UNIVERSIDADE DE SÃO PAULO

Bayesian estimation in the study of tamped Random variables

JOSEMAR RODRIGUES

ERIC FERNANDO RODRIGUES

Nº 10

NOTAS

Instituto de Ciências Matemáticas de São Carlos

Instituto de Ciências Matemáticas de São Carlos

ISSN - 0103-2577

Bayesian estimation in the study of tamped Random variables

JOSEMAR RODRIGUES
ERIC FERNANDO RODRIGUES

Nº 10

N O T A S D O I C M S C
Série Estatística

São Carlos
abr. / 1994

Bayesian Estimation in the Study of Tampered Random Variables

Josemar Rodrigues
ICMSC-USP-13560-970-São Carlos-SP-Brasil
and
Eric Fernando Rodrigues
DEs-UFSCar-13565-905-São Carlos-SP-Brasil

Abstract

This paper is concerned with a Bayesian analysis of the tampered exponential model originally introduced by Goel (1971). This model can be applied in various situations where the job is interrupted by introducing some external factor and the remaining time to finish it is changed by an unknown constant α. This constant is called tampering coefficient. The motivation to formulate this model was the desire to study the effectiveness of an injection (the external factor) which is given to the pacient to reduce the time to delivery of the baby. The posterior distribution of α is obtained and the statistical analysis of the effectiveness of the external factor is discussed from a decision point of view.

Key words: Bayesian estimator; tampered variables; tampering coefficient.

1 Introduction

Let X be a random variable with an exponential distribution with density

$$f(x \mid \theta) = \theta \exp\{-\theta x\}, \quad x > 0, \quad \theta > 0, \tag{1}$$

θ unknown parameter. In some experiments, however, we may not be able to observe X because of practical limitations and instead we observe a random variable, for some specified x, defined as follows:

$$Y = \begin{cases} X, & \text{for } X < x \\ x + \alpha(X - x), & \text{for } X \geq x \end{cases} \tag{2}$$

If $X \geq x$, we accelerate or decelerate the process at instant x by some unknown factor α. Thus a sample of n observations $Y_1, \ldots, Y_n$ is obtained on the random variable Y corresponding to preassigned tampering points $x_1, \ldots, x_n$. If the variable Y is less than the tampering points x, then Y is untampered variable. Otherwise, Y is called tampered variable (Goel, 1971). In general, an untampered observation comes from a test item that failed under standard conditions and a tampered observation comes from a test item that failed under the higher stress level. We suppose that, if the value X is greater or equal to specified value x, then we introduce in the job some external factor and the process is continued until to finish. The parameter α is a measure of the effectiveness of this external factor. In this paper, we are interested in estimating the parameter α from the Bayesian viewpoint. When working with the accelerated lifetime models, usually $0 \leq \alpha < 1$. Since this restriction in the possible values of α is incorporated in a natural way as a part of the prior information (Armero an Bayarri, 1992), we consider in this paper $\alpha > 0$.

The motivation for this problem was the desire to study the effectiveness of an injection (external factor) which is given to the pacient to reduce the labor duration. The labor duration is the duration from the onset of labor (rupture of the membrane) to delivery of the baby. Since pacients takes a different amount of time to reach the hospital, the injection is given after different times from the onset of the labor. In this experiment, the observations are the labor duration before the injection was given as well the labor duration after the injection at different critical points.

This paper is organized as follows: In Section 1 we comment and formulate the model. In Section 2 we introduce a new family of distributions (called Kummer distribution) that will be need in the rest of the paper. In Section 3 we derive the the posterior distribution of the tampering coefficient with respect to a conjugate prior. Finally, the statistical anlysis from a decision point of view of the tamperimg coefficient is discussed in Section 4 and a simulation study is presented in Section 5.

2 The Kummer distribution

Before presenting the statistical analysis of the model (2), we introduce a new family of continuous distributions that appears frequently in this paper. The results of this section are an adaptation of more general results given by Armero and Bayarri (1994).

Defintion 1: A random variable T has a Kummer distribution with parameters α, β, γ $(0 < \alpha < \beta, \gamma > 0)$ if it has a continuous distribution whose p.d.f. is

$$Ku(t \mid \alpha, \beta, \gamma) = C \frac{t^{\alpha-1}}{(1+\gamma t)^{\beta}}, \quad t > 0 \tag{3}$$

and $Ku(t \mid \alpha, \beta, \gamma) = 0$ otherwise, where the proportionality constant C is that

$$C^{-1} = \frac{\Gamma(\alpha)}{\gamma^{\alpha}} U(\alpha, \alpha + 1 - \beta). \tag{4}$$

The function $U(a.b)$ is a particular case of Kummer's function (see for instance, Abramowitz and Stegun, 1964) with integral representation, for $a > 0, 0 < b < 1$ given by

$$\Gamma(a) U(a, b) = \int_0^{\infty} x^{a-1}(1+x)^{b-a-1} dx. \tag{5}$$

But the integral on RHS of (5) does converge and it is given by $\Gamma(a)\Gamma(1-b)/\Gamma(a+1-b)$. Hence, we have that

$$U(a, b) = \frac{\Gamma(1-b)}{\Gamma(a+1-b)}, \quad a > 0, \quad 0 < b < 1. \tag{6}$$

We have called the distribution in Definition 1 the Kummer distribution because it is derived from a particular case of Kummer's function. It generalizes the F-distribution. Clearly, $F(\nu_1, \nu_2) = Ku(\frac{\nu_1}{2}, \frac{\nu_1+\nu_2}{2}, \frac{\nu_1}{\nu_2})$ and

$$\frac{(\beta-\alpha)\gamma}{\alpha} T \sim F(2\alpha, 2\beta - 2\alpha), \tag{7}$$

for $\alpha < \beta$. The moments of $Ku(t \mid \alpha, \beta, \gamma)$ distribution can easily be computed from (7) and the moments of the F-distribution and are given by

$$E[T^k] = \frac{\Gamma(\alpha + k)}{\Gamma(\alpha)\gamma^k} \cdot \frac{\Gamma(\beta - \alpha - k)}{\Gamma(\beta - \alpha)}. \tag{8}$$

3 Posterior Distribution

A sample of n observations $Y_1, \ldots, Y_n$ is obtained on the variable Y corresponding to the known fixed values $x_1, \ldots, x_n$. The statistical problem involved, under model (1) and (2), is the estimation of α for given values $x_1, \ldots, x_n$. In fact, if m denotes the number of tampered observvations among $Y_1, \ldots, Y_n$ and A denotes the set of indices $i \in \{1, \ldots, n\}$ for which Y_i is a tampered observation, then it can be shown that the likelihood is

$$L(\theta, \beta) = \theta^n \exp\{-\theta[\sum_{i \in A^c} T_i + \sum_{i \in A} x_i + \beta \sum_{i \in A}(Y_i - x_i)]\}\beta^m, \tag{9}$$

where $\beta = 1/\alpha$. It is convenient work with the parameter β rather than directly with the parameter α. A conjugate prior density based on the form (9) is

$$p(\theta, \beta) \propto \theta^{n_0 - 1} \exp\{-\theta(a_0 + \beta b_0)\}\beta^{m_0 - 1}, \quad m_0 < n_0. \tag{10}$$

This prior defines a proper density which we will denote in this paper by $(\theta, \beta) \sim GaKu(n_0, a_0, b_0, m_0)$, for $n_0 > 0$, $a_0 > 0$, $b_0 > 0$, $m_0 > 0$. This notation means that (θ, β) has a Gamma-Kummer distribution. The name comes from the fact that, if $(\theta, \beta) \sim GaKu(n_0, a_0, b_0, m_0)$ then

$$\begin{aligned} p(\theta \mid \beta) &= Ga(\theta \mid n_0, a_0 + \beta b_0) \\ &= \frac{(a_0 + \beta b_0)^{n_0}}{\Gamma(n_0)} \theta^{n_0 - 1} \exp\{-\theta(a_0 + \beta b_0)\}, \end{aligned} \tag{11}$$

and for $a > 0$, the marginal distribution of β is

$$\begin{aligned} p(\beta) &= Ku(\beta \mid m_0, n_0, \delta) \\ &= \frac{\delta^{m_0} \beta^{m_0 - 1}}{\Gamma(m_0) U(m_0, m_0 - n_0 + 1)(1 + \delta\beta)^{n_0}} \\ \delta &= \frac{b_0}{a_0}, \quad \beta > 0. \end{aligned} \tag{12}$$

If the prior distribution of (θ, β) is $GaKu(n_0, a_0, b_0, m_0)$, then it follows from (9) an (10) that the posterior density is given by

$$p(\theta, \beta) \propto L(\theta, \beta) p(\theta, \beta) = GaKu(n_1, a_1, b_1, m_1)$$

where

$$
\begin{aligned}
n_1 &= n_0 + n \\
a_1 &= a_0 + \sum_{i \in A^c} T_i + \sum_{i \in A} x_i \\
b_1 &= b_0 + \sum_{i \in A} (Y_i - x_i) \\
m_1 &= m_0 + m.
\end{aligned} \tag{13}
$$

Therefore, the Gamma-Kummer distribution is conjugate for the experiment considered.

For assessment purposes we find it easiest to think in terms of the following parametrization:

$$
\begin{cases} \theta = \theta \\ \eta = \theta\beta \end{cases} \tag{14}
$$

The above parametrization factorizes the likelihood function (9) so that

$$
L(\theta, \eta) = A(\theta).B(\eta),
$$

where

$$
\begin{aligned}
A(\theta) &= \theta^{n-m} \exp\{-\theta[\sum_{i \in A^c} T_i + \sum_{i \in A} x_i]\}, \quad \text{and} \\
B(\eta) &= \eta^n \exp\{-\eta \sum_{i \in A} (Y_i - x_i)\}.
\end{aligned} \tag{15}
$$

The separability of (15) reflects, following Basu (1997), two independent sources of information: $A(\theta)$ which is informative exclusively about θ and $B(\eta)$ which provides information about η. It is interesting to note that the contribution of $A(\theta)$ is the same as that of an ordinary life test based on the exponential distribution censored at x_i. If we are interested in θ, an usual Bayesian analysis based on $A(\theta)$ can be easily performed (see for example Martz an Waller, 1982). Since the likelihood is separable, we recognize the parameter θ and η are unrelated (Basu, 1977). In this situation, it is quite natural to assume a conjugate gamma prior for θ and η, or.

$$
p(\theta, \eta) = Ga(\theta \mid \alpha_1, \beta_1).Ga(\eta \mid \alpha_2, \beta_2) \tag{16}
$$

which results from (14) in

$$\begin{aligned} p(\theta,\beta) &= GaKu(n_0,a_0,b_0,m_0), \quad \text{where} \\ n_0 &= \alpha_1+\alpha_2 \\ a_0 &= \beta_1 \\ b_0 &= \beta_2 \\ m_0 &= \alpha_2. \end{aligned} \tag{17}$$

The values $\alpha_1, \alpha_2, \beta_1$ and β_2 can be easily elicitated from (16) by using the following equations:

$$\begin{aligned} \alpha_1 &= \frac{\alpha_0^2}{\sigma_0^2} \quad , \beta_1 = \frac{\alpha_0}{\sigma_o^2} \\ \alpha_2 &= \frac{\beta_0^2}{\sigma_0^2} \quad , \beta_2 = \frac{\beta_0}{\sigma_0^2}, \end{aligned} \tag{18}$$

where α_0= the prior mean of θ, β_0= the prior mean of η and σ_0^2 is the prior variance of θ and η.

We recall that our interest is to get information about β, considering θ as a nuisance parameter. In fact, from (9) and (17), it can be shown that the marginal posterior of β is

$$p(\beta \mid \text{data}) = Ku[\beta \mid \alpha_1+m, \alpha_1+\alpha_2+n, \frac{\beta_2+\sum_{i\in A}(Y_i-x_i)}{\beta_1+\sum_{i\in A^c}T_i+\sum_{i\in A}x_i}]. \tag{19}$$

or from (7), we have that

$$\frac{(\alpha_1+n-m)[\beta_2+\sum_{i\in A}(Y_i-x_i)]}{(\alpha_2+m)[\beta_1+\sum_{i\in A^c}T_i+\sum_{i\in A}x_i]}\beta \mid \text{data} \sim F[2(\alpha_2+m), 2(\alpha_1+n-m)].$$

It should be noted that this posterior does not depend on the values of the tampering points corresponding to the untampered observations. Hence, it does not depend on the method by which these points were chosen. Taking in (3) $\alpha = m_0+m$, $\beta = n_0+m$ and

$$\gamma = \frac{\beta_2+\sum_{i\in A}(Y_i-x_i)}{\beta_1+\sum_{i\in A^c}T_i+\sum_{i\in A}x_i}, \tag{20}$$

it can be shown from (8) that Bayes estimator of β with respect to the square loss function is given by

$$\hat{\beta} = \frac{\beta_1 + \sum_{i\in A^c} T_i + \sum_{i\in A} x_i}{\beta_2 + \sum_{i\in A}(Y_i - x_i)} \cdot \frac{\alpha_2 + m + 1}{\alpha_1 + n - m}. \tag{21}$$

The posterior mode can be easily obtained from (19) and is given by

$$\hat{\beta}_m = \frac{(\alpha_1 + n - m)(\alpha_2 + m - 1)}{(\alpha_2 + m + 1)(\alpha_1 + n - m + 1)}\hat{\beta}. \tag{22}$$

The optimal choice of the values $x_1, \ldots, x_n$ is presented by Goel (1971) for various types of observational costs. It is shown, that ,for many cost functions, the optimal design uses only tampering points $x_i = 0$ and $x_i = \infty$ and the number of tampered observations ,or, the number of $x_i = 0$ is explicity derived.

4 Test for acceleration or deceleration

In many practical situations involving lifetime observations,it can be interesting to study whether acceleration can or can not be attained.In the case of model (2) a test for acceleration or deceleration can be stated as testing

$$\mathrm{H}_0 : \beta \leq 1 \quad \text{against} \quad \mathrm{H}_1 : \beta > 1. \tag{23}$$

From the Bayesian point of view, this test is easy to formulate in terms of the posterior distribution of β (19). From the inferential point of view, we choose between H_0 or H_1 based on the posterior probabilities:

$$\begin{aligned} P[\mathrm{H}_0 \mid \text{data}] &= P[\beta \leq 1 \mid \text{data}] \quad \text{and} \\ P[\mathrm{H}_1 \mid \text{data}] &= P[\beta > 1 \mid \text{data}] = 1 - P[\beta \leq 1 \mid \text{data}]. \end{aligned} \tag{24}$$

In general, this problem can be considered as decision problem with actions a_0 an a_1, where a_i denotes acceptance of the hypothesis H_i ,$i = 0, 1$ and a loss function $L(a_i, \beta)$ which represents the loss when the value of the tampering coefficient is β and the statistician takes the action a_i. From the posterior distribution of β (19). the posterior expected loss of action a_i is

$$l(a_i) = E_{\beta|\text{data}}[L_i(a_i, \beta)], \quad i = 0, 1, \tag{25}$$

and the Bayes action will be that for which the posterior loss is the smallest. For this testing problem, we assume the folowing loss function:

$$\begin{aligned} L(a_0,\beta) &= 0 \quad \text{and} \quad L(a_1,\beta) = c_1, \text{for} \quad \beta < 1 \\ L(a_0,\beta) &= c_0 \quad \text{and} \quad L(a_1,\beta) = 0, \text{for} \quad \beta \geq 1. \end{aligned} \tag{26}$$

From this loss function we obtain :

$$\begin{aligned} l(a_0) &= c_0 P[\beta \geq 1 \mid \text{data}] \\ l(a_1) &= c_1 P[\beta < 1 \mid \text{data}] \end{aligned} \tag{27}$$

and a_0 is the Bayes action if

$$l(a_0) < l(a_1) \tag{28}$$

which is equivalent to

$$P[\beta < 1 \mid \text{data}] > \frac{c_0}{c_1 + c_0}. \tag{29}$$

From (29) and (7) we have that a_0 is the Bayes action if

$$P[F(\nu_1,\nu_2) < B \mid \text{data}] > \frac{c_0}{c_1 + c_0}, \tag{30}$$

where

$$\begin{aligned} B &= \frac{(\alpha_1 + n - m)[\beta_2 + \sum_{i\in A}(Y_i - x_i)]}{(\alpha_2 + m)[\beta_1 + \sum_{i\in A^c} T_i + \sum_{i\in A} x_i]}, \\ \nu_1 &= 2(\alpha_1 + m) \quad \text{and} \quad \nu_2 = 2(\alpha_2 + n - m) \end{aligned}$$

and $F(\nu_1, \nu_2)$ is a random variable with F-distribution with ν_1 and ν_2 degrees of freedom.

5 Simulation Study

In this section, we illustrate the performance of the posterior mean and posterior mode based on a simulation of size 1000. The simulation was carried out on a 286 micro-computer using the Pascal language to generate pseudo -random numbers from the exponential model (1) with $\theta = 1.0$ and $\beta = 2.0$.. The tampering points were chosen to be (for $n = 10$):

$$\begin{aligned} x_1 &= 1.0, \quad x_2 = 1.5, \quad x_3 = 1.0, \quad x_4 = 1.8, \quad x_5 = 1.5, \\ x_6 &= 1.3, \quad x_7 = 0.9, \quad x_8 = 0.8, \quad x_9 = 0.7, \quad x_{10} = 1.0 \end{aligned}$$

Table 1 shows the good performance of $\hat{\beta}$ for various values of the prior means α_0, β_0 and the prior variance σ_0^2 . Table 2 shows the performance of the posterior mode $\hat{\beta}_m$ which is very close to the posterior mean for small prior variance σ_0^2. Figura 1 and 2 illustrate the behavior of the Kummer posterior ditribution of β for $n = 10, \alpha_0 = 1, \beta_0 = 2$ and $\sigma_0^2 = 1.0$ and $\sigma_0^2 = 0.001$, respectively.

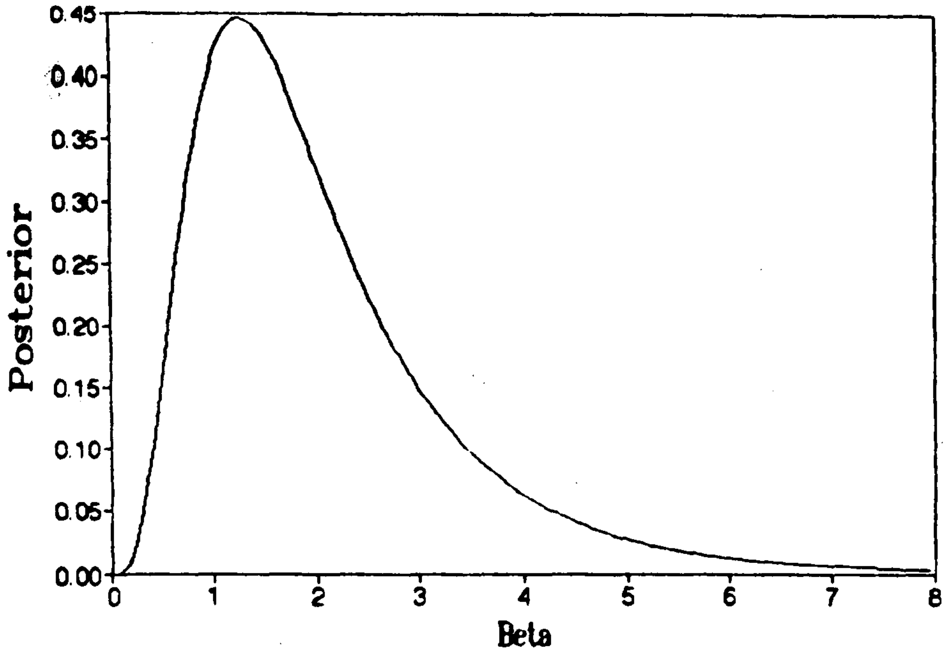

Figure 1: Posterior distribution of β for $\alpha_0 = 1.0$,$\beta_0 = 2.0$ and $\sigma_0^2 = 1.0$

Table 1: One thousand sets of simulated data for $\theta = 1.0, \beta = 2.0$ and $n = 10$. Average of the posterior mean (21) for different values of α_0, β_0 and σ_0^2.

		$\sigma_0^2 = 0.001$				$\sigma_0^2 = 1.0$			
		α_0				α_0			
		0.9	1.0	1.1	3.0	0.9	1.0	1.1	3.0
	1.9	2.1065	1.8979	1.7265	0.6342	2.0022	1.9885	1.9628	1.2604
β_0	2.0	2.2173	1.9975	1.8170	0.6675	2.0774	1.9899	1.9852	1.2916
	2.1	2.3280	2.0970	1.9076	0.7008	2.0561	2.0401	2.0186	1.2951
	4.0	4.4313	3.9923	3.6319	1.3343	2.9580	2.8883	2.8635	1.8446

Table 2: One thousand sets of simulated data for $\theta = 1.0, \beta = 2.0$, and $n = 10$. Average of the posterior mode (22) for different values of α_0, β_0 and σ_0^2.

		$\sigma_0^2 = 0.001$				$\sigma_0^2 = 1.0$			
		α_0				α_0			
		0.9	1.0	1.1	3.0	0.9	1.0	1.1	3.0
	1.9	2.0992	1.8922	1.7220	0.6337	1.1191	1.1294	1.1266	0.8710
β_0	2.0	2.2097	1.9916	1.8122	0.6670	1.1671	1.1256	1.1373	0.8935
	2.1	2.3200	2.0908	1.9026	0.7003	1.1487	1.1549	1.1611	0.8943
	4.0	4.4161	3.9804	3.6223	1.3332	1.6584	1.6317	1.6450	1.2785

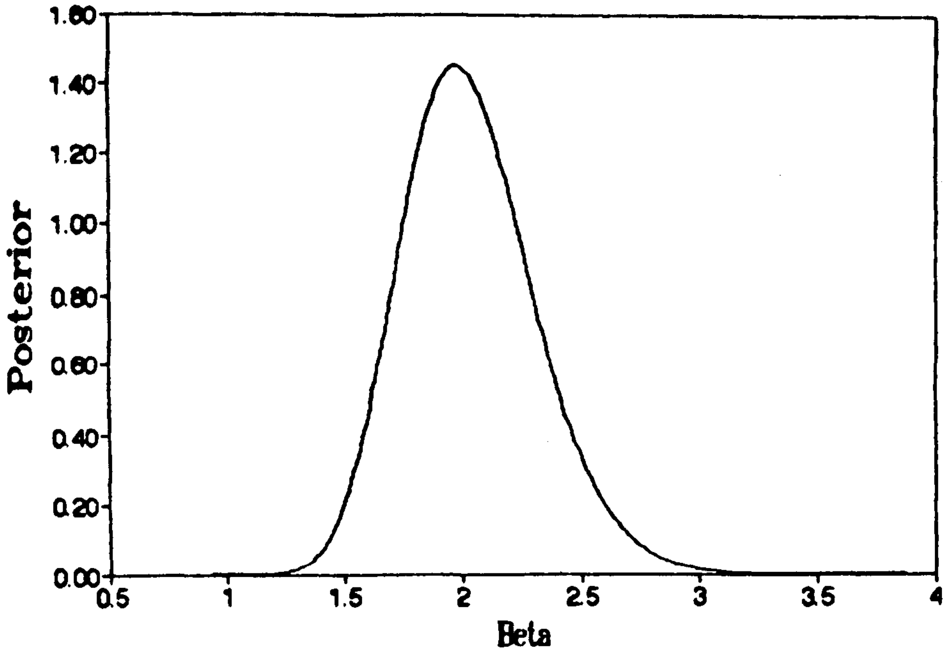

Figura 2: Posterior distribution of β for $\alpha_0 = 1.0$, $\beta_0 = 2.0$ and $\sigma_0^2 = 0.001$.

References

[1] Abramowitz, M.; Stegun, I.A. (1965). *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, Dover Publications, Inc., New York

[2] Armero, C. and Bayarri, M.J. (1992). Bayesian Prediction in M/M/1 Queues. Technical Report Nr. 9-92 - Universitat de Valencia.

[3] Basu, D. (1997). On the elimination of nuisance parameters, *JASA*, v.22. no. 358.355-365.

[4] Goel, P.K. (1971). Some estimation problems in the study of tampered random variables, Technical Report n0. 60, Department of Statistics. Carnigie-Mellon

University (Pittsburg, Penn.).

[5] Martz, H. and Waller, R.A. (1982). *Bayesian analysis*, Wiley.

NOTAS DO ICMSC

Serie Estatistica

Nº 009/94 ACHCAR, J.A.; FOGO, J.C. An useful reparametrization for the reliability in the Weibull case

Nº 008/94 RODRIGUES, J.; LEITE, J.G. A note on Bayesian analysis in M/M/1 queues from confidence intervals

Nº 007/94 ACHCAR, J.A.; DAMACENO, V.L. An useful reparametrization for the survival function considering an exponentialregression model

Nº 006/94 RODRIGUES, J. Bayesian estimation of a normal mean parameter using linex loss function and robustness considèrations

Nº 005/93 ACHCAR, J.A.; FOGO, J.C. Some useful reparametrizations for t reliability function considering an exponential regession model

Nº 004/93 LEME, E.C. A m.v.u. estimator v.s. a m.l. estimator: a way o comparing their precisions by using the Pareto distribution

Nº 003/93 ACHCAR, J.A. Some practical aspects of approximate bayesian inference

Nº 002/93 RODRIGUES, J.; BABA, M.Y. Bayesian estimation of a simple regression model with measurement errors

Nº 001/93 ACHCAR, J.A. Some aspects of reparametrization for statistic models