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Accurate Inferences for the Reliability Function Considering Accelerated Life Tests.

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Abstract

With accelerated life tests assuming the power rule model and a Weibull distribution for the lifetimes of the units, we consider some aspects of accurate inferences on the reliability function at time t_0 in a specified stress level V_p. We use a reparametrization proposed by Guerrero and Johnson (1982) exploring a nonnormality measure for likelihood funtions and posterior densities introduced by Kass and Slate (1992).

Keywords: Accelerated life tests, reliability function, reparametrization.

1. Introduction

Accelerated life tests are used in industrial applications when product life under normal operating conditions is very long but we need to obtain measures of reliability of units under the usual stress level (see for example, Mann, Schafer and Singpurwalla, 1974; Nelson, 1990). Usually, statisticians consider the asymptotical normality of the maximum likelihood estimators to get inferences for the reliability function at time t_0 assuming different parametric models and censored observations (see for example, Lawless, 1982). These asymptotical results could be very poor, considering small or moderate sample sizes.

To improve these approximate inference results, we could consider an appropriate reparametrization to get good normality of the likelihood function (see for example, Anscombe, 1964; or Sprott, 1973, 1980).

Good parametrization also is very important to get accurate Bayesian inferences considering numerical or approximation methods for posterior moments or posterior densities of interest (see for example, Achcar and Smith, 1990; Kass and Slate, 1992; or Hills and Smith, 1993).

Consider T a random variable denoting the life time of a unit with a Weibull density,

$$f(t;\alpha,\beta)=\frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1}\exp\left\{-\left(\frac{t}{\alpha}\right)^{\beta}\right\} \tag{1}$$

where $t \geq 0$; $\alpha,\beta > 0$ and assume a stress variable V affecting the scale parameter α, but with common shape parameter β for all stress levels.

With k levels of a stress variable V, assume the power rule model,

$$\alpha_i=\frac{\delta}{V_i^{\gamma}} \tag{2}$$

where $i = 1,2,\ldots,k$ and δ and γ are unknown parameters.

Also, assume a type II censoring mechanism, that is, the experiment terminates when we observe d_i failures for each stress level V_i. Thus, with n_i units at the beginning of each test with stress V_i, we have the ordered uncensored observations given by $t_{i(1)},t_{i(2)},\ldots,t_{i(d_i)}$ and n_i-d_i censored observations equal to $t_{i(d_i)}$, $i=1,2,\ldots,k$.

Considering the data of k stress levels $V_1, V_2, \ldots, V_k$ taken at random, the likelihood function for β, γ and δ is given by

$$L(\beta,\gamma,\delta)=\frac{\beta^d}{\delta^{d\beta}}\left(\prod_{i=1}^{k}V_i^{d_i\beta\gamma}\right)\left(\prod_{i=1}^{k}\prod_{j\in D_i}t_{i(j)}^{\beta-1}\right)\exp\left\{-\sum_{i=1}^{k}\frac{T_i(\beta)V_i^{\gamma\beta}}{\delta^{\beta}}\right\} \tag{3}$$

where $d=\sum_{i=1}^{k}d_i$, D_i is the set of uncensored observations at stress level V_i and $T_i(\beta)=\sum_{j=1}^{d_i}t_{i(j)}^{\beta}+(n_i-d_i)t_{i(d_i)}^{\beta}$.

Considering a specified stress level V_p, the reliability function at time t_0 is given by

$$R_p(t_0)=\exp\left\{-\left(\frac{t_0}{\alpha_p}\right)^{\beta}\right\}. \tag{4}$$

From (2) and (4), we have $\delta=t_0V_p^{\gamma}\left(-\ln r_p\right)^{-1/\beta}$, $0\le r_p=R_p(t_0)\le 1$, and the logarithm of the likelihood function for r_p, β and γ is given by

$$\begin{aligned} l\left(r_p,\gamma,\beta\right)&=d\ln\beta-d\beta\ln t_0+\beta\gamma\,\mathbf{K}_1+(\beta-1)\mathbf{K}_2+d\ln\left(-\ln r_p\right)+\\ &+\left(\ln r_p\right)\sum_{i=1}^{k}T_i(\beta)\left[\frac{Q_i(\gamma)}{t_0}\right]^{\beta}, \end{aligned} \tag{5}$$

where $\mathbf{K}_1=\sum_{i=1}^{k}d_i\ln V_i-d\ln V_p$, $\mathbf{K}_2=\sum_{i=1}^{k}\sum_{j\in D_i}\ln t_{i(j)}$ and $Q_i(\gamma)=\left(V_i/V_p\right)^{\gamma}$.

Standard classical methods for inferences on r_p, β and γ usually requires the use of iterative methods to obtain the maximum likelihood estimators and the use of asymptotical results.

2. An useful reparametrization for the reliability function at time t_0.

One way to improve the normality of the maximum likelihood estimator for the reliability function $R_p(t_0)$ at time t_0 and in a specified stress level V_p, is to consider different parametrizations or tranformations of $R_p(t_0)$.

To obtain an invertible family of tranformations which includes the logit tranformation $\ln\left[R_p/(1-R_p)\right]$, Guerrero and Johnson (1982), suggest the tranformation

$$\phi^{*}_{GJ}(\lambda)=\left[\left(\frac{R_p}{1-R_p}\right)^{\lambda}-1\right]\Big/\lambda \tag{6}$$

For a given λ, we can consider a modified form of Guerrero and Johnson tranformation given by

$$\phi_{GJ}(\lambda)=\left(\frac{R_p}{1-R_p}\right)^{\lambda}-1 \tag{7}$$

which should not produce different results as considering (6).

The transformation (7) has inverse given by

$$R_p=\frac{(\phi_{GJ}+1)^{1/\lambda}}{1+(\phi_{GJ}+1)^{1/\lambda}} \tag{8}$$

To consider the reparametrization (7), we should have an appropriate value of λ that gives good normality for the likelihood function of ϕ_{GJ}. One way to find this value assuming β and γ known, is to choose λ in (7) that gives third derivatives of the logarithm of the likelihood function $l[\phi_{GJ}(\lambda)]$ at the maximum likelihood estimator $\hat{\phi}_{GJ}(\lambda)$ in a standardized form,

$$STD\left[\hat{\phi}_{GJ}(\lambda)\right]=\left|\,l'''\left(\hat{\phi}_{GJ}(\lambda)\right)\left[-l''\left(\hat{\phi}_{GJ}(\lambda)\right)\right]^{-3/2}\right| \tag{9}$$

close to zero (see for example, Sprott, 1973; or Kass and Slate, 1992).

When all parameters are unknown, we could search for a joint transformation of $R_p(t_0)$, β and γ that gives joint normality for the likelihood function. One way to find this transformation is to explore some third derivative summaries that generalizes the standardized third derivatide (9) used in the one-paramater case (see Kass and Slate, 1992).

One of these measures is given by

$$m^2\overline{B}^2 = \sum_{a,b,c,d,e,f} d_{abc} d_{def} b_{ab} b_{de} b_{cf} \tag{10}$$

where b_{ab} are the elements of the inverse of the information matrix, d_{abc} denotes the third derivatives of the logarithm of the likelihood function locally at the maximum likelihood estimators and m is the number of parameters.

An alternative way is to search for an appropriate value of λ in reparametrization (7) for the reliability function $R_p(t_0)$ that gives close normality for the profile likelihood function of $\phi_{GJ}(\lambda)$.

3. The exponential case.

Assuming $\beta = 1$ and the reparametrization (7) for the reliability function $R_p(t_0)$ at time t_0 and stress level V_p, the logarithm of the likelihood function for ϕ_{GJ} and γ is given (from (5)) by

$$l(\phi_{GJ},\gamma) = -d \ln t_0 + \gamma\, \mathbf{K}_1 + d \ln[B(\phi_{GJ})] - \frac{B(\phi_{GJ})}{t_0} \sum_{i=1}^{k} T_i Q_i(\gamma) \tag{11}$$

where $\mathbf{K}_1$ and $Q_i(\gamma)$ are defined in (5), $T_i = \sum_{j=1}^{d_i} t_{i(j)} + (n_i - d_i) t_{i(d_i)}$ and $B(\phi_{GJ}) = \ln\left[1 + (1+\phi_{GJ})^{-1/\lambda}\right]$.

Assuming γ unknown, we should search for an appropriate value of λ in parametrization (7) such that

$$STD\left(\hat{\phi}_{\alpha J}\right)=\left| d^{-1/2}\left(2-\frac{3B''\left(\hat{\phi}_{\alpha J}\right)B\left(\hat{\phi}_{\alpha J}\right)}{\left[B'\left(\hat{\phi}_{\alpha J}\right)\right]^{2}}\right)\right| \tag{12}$$

is close to zero ($\hat{\phi}_{\alpha J}$ maximizes the profile likelihood function for $\phi_{\alpha J}$).

Assuming γ known, we obtain from $STD\left(\hat{\phi}_{\alpha J}\right)=0$ $\left(\text{where } \hat{\phi}_{\alpha J}=\left(\hat{r}_p^{-1}-1\right)^{-\lambda}-1;\right.$ $\left.\hat{r}_p=\exp\left\{-dt_0\Big/\sum_{i=1}^{k}T_iQ_i\right\}\right)$ the appropriate value of λ given by

$$\lambda=\left(\frac{2T}{3dt_0}+1\right)\left(1-e^{-dt_0/T}\right)-1 \tag{13}$$

where $T=\sum_{i=1}^{k}T_iQ_i$, $T_i=\sum_{j=1}^{d_i}t_{i(j)}+(n_i-d_i)t_{i(d_i)}$ and $Q_i=V_i^{\gamma}/V_p^{\gamma}$.

4. An Example.

In table 1, we have a generated type II censoring data set considering the power rule model (2) with $\gamma=0.8$, $\delta=500$ and an exponential distribution $(\beta=1 \text{ in } (1))$. From table 1 we have $d=\sum_{i=1}^{5}d_i=65$.

Table 1: Generated data set with $\gamma=0.8$ and $\delta=500$.

i	V_i	n_i	d_i	$t_{i(j)}$
1	10	30	5	6, 8, 10, 12, 14
2	20	30	8	4, 5, 5, 6, 8, 8, 9, 14
3	30	30	12	2, 3, 3, 5, 6, 7, 7, 8, 8, 9, 10, 17
4	40	30	18	3, 3, 4, 5, 6, 6, 8, 9, 10, 10, 12,12, 13, 14, 14, 15, 24
5	50	30	22	2, 3, 4, 5, 5, 8, 8, 8, 9, 10, 12, 13,14, 14, 15, 18, 18, 18, 19, 20, 20,27

The maximum likelihood estimators for γ and δ are given by $\hat{\gamma} = 0.7995$ and $\hat{\delta} = 500.5555$. The maximum likelihood estimator for the reliability function $R_1(t_0)$ at time $t_0 = 200$ considering the usual stress level $V_1 = 10$ is given by $\hat{r}_1 = 0.0806$.

To improve the joint normality of the maximum likelihood estimators, we consider the transformation (from (7)) $\phi_{GJ}(\lambda)$ and γ where the appropriate value of λ minimizes the third derivative summary (10). In table 2, we have the values for (10) considering $t_0 = 10,\ 100,\ 200$ and 400 in both parametrizations $(R_1(t_0), \gamma)$ and $(\phi_{GJ}(\lambda), \gamma)$.

Table 2: Values for $m^2\overline{B}^2$ (see (10)) and λ $(V_1 = 10)$.

t_0	$\hat{R}_1(t_0)$	$m^2\overline{B}^2(\hat{r}_1, \hat{\gamma})$	$\hat{\phi}_{GJ}(\lambda)$	$m^2\overline{B}^2(\hat{\phi}_{GJ}, \hat{\gamma})$	λ
10	0.8817	0.8589	0.1918	0.0014	0.0874
100	0.3201	0.4413	-0.2563	0.0014	0.3201
200	0.0806	5.9022	-0.4739	0.0014	0.2639
400	0.0065	35.5343	-0.5263	0.0014	0.1486

In figures 1 and 2, we have contour plots for the likelihood functions in both parametrizations. We observe a good improvement in the normality of the likelihood function for $\phi_{GJ}(\lambda)$ and γ, specially for $t_0 = 400$, since we have eliptical form for the contour plot (see figure 2).

Figure 1: Contour plots for the likelihood function $L(r_1, \gamma)$ $(V_1 = 10)$.

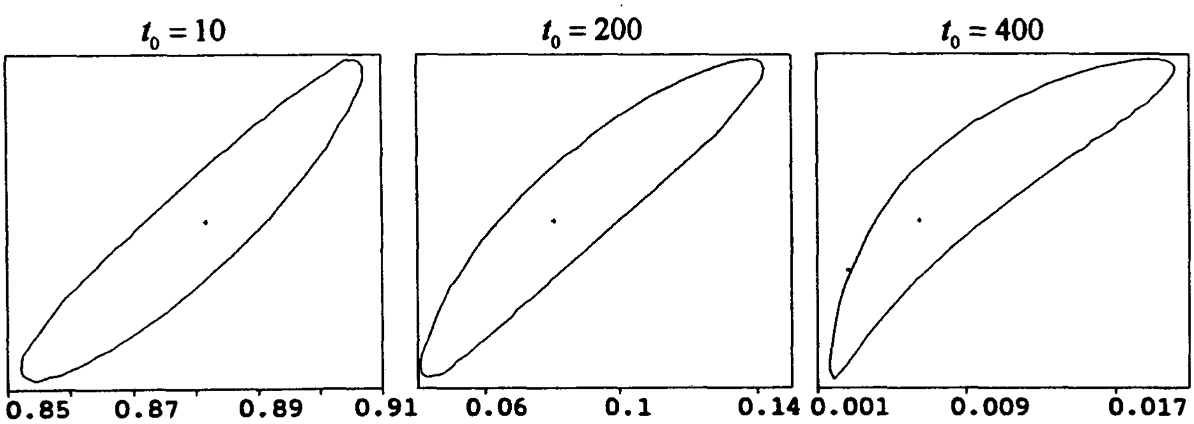
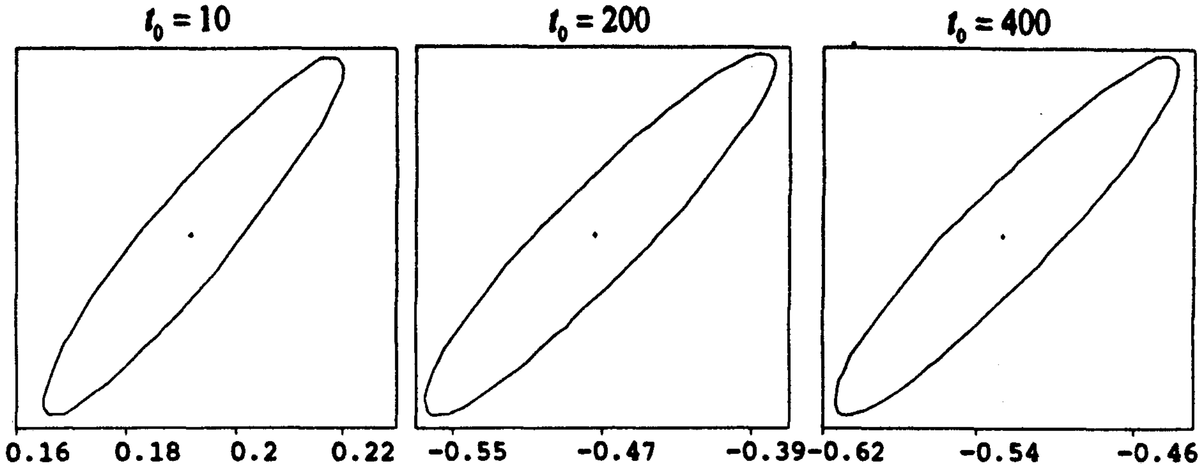


Figure 2: Contour plots for the likelihood function $L(\phi_{GJ}, \gamma)$ $(V_1 = 10)$.



In figures 3 and 4, we have plots for the profile likelihood functions $L_{\hat{\gamma}}(r_1)$ and $L_{\hat{\gamma}}(\phi_{GJ}(\lambda))$, where $\hat{\gamma}$ is the maximum likelihood estimator of γ given r_1 or $\phi_{GJ}(\lambda)$, respectively, with the appropriate values of λ given by (9). We observe good normality for the profile likelihood in parametrization $\phi_{GJ}(\lambda)$ (see figure 4).

Figure 3: Profile likelihood function $L_{\hat{\gamma}}(r_1)$ $(V_1 = 10)$.

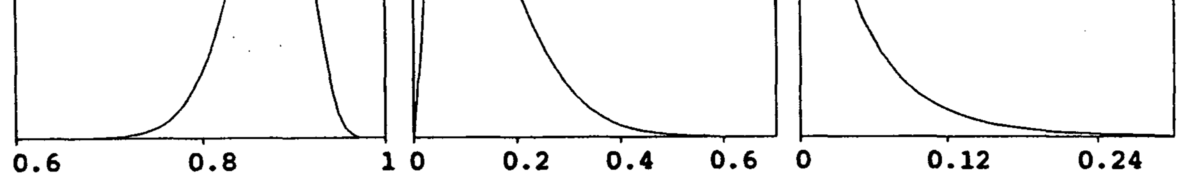
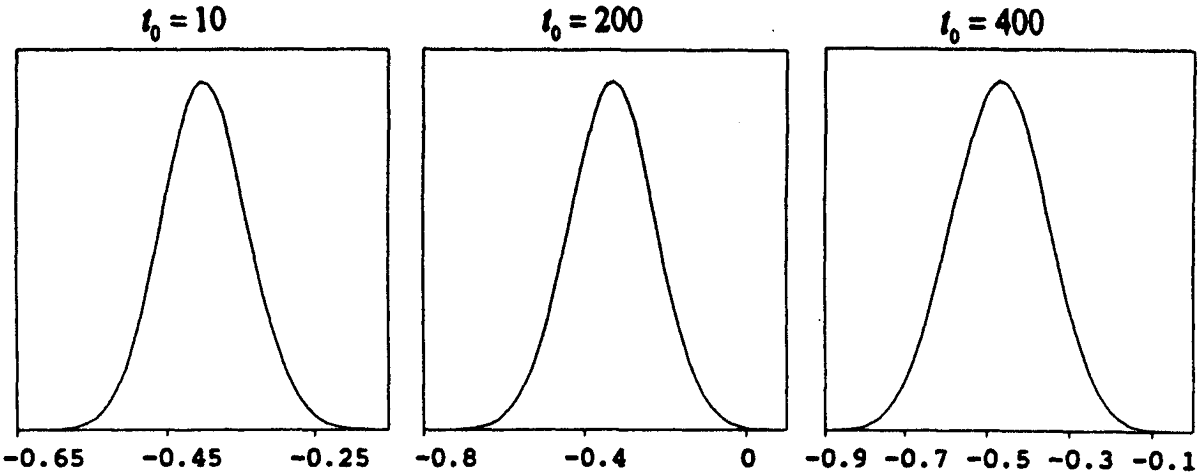


Figure 4: Profile likelihood function $L_\gamma(\phi_{GJ}(\lambda))$ $(V_1 = 10)$.



We also could check the normality of the profile likelihood function in parametrization $\phi_{GJ}(\lambda)$ considering the t-plot (see Hills and smith, 1993) $T(\phi_{GJ})$ against some values of ϕ_{GJ}, where,

$$T(\phi_{GJ}) = \text{sgn}(\phi_{GJ} - \hat{\phi}_{GJ})\left\{-2l_\gamma(\phi_{GJ}) + 2l_\gamma(\hat{\phi}_{GJ})\right\}^{1/2} \tag{14}$$

$\hat{\phi}_{GJ}$ is the maximum of the profile likelihood function $L_\gamma(\phi_{GJ}(\lambda))$, with the appropriate values of λ given by (9). Since we observe straight lines (see figure 5), we conclude by the normality of the profile likelihood function for $\phi_{GJ}(\lambda)$. In the original parametrization, the plots of $T(R_1(t_0))$ against $R_1(t_0)$ are markedly curved (see figure 6), which indicates the nonnormality of the profile likelihood function for $R_1(t_0)$, with $t_0 = 10$, 200 and 400.

Assuming $\gamma = 0.7995$ known, we have in table 3 approximate 95% confidence intervals for $R_1(t_0)$ with $t_0 = 10$, 100, 200 and 400 considering the asymptotical normality for the maximum likelihood estimators in both parametrizations $R_1(t_0)$ and $\phi_{GJ}(\lambda)$ and exact 95% confidence intervals for $R_1(t_0)$ based on the fact that $2T(-\ln r_1)/t_0$ has an exact chi-square distribution with $2d$ degrees of freedom. We observe good inference results considering the parametrization $\phi_{GJ}(\lambda)$ with the appropriate values of λ, especially for extreme values of t_0.

Figure 5: T-plot of the profile likelihood function $L_{\hat{\gamma}}(\phi_{GJ}(\lambda))$ $(V_1 = 10)$.

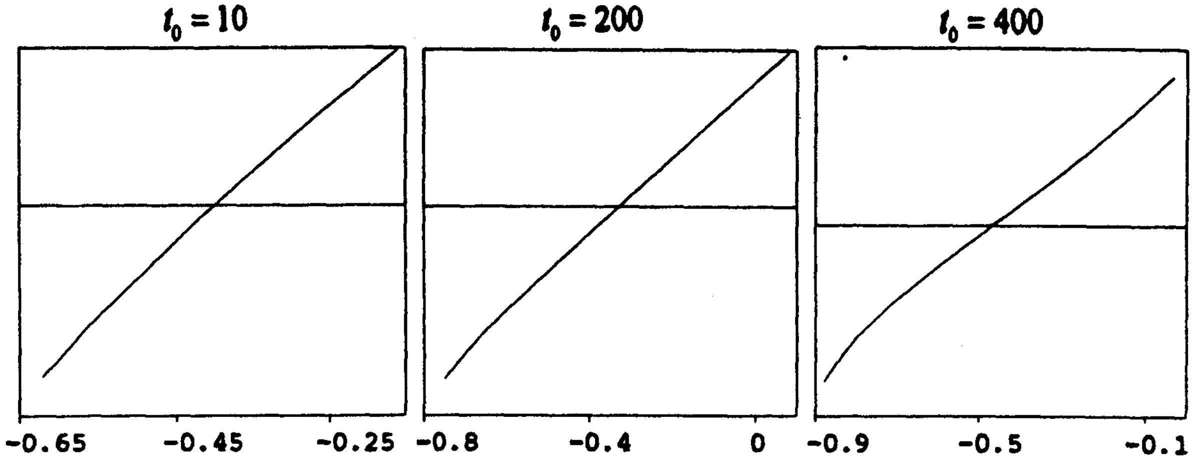


Figure 6: T-plot of the profile likelihood function $L_{\hat{\gamma}}(r_1)$ $(V_1 = 10)$.

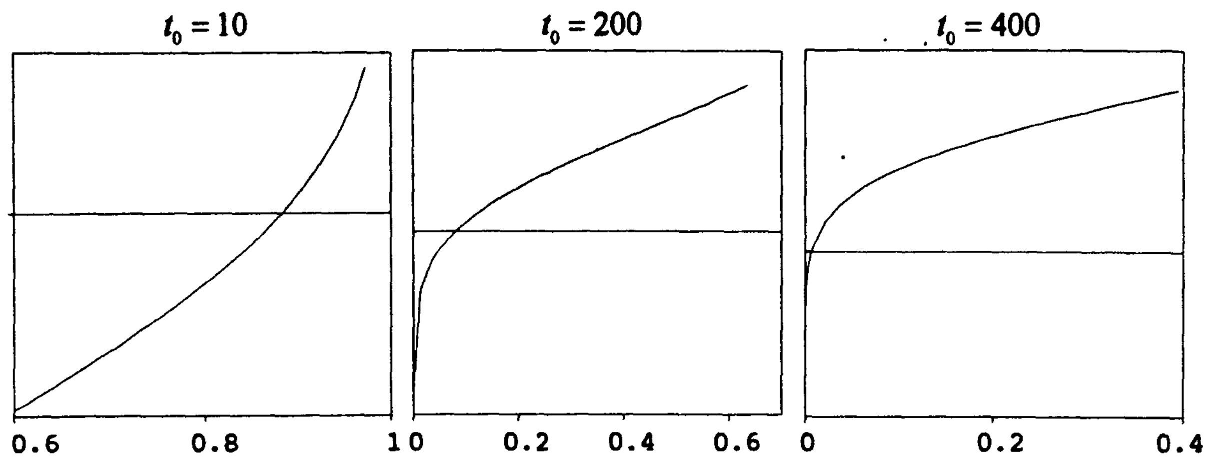


Table 3: 95% confidence intervals for $R_1(t_0)$ $(V_1 = 10)$.

t_0	Using exact distribution for $2T(-\ln r_1)/t_0$	Asymptotical normality for $\hat{R}_1(t_0)$	λ given by (13)	Asymptotical normality for $\hat{\phi}_{GJ}(\lambda)$
10	(0.8536 , 0.9074)	(0.8547 , 0.9086)	-0.2553	(0.8530 , 0.9069)
100	(0.2054 , 0.3761)	(0.1971 , 0.3709)	0.0952	(0.2040 , 0.3761)
200	(0.0422 , 0.1433)	(0.0313 , 0.1300)	0.1628	(0.0415 , 0.1416)
400	(0.0018 , 0.0205)	(-0.0015 , 0.0145)	0.1250	(0.0017 , 0.0202)

In table 4, we have Laplace's approximations (see for example, Tierney and Kadane, 1986) for the posterior mean $E(R_1(t_0)/\text{data})$ with $\beta = 1$ and $\gamma = 0.7995$ known, considering a Jeffreys noninformative prior density for $R_1(t_0)$, $\pi_0(r_1) \propto 1/(-\ln r_1)\, r_1$, $0 \le r_1 \le 1$ (see for example, Martz and Waller, 1982). Since in this case, we have an exact expression for $E(R_1(t_0)/\text{data})$, we observe very accurate approximate Bayesian inferences for $R_1(t_0)$ considering the parametrization $\phi_{\omega}(\lambda)$ with λ given by (13).

Table 4: Posterior means for $R_1(t_0)$ $(V_1 = 10)$.

t_0	Exact	Laplace's Approximations			
		Reparametrization $R_1(t_0)$	Percentage errors	Reparametrization $\phi_{\omega}(\lambda)$	Percentage errors
10	0.8818	0.8818	(0.000)	0.8818	(0.000)
100	0.2874	0.2874	(0.009)	0.2874	(0.002)
200	0.0846	0.0848	(0.203)	0.0845	(0.018)
400	0.0078	0.0080	(2.540)	0.0078	(0.052)

5. Concluding remarks.

The use of transformation (7) for the reliability function at a specified time t_0 and in a fixed stress level V_p considering an appropriate value for λ obtained from the standardized third derivative summaries (9) or (10) could be of great practical interest to improve the accuracy of the asymptotical inferences. As it was observed in this paper, the invertible reparametrization (7) is easily obtained for each application considering accelerated life test models. We also could extend these results to other stress-response models used in engineering applications, like the Arrhenius or Eyring models (see for example, Nelson, 1990), and to consider other parametrical distributions for the lifetimes of the units.

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