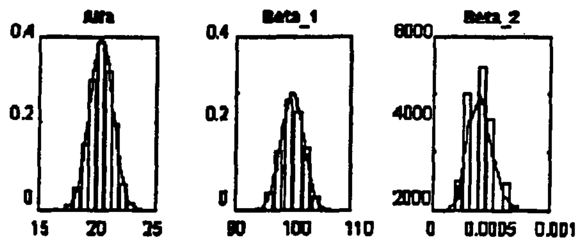

Figure 1. Approximate marginal posterior densities (superposition of a Musa-Okumoto and an exponential process).

In figure 2, we observe the convergence of the Gibbs algorithm with different initial values for the parameters.

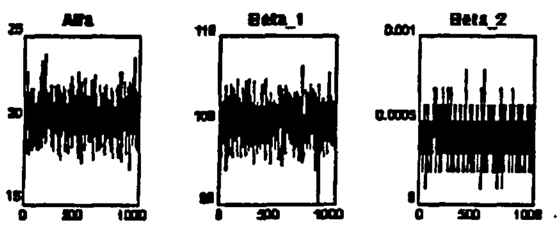

Figure 2. Convergence of the Gibbs algorithm (superposition of a Musa-Okumoto and an exponential process).

Now, let us consider the superposition of the Musa-Okumoto process and an exponential process in the presence of a covariate x, assuming values 2,3 and 4, with intensity function (11). For a Bayesian analysis consider the prior distributions (13) with $a_1 = 400, b_1 = 20, a_2 = 2500, b_2 = 25, a_3 = 0.25, b_3 = 500, c_1 = 0.3, c_2 = 0.5, d_1 = -1$ *and* $d_2 = 2$. From the conditional distributions for $I, \alpha, \beta_1, \beta_2, \gamma$ *and* τ given in (15), we generate 10 separate Gibbs chains each of which ran for 1200 iterations. We eliminated the first 200 elements of each chain, and we selected every 10th element in each chain. Thus we obtained a sample of size 1000.

In table 3, we have the obtained posterior summaries of the parameters $\alpha, \beta_1, \beta_2, \gamma$ *and* τ, and in figure 3, we have the plots of the approximate marginal densities considering S=1000 Gibbs samples.

We also have in Table 3, the estimated potential scale reductions $\widehat{R}$. We observe convergence for all parameters.

Parameter	Mean	S.D.	95% *Credible* Interval	$\widehat{R}$
α	21.4154	1.0056	(19.5014;23.4389)	1.0019
β_1	98.7085	2.0312	(95.8152;103.6534)	0.9995
β_2	0.00031753	0.000468	(0.00000023;0.0017)	1.0042
γ	-0.9838	0.2984	(-1.5523;-0.3891)	1.0002
τ	2.0780	0.2899	(1.5553;2.6867)	1.0021

Table 3. Posterior summaries (superposition of Musa-Okumoto and an exponencial process in the presence of a covariate x

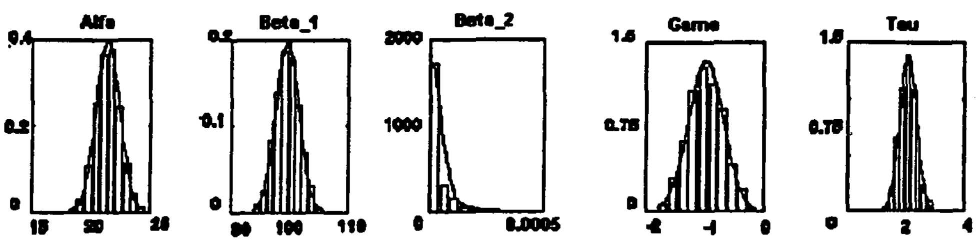

Figure 3. Approximate marginal posterior densities (superposition of a Musa-Okumoto and an exponential process in the presence of a covariate **x**.

Observe that for each level of the covariate x, we have a special case of the superposition of a Musa and Okumoto process and an exponential process. In Table 4, we have Monte Carlo estimates for the posterior means of $m_j(t), j = 1,2,3$ (mean value function at time t and covariate x_j), based on the S=1000 Gibbs samples.

t_i	x_i	Original Model	With Covariate	Without Covariate	t_i	x_i	Original Model	With Covariate	Without Covariate
1.6590	3	0.3269	0.3551	0.3345	165.0632	4	19.4844	21.0385	25.1622
3.4442	2	0.6452	0.7050	0.6896	172.1504	3	19.9393	21.5290	26.1696
4.3809	2	0.8171	0.8927	0.8740	175.2387	3	20.1652	21.7714	26.6104
22.1900	4	4.0046	4.3398	4.1646	182.6241	3	20.6957	22.3403	27.6696
25.2149	4	4.4933	4.8687	4.6892	184.8264	4	20.9229	20.5855	27.9870
37.6644	3	6.3526	6.8870	6.7674	186.5892	4	21.0463	22.7182	28.2415
39.9029	3	6.6733	7.2340	7.1290	190.4671	3	21.2444	22.9285	28.8029
43.4150	3	7.1663	7.7673	7.6899	193.0431	2	20.9250	22.5130	29.1772
43.5898	2	6.9152	7.5298	7.7177	195.1684	2	21.0725	22.6674	29.4868
59.3859	3	9.2667	10.0374	10.1568	198.6102	3	21.7989	23.5228	29.9897
61.9592	2	9.2317	10.0377	10.5435	224.8489	4	23.5538	25.4131	33.8929
64.1077	3	9.8476	10.6648	10.8644	227.2276	4	23.6999	25.5699	34.2533
70.5231	4	10.6654	11.5379	11.8132	230.9596	4	23.9269	25.8138	34.8211
73.6860	2	10.5823	11.4959	12.2762	235.2451	2	23.7026	25.4052	35.4767
82.1966	3	11.9293	12.9114	13.5089	247.9304	4	24.9279	26.8889	37.4408
88.3958	2	12.1594	13.1941	14.3967	252.9478	4	25.2146	27.1967	38.2276
89.2106	2	12.2433	13.2843	14.5128	253.5369	2	24.8207	26.5584	38.3203
92.9258	3	13.0692	14.1404	15.0411	256.0304	2	24.9697	26.7115	38.7139
95.6049	4	13.4084	14.4961	15.4206	272.1693	4	26.2765	28.3367	41.2969
97.4439	3	13.5305	14.6375	15.6806	283.5702	4	26.8809	28.9853	43.1605
105.7311	3	14.3501	15.5205	16.8475	288.3712	4	27.1300	29.2527	43.9552
106.8248	3	14.4559	15.6343	17.0010	291.4028	3	27.2508	29.3471	44.4601
122.2828	2	15.3952	16.6610	19.1629	293.3793	2	27.1135	28.9001	44.7905
131.7938	2	16.2221	17.5425	20.4903	297.1773	4	27.5792	29.7347	45.4284
135.8979	3	17.0806	18.4587	21.0634	298.3395	4	27.6378	29.7975	45.6244
140.5997	2	16.9609	18.3284	21.7206	306.5081	3	28.0181	30.1639	47.0121
145.1028	2	17.3294	18.7198	22.3510	314.0550	2	28.2383	30.0365	48.3102
145.4650	4	17.9481	19.3856	22.4017	329.2328	4	29.1347	31.4031	50.9689
151.4792	3	18.3588	19.8323	23.2456	345.7896	4	29.8935	32.2167	53.9441
160.3269	2	18.5326	19.9945	24.4920	355.1561	2	30.3675	32.1632	55.6626

Table 4. Monte Carlo estimates for $E(m(t_i)|\mathcal{D})$

In Table 4, we also have Monte Carlo estimates for $E(m(t)|\mathcal{D}_t)$ considering the superposition of a Musa-Okumoto and an exponential process not including the covariate x.

To compare the Bayes estimates of the intensity function on the simulated data of Table 1, using the superposition model with $\lambda(t)$ given in (21) assuming $\alpha = 20, \beta_1 = 100, \beta_2 = 0.0005, \gamma = -1.0$ *and* $\tau = 2$, we have in Table 5, Monte Carlo estimates of $E(\lambda(t)|\mathcal{D}_t)$ based on $S = 1000$ Gibbs samples considering the superposition of a Musa-Okumoto and an exponential process including and not including the covariate x.

We observe better inference results considering the inclusion of the covariate x, especially for the levels 3 and 4 of the covariate x. In figure 4, we have plots of $m(t)$ considering the original model used to simulate the data of Table 1, the Bayes estimate not including the covariate x and the Bayes estimate including the covariate x.

t_i	x_i	Original Model	With Covariate	Without Covariate	t_i	x_i	Original Model	With Covariate	Without Covariate
1.6590	3	0.1954	0.2123	0.2003	165.0632	4	0.0755	0.0812	0.1418
3.4442	2	0.1843	0.2013	0.1976	172.1504	3	0.0736	0.0789	0.1426
4.3809	2	0.1826	0.1994	0.1962	175.2387	3	0.0728	0.0781	0.1430
22.1900	4	0.1635	0.1770	0.1749	182.6241	3	0.0709	0.0760	0.1440
25.2149	4	0.1596	0.1727	0.1720	184.8264	4	0.0709	0.0755	0.1443
37.6644	3	0.1444	0.1563	0.1623	186.5892	4	0.0698	0.0751	0.1445
39.9029	3	0.1421	0.1538	0.1608	190.4671	3	0.0690	0.0740	0.1451
43.4150	3	0.1387	0.1500	0.1586	193.0431	2	0.0696	0.0729	0.1455
43.5898	2	0.1337	0.1450	0.1585	195.1684	2	0.0692	0.0724	0.1459
59.3859	3	0.1248	0.1349	0.1508	198.6102	3	0.0672	0.0720	0.1464
61.9592	2	0.1191	0.1287	0.1498	224.8489	4	0.0616	0.0662	0.1513
64.1077	3	0.1213	0.1310	0.1490	227.2276	4	0.0612	0.0657	0.1518
70.5231	4	0.1172	0.1265	0.1469	230.9596	4	0.0605	0.0650	0.1526
73.6860	2	0.1114	0.1202	0.1459	235.2451	2	0.0624	0.0646	0.1535
82.1966	3	0.1093	0.1179	0.1439	247.9304	4	0.0575	0.0618	0.1563
88.3958	2	0.1032	0.1110	0.1426	252.9478	4	0.0567	0.0609	0.1574
89.2106	2	0.1028	0.1105	0.1425	253.5369	2	0.0599	0.0616	0.1576
92.9258	3	0.1032	0.1113	0.1419	256.0304	2	0.0596	0.0612	0.1582
95.6049	4	0.1022	0.1102	0.1415	272.1693	4	0.0538	0.0578	0.1621
97.4439	3	0.1009	0.1088	0.1413	283.5702	4	0.0522	0.0560	0.1649
105.7311	3	0.0969	0.1044	0.1404	288.3712	4	0.0516	0.0553	0.1662
106.8248	3	0.0964	0.1038	0.1403	291.4028	3	0.0517	0.0551	0.1670
122.2828	2	0.0886	0.0946	0.1396	293.3793	2	0.0479	0.0562	0.1675
131.7938	2	0.0853	0.0909	0.1396	297.1773	4	0.0504	0.0541	0.1685
135.8979	3	0.0847	0.0910	0.1397	298.3395	4	0.0502	0.0540	0.1688
140.5997	2	0.0825	0.0877	0.1399	306.5081	3	0.0499	0.0531	0.1710
145.1028	2	0.0812	0.0862	0.1401	314.0550	2	0.0535	0.0538	0.1731
145.4650	4	0.0815	0.0877	0.1401	329.2328	4	0.0467	0.0501	0.1774
151.4792	3	0.0795	0.0854	0.1405	345.7896	4	0.0450	0.0498	0.1849
160.3269	2	0.0770	0.0814	0.1413	355.1561	2	0.0503	0.0498	0.1849

Table 5. Monte Carlo estimates for $E(\lambda(t)|\mathcal{D})$

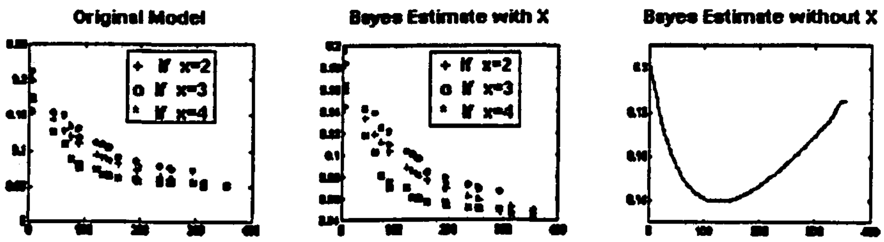

Figure 4. Comparison of the original intensity function $\lambda(t)$ and the Bayes estimates with a covariate x and without it.

Considering M_1 for the model with a covariate x and M_2 for the model not including the covariate, we find Monte Carlo estimates for V_M (see (20)) considering the generated Gibbs samples. Since $V_{M_2}/V_{M_1} < 1$, we conclude that the best fit for the software reliability data of Table 1 is given by the superposition of a Musa-Okumoto process and an exponential process including a covariate x.

6 Concluding Remarks

The use of Markov Chain Monte Carlo methods is a suitable way to get inferences for the superposition of nonhomogeneous Poisson process, considering different choices for the intensity functions. These models give a great flexibility to fit counting data in software reliability or other applications of nonhomogeneous Poisson process.

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