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Abstract

In this paper, we present a simple method to get appropriate reparametrization for the reliability function at time t_0 considering censored lifetime data and a Weibull distribution. With the obtained reparametrization, we get very accurate aproximate inference results based on the "normality" of the likelihood or posterior density for the reliability function.

Keywords: Reliability function, reparametrization, Weibull distribution, censored data.

1 Introduction

Usually, inferences on $R(t_0) = P(T > t_0)$, the reliability function at time t_0, assuming different parametrical models and censored lifetime observations are obtained by using asymptotical methods (see foe example, Lawless, 1982). One of these asymptotical results is given by the asymptotical normality of the maximum likelihood estimators. Under the Bayesian approach, we get marginal posterior densities or posterior moments for $R(t_0)$ based on numerical or approximation methods. These results, usually depend on an appropriate transformation of $R(t_0)$, to get accurate results. One way to find an appropriate reparametrization, is to search for an one-to-one transformation of $R(t_0)$ that gives close "normality" for the likelihood function (see for example, Anscombe, 1964; Sprott, 1973, 1980; Kass and Slate, 1992; or Hills and Smith, 1993).

Assuming a Weibull distribution for the lifetimes in a reliability experiment, we explore the use of a transformation for proportions introduced by Guerrero and Johnson (1982) and a measure to nonnormality of likelihood functions given by the standardized form of the third derivative of the logarithm of the likelihood function (see Sprott, 1973; or, Kass and Slate, 1992).

We also check the adequability of the proposed reparametrization, by using a t-plot proposed by Hills and Smith (1993).

With the proposed reparametrization of $R(t_0)$, we get in a simple way, very accurate inference results for the reliability function at time t_0.

2 A Reparametrization for $R(t_0)$

Some parametric families of transformations for proportions (see for example, Atkinson, 1985) could be used to improve the "normality" of the likelihood for $R(t_0)$.

Among these transformations, an invertible family of transformation which includes the logit parametrization $\phi_L = ln[R/(1-R)]$, where $R = R(t_0)$ is given (see Guerrero and Johnson, 1982) by

$$\phi^{*}_{GJ}(\lambda) = \left\{\left(\frac{R}{1-R}\right)^{\lambda} - 1\right\} \Big/ \lambda \, . \tag{1}$$

For a given λ, we can consider a modified form of (1) given by

$$\phi_{GJ}(\lambda) = \left(\frac{R}{1-R}\right)^{\lambda} - 1 \, , \tag{2}$$

which should not produce different results as considering (1).

The great advantage of transformation (2), is that it is readily inverted. With $\phi_{GJ} = \phi_{GJ}(\lambda)$, we obtain

$$R = \frac{(\phi_{GJ}+1)^{1/\lambda}}{1+(\phi_{GJ}+1)^{1/\lambda}} \, . \tag{3}$$

To find an appropriate value of λ that gives good "normality" of the likelihood function for $\phi_{GJ}(\lambda)$, we choose λ in (2) that gives third derivative of the logarithm of the likelihood function $l(\phi_{GJ}(\lambda))$ at the maximum likelihood estimator $\hat{\phi}_{GJ}(\lambda)$ in a standardized form,

$$STD\left(\hat{\phi}_{GJ}(\lambda)\right) = \left| l'''\left(\hat{\phi}_{GJ}(\lambda)\right)\left(-l''\left(\hat{\phi}_{GJ}(\lambda)\right)\right)^{-3/2} \right| \tag{4}$$

close to zero (see Sprott, 1973; or Kass and Slate, 1992).

3 Reliability Function at Time t_0 Considering a Weibull Distribution and Censored Data

Suppose there is a random sample of n units with lifetimes $T_1, T_2, \ldots, T_n$, but that associated to each unit is also a fixed censoring time $L_i > 0$ (type I censored data). We observe T_i only if $T_i \leq L_i$ and the data consists of pairs $(T_i, \delta_i), i = 1, \ldots, n$ where $T_i = min(T_i, L_i)$ and $\delta_i = 1$ if $t_i = T_i$ or $\delta_i = 0$ if $t_i = L_i$.

Considering the Weibull distribution with density,

$$f(t;\alpha,\beta) = \frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1} exp\left\{-\left(\frac{t}{\alpha}\right)^{\beta}\right\} \tag{5}$$

where $t > 0; \alpha, \beta > 0$ and type I censored data, the log-likelihood function for the reliability function at time $t_0, R = R(t_0) = exp\left\{-\left(\frac{t_0}{\alpha}\right)^{\beta}\right\}$ and β, is given by

$$\begin{aligned} l(R,\beta) &= dln\beta - d\beta lnt_0 + dln(-lnR) + \\ &+ (\beta - 1)\textstyle\sum_{i\in D} lnt_i + \frac{T(\beta)}{t_0^{\beta}} lnR \end{aligned} \tag{6}$$

where $T(\beta) = \sum_{i=1}^{n} t_i^{\beta}, d = \sum_{i=1}^{n} \delta_i$ is the observed number of lifetimes and D denotes the set of units for whom lifetimes are uncensored.

The maximum likelihood estimator for $R(t_0)$ is given by

$$\hat{R}(t_0) = exp\left\{-\frac{dt_0^{\hat{\beta}}}{\sum_{i=1}^{n} t_i^{\hat{\beta}}}\right\}, \tag{7}$$

where $\hat{\beta}$ satisfies,

$$\frac{d}{\hat{\beta}} - d\ln t_0 + \sum_{i\in D} \ln t_i - \frac{d\sum_{i=1}^{n} t_i^{\hat{\beta}}(\ln t_i - \ln t_0)}{\sum_{i=1}^{n} t_i^{\hat{\beta}}} = 0 \tag{8}$$

For large sample sizes, we can get inferences on β and $R(t_0)$ based on the usual normal limiting distribution for the maximum likelihood estimators considering the observed information matrix (see for example, Lawless, 1982). For small or moderate sample sizes, this asymptotic distribution could not be appropriate.

With type II censored data, the form of likelihood function (6) is the same, but d is fixed and $T(\beta) = \sum_{i=1}^{d} t_{(i)}^{\beta} + (n-d)t_{(d)}^{\beta}$, where $t_{(1)}, \ldots, t_{(d)}$ are the first d ordered observations of a random sample of size n from the Weibull density (5).

4 The Guerrero - Johnson Transformation for $R(t_0)$ with β Known

Assuming β known, the logarithm of the likelihood function for $R(t_0)$ is given by,

$$l(R) \propto d\ln(-\ln R) + \frac{T}{t_0^{\beta}}\ln R\ , \tag{9}$$

where $T = \sum_{i=1}^{n} t_i^{\beta}$.

The maximum likelihood estimator $\hat{R}(t_0) = exp\left\{-dt_0^{\beta}/T\right\}$ has an asymptotic normal distribution based on the observed Fisher information, given by

$$\hat{R} \overset{a}{\sim} N\left\{R; \frac{\hat{R}^2\left(\ln\hat{R}\right)^2}{d}\right\}\ . \tag{10}$$

In the original parametrization $R(t_0)$, the standardized third derivative (4) of the logarithm of the likelihood function at $\hat{R}$ is given by

$$STD\left(\hat{R}\right) = \left|d^{-1/2}\left(3ln\hat{R} + 2\right)\right| \tag{11}$$

Observe that if t_0 is large, that is, $R(t_0)$ is small, we could have large values for $STD\hat{R}$), which indicates bad "normality" for the likelihood function.

To improve the "normality" of the likelihood function, we could consider the reparametrization (2).

The logarithm of the likelihood function for $\phi_{GJ}(\lambda)$ is given by

$$l\left(\phi_{GJ}\right) \propto dlnB\left(\phi_{GJ}\right) - \frac{T}{t_0^\beta}B\left(\phi_{GJ}\right) \tag{12}$$

where $T = \sum_{i=1}^n t_i^\beta, B\left(\phi_{GJ}\right) = ln\left[1 + \left(\phi_{GJ} + 1\right)^{-1/\lambda}\right]$.

At the maximum likelihood estimator $\hat{\phi}_{GJ} = \left[e^{dt_0^\beta/T} - 1\right]^{-\lambda} - 1$, the standardized third derivative (4) of $l\left(\phi_{GJ}\right)$ locally at $\hat{\phi}_{GJ}$, is given by

$$STD\left(\hat{\phi}_{GJ}\right) = \left|d^{-1/2}\left(2 - \frac{3B\left(\hat{\phi}_{GJ}\right)B''\left(\hat{\phi}_{GJ}\right)}{\left(B'\left(\hat{\phi}_{GJ}\right)\right)^2}\right)\right| . \tag{13}$$

Therefore, we find an appropriate value for λ such that $STD\left(\hat{\phi}_{GJ}\right) = 0$, given by

$$\lambda = \left(\frac{2T}{3dt_0^\beta} + 1\right)\left(1 - e^{-dt_0^\beta/T}\right) - 1 . \tag{14}$$

Whit this value of λ, we can consider the asymptotic normality of $\hat{\phi}_{GJ}(\lambda)$,

$$\hat{\phi}_{GJ}(\lambda) \overset{a}{\sim} N\left\{\phi_{GJ}(\lambda); \frac{d\lambda^2 t_0^{2\beta} e^{2dt_0^\beta/T}}{T^2\left(e^{dt_0^\beta/T}-1\right)^{2(\lambda+1)}}\right\} \tag{15}$$

to get better inferences on $R(t_0)$.

5 The Guerrero - Johnson Transformation for $R(t_0)$ with β Unknown

When β is unknown, we should search for a joint transformation of β and $R(t_0)$ that gives joint "normality" for the likelihood function. Since this transformation is not easily obtained and our parameter of interest is $R(t_0)$, an alternative way is to search for a reparametrization of $R(t_0)$ that gives close "normality" for the profile likelihood.

The profile likelihood function for $R(t_0)$ is given by

$$L_{\hat{\beta}}(R) = exp\left\{l\left(R,\hat{\beta}\right)\right\} \tag{16}$$

where $l(R,\beta)$ is given in (6) and for each value of R, we find $\hat{\beta}$ such that

$$(lnR)\sum_{i=1}^{n}\frac{t_i^{\hat{\beta}}}{t_0^{\hat{\beta}}}ln\left(\frac{t_i}{t_0}\right)+\frac{d}{\hat{\beta}} = dlnt_0-\sum_{i\in D}lnt_i\,.$$

To improve the "normality" of the profile likelihood, we also could use the transformation (2). As a special case, we could consider the transformation ϕ_{GJ} and $\theta = ln\beta$. The logarithm of the profile likelihood function for ϕ_{GJ} is (from (6)) given by,

$$
\begin{aligned}
l_{\widehat{\theta}}(\phi_{GJ}) &= d\widehat{\theta} - de^{\widehat{\theta}} ln t_0 + d ln B(\phi_{GJ}) + \\
&+ \left(e^{\widehat{\theta}} - 1\right) \sum_{i \in D} ln t_i - \frac{T(\widehat{\theta})}{t_0^{e^{\widehat{\theta}}}} B(\phi_{GJ}) \ ,
\end{aligned}
\qquad (17)
$$

where $T(\widehat{\theta}) = \sum_{i=1}^{n} t_i^{e^{\widehat{\theta}}}$ and $\widehat{\theta}$ is the maximum likelihood estimator of θ for each fixed value of ϕ_{GJ}.

In practical work, we should search for a value of λ such that the third derivative of the logarithm of the profile likelihood (17) at the maximum $\widehat{\phi}_{GJ}$ (see (4)) is close to zero.

6 Some Examples

6.1 An Example Considering an Exponential Distribution

Consider a type II censoring data set consisting of $n = 12$ units where the experiment terminated when it was observed $d = 8$ failures (data set introduced by Lawless, 1982, p. 103). The observed lifetimes (in hours) are given by 31,58,157,185,300,470,497 and 673. Assuming the exponential density given in (5) with $\beta = 1$, we have $T = \sum_{i=1}^{8} t_{(i)} + 4t_{(8)} = 5063$. The maximum likelihood estimator for the reliability function at time $t_0 = 5$ is given by $\widehat{R}(5) = 0.9921$. From the normal limiting distribution (11) for $\widehat{R}(5)$, we find a 95% confidence interval for $R(5)$ given by (0.9867;0.9976).

It is interesting to observe that $2T/\theta$ has an exact chi-square distribution with 2d degrees of freedom. An exact 95% confidence interval for θ is given by (351.6;1465.4), which corresponds to a 95% confidence interval for $R(5)$ given by (0.9859;0.9966).

Considering the Guerrero-Johnson transformation (2), we could improve the "normality" of the likelihood function considering an appropriate value for λ in $\phi_{GJ}(\lambda)$. With $t_0 = 5$ and $\beta = 1$, we find from (14), $\lambda = -0.3281$.

From the normal limiting distribution (15) for the maximum likelihood estimator $\hat{\phi}_{GJ}(-0.3281) = -0.7955$, we find an approximate 95% confidence interval for $\phi_{Gj}(-0.3281)$ given by $(-0.8422; -0.7487)$, which corresponds to a better 95% confidence interval for $R(5)$ given by $(0.9854; 0.9964)$.

We also could check the "normality" of the likelihood function in the parametrization $\phi_{GJ}(-0.3281)$ considering the t-plot (see Hills and Smith, 1993) $T(\phi_{GJ})$ against some values of ϕ_{GJ}, where

$$T(\phi_{GJ}) = sgn\left(\phi_{GJ} - \hat{\phi}_{GJ}\right)\left\{-2l(\phi_{GJ}) + 2l\left(\hat{\phi}_{GJ}\right)\right\}^{1/2} \tag{18}$$

and $\hat{\phi}_{GJ}$ is the maximum likelihood estimator of ϕ_{GJ}.

Since we observe a straight line (see figure 1), we conclude by the "normality" of the likelihood function for $\phi_{GJ}(-0.3281)$. In the original parametrization $R(5)$, the plot of $T(R(5))$ against $R(5)$ is markedly curved (see figure 2), which indicates the nonnormality of the likelihood function for $R(5)$.

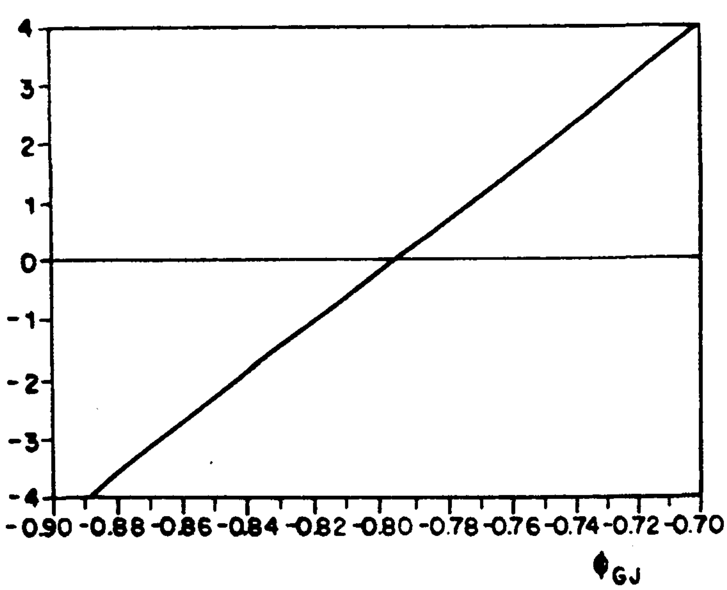

Figure 1 - t-plot for $\phi_{GJ}(-0.3281)$

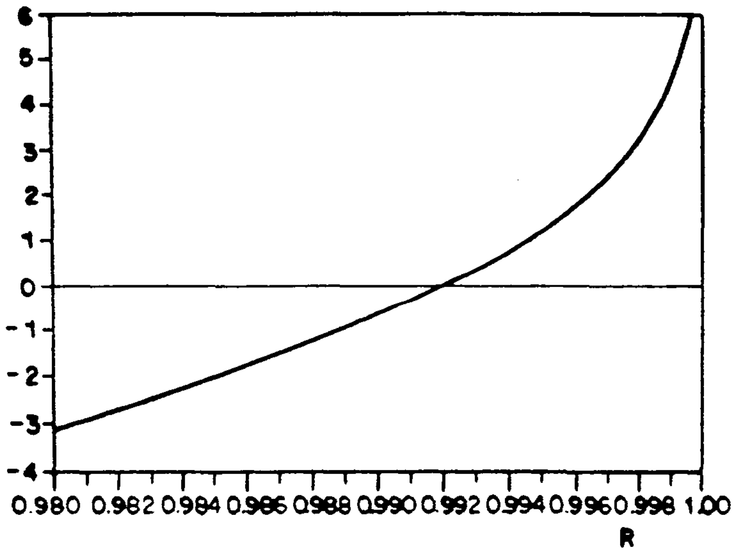

Figure 2 - t-plot for $R(5)$

In table 1, we have exact and approximate 95% confidence intervals for $R(t_0)$ with $t_0 = 5, 30, 500$ and 2000, respectively, considering the parametrizations $R(t_0)$ and $\phi_{GJ}(\lambda)$.

t_0	Using Exact Distribution for $2T/\theta$	Asymptotical Normality for $\widehat{R}(t_0)$	λ Given in (14)	Asymptotical Normality for $\widehat{\phi}_{GJ}(\lambda)$
5	(0.9859;0.9966)	(0.9867;0.9976)	-0.3281	(0.9854;0.9964)
30	(0.9182;0.9797)	(0.9224;0.9850)	-0.3026	(0.9155;0.9786)
500	(0.2412;0.7109)	(0.2054;0.7023)	0.0071	(0.2332;0.6930)
2000	(0.0034;0.2554)	(-0.0505;0.1353)	0.1596	(0.0026;0.2373)

Table 1. 95% Confidence Intervals for $R(t_0)$

We also could consider the proposed reparametrization to get a Bayesian analysis for the reliability at time t_0. If the posterior density for $R(t_0)$ is close to "normality", we could find in a simple way, Bayesian intervals for $R(t_0)$ considering normal approximations for the posterior density.

In the original parametrization $R(t_0)$, an informative prior density considered in the literature (see for example, Martz and Waller, 1982) is given by a Beta density,

$$g(r; \alpha_0, \beta_0) = \frac{1}{B(\alpha_0, \beta_0)} r^{\alpha_0 - 1} (1 - r)^{\beta_0 - 1} \tag{19}$$

where $0 \leq r \leq 1; \alpha_0, \beta_0 > 0$ and $B(\alpha_0, \beta_0) = \dfrac{\Gamma(\alpha_0)\Gamma(\beta_0)}{\Gamma(\alpha_0 + \beta_0)}$.

With $\alpha_0 = \beta_0 = 1$ in (19), we have an uniform prior density for $R(t_0)$. In this case, the posterior density for $R(t_0)$, considering type II censored data is given by

$$g_1(r|data) = \frac{\left(1 + \frac{T}{t_0}\right)^{d+1} (-lnr)^d r^{T/t_0}}{\Gamma(d+1)}, \tag{20}$$

where $0 \leq r \leq 1, T = \sum_{i=1}^{d} t_{(i)} + (n - d)t_{(d)}$ and d is the fixed number of observed failures.

Considering the Guerrero-Johnson transformation (2) with a specified value of λ, the posterior density for $\phi_{GJ}(\lambda)$ is given (from (20)) by,

$$\begin{aligned} g_1(\phi_{GJ} \mid data) &= \frac{(1 + T/t_0)^{d+1}}{\lambda\Gamma(d+1)} \left\{ln\left[(\phi_{GJ} + 1)^{-1/\lambda} + 1\right]\right\}^n \times \\ &\times \frac{(\phi_{GJ} + 1)^{\frac{1}{\lambda}\left(\frac{T}{t_0}+1\right)-1}}{\left\{1 + (\phi_{GJ} + 1)^{1/\lambda}\right\}^{T/t_0+2}}, \end{aligned} \tag{21}$$

where $-1 < \phi_{GJ} < \infty$ for $\lambda \neq 0$.

With $t_0 = 2000$ hours and assuming an uniform prior density for $R(t_0)$, we have in figures 3 and 4, the graphs of the posterior densities for $R(2000)$ and $\phi_{GJ}(0.1596)$, respectively. We observe good "normality" for the posterior density (21) in the parametrization $\phi_{GJ}(0.1596)$.

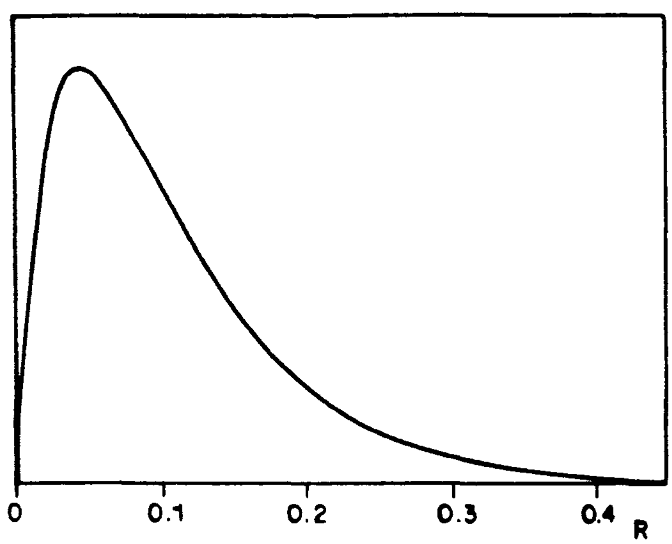

Figure 3 - Posterior density for $R(2000)$.

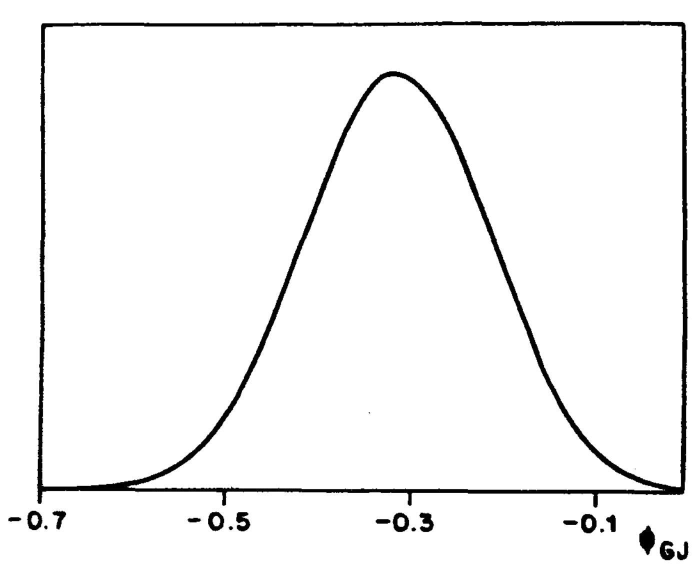

Figure 4 - Posterior density for $\phi_{GJ}(0.1596)$.

Considering the Jeffreys noninformative prior density for $R(t_0), g(r) \propto 1/[r(-lnr)], 0 \leq r \leq 1$ (see for example, Martz and Waller, 1982), the posterior density for $R(t_0)$ is given by

$$g_2(r|data) = \frac{(-lnr)^{d-1} T^d r^{\frac{T}{t_0}-1}}{t_0^d \Gamma(d)}, \tag{22}$$

where $0 \leq r \leq 1$.

The posterior density for $\phi_{GJ}(\lambda)$ is (from (22)) given by

$$\begin{aligned} g_2(\phi_{GJ}|data) &= \frac{T^d}{\lambda t_0^d \Gamma(d)} \left\{ ln \left[(\phi_{GJ}+1)^{-1/\lambda} + 1 \right] \right\}^{d-1} \times \\ &\times \frac{(\phi_{GJ}+1)^{\frac{T}{\lambda t_0}-1}}{\left\{ 1 + (\phi_{GJ}+1)^{1/\lambda} \right\}^{\frac{T}{t_0}+1}}, \end{aligned} \tag{23}$$

where $-1 < \phi_{GJ} < \infty$ for $\lambda \neq 0$.

In table 2, we have 95% Bayesian intervals for $R(t_0)$, with $t_0 = 5, 30, 500$ and 2000, respectively, considering the parametrizations $R(t_0)$ and $\phi_{GJ}(\lambda)$. We observe very accurate results, assuming the parametrization $\phi_{GJ}(\lambda)$, by comparing numerically integrated intervals using Simpson's rule with the approximate Bayesian intervals (see table 2).

	t_0	Using Simpson's Rule	Normal approximation for $g_i(r/data)$	λ	Normal approximation for $g_i(\phi_{GJ}/data)$
$(i = 1)$	5	0.98457;0.99595	0.98670;0.99756	-0.3281	0.98453;0.99598
Uniform	30	0.91133;0.97605	0.92238;0.98503	-0.3026	0.91132;0.97618
Prior for	500	0.24249;0.69081	0.20537;0.70228	0.0071	0.24759;0.68444
$R(t_0)$	2000	0.01152;0.31180	-0.05047;0.13531	0.1596	0.01146;0.31121
$(i = 2)$	5	0.98586;0.99660	0.98801;0.99820	-0.3281	0.98581;0.99663
Jeffreys	30	0.91809;0.97974	0.92949;0.98878	-0.3026	0.91797;0.97985
Prior for	500	0.24067;0.71100	0.20051;0.72826	0.0071	0.24586;0.70294
$R(t_0)$	2000	0.00335;0.25555	-0.02470;0.04540	0.1596	0.00305;0.26065

Table 2 - 95% Bayesian intervals for $R(t_0)$

The use of reparametrization $\phi_{GJ}(\lambda)$ also gives good improvements in the accuracy of Laplace approximations for the posterior momemts $E(R(t_0)/data)$ (see for example, Tierney and Kadane, 1986). In table 3, we have Laplace approximations for $E(R(t_0)/data)$ considering both parametrizations $R(t_0)$ and $\phi_{GJ}(\lambda)$. Observe that, in this case, the exact posterior moments for $R(t_0)$ are given by

$$E_1(R|data) = \left(\frac{1 + T/t_0}{2 + T/t_0}\right)^{d+1} , \tag{24}$$

considering the uniform prior for $R(t_0)$ and,

$$E_2(R|data) = (1 + t_0/T)^{-d} , \tag{25}$$

considering the Jeffreys prior for $R(t_0)$.

	t_0	Exact	Parametrization $R(t_0)$	λ	Parametrization $\phi_{GJ}(\lambda)$
$(i = 1)$	5	0.99116	0.99116 (0%)	-0.3281	0.99116 (0%)
Uniform	30	0.94851	0.94848 (0.003%)	-0.3026	0.94852 (0.001%)
Prior for	500	0.46089	0.45997 (0.20%)	0.0071	0.46109 (0.043%)
$R(t_0)$	2000	0.10604	0.12229 (15.32%)	0.1596	0.10530 (0.698%)
$(i = 2)$	5	0.99213	0.99311 (0.099%)	-0.3281	0.99213 (0%)
Jeffreys	30	0.95384	0.95949 (0.592%)	-0.3026	0.95384 (0%)
Prior for	500	0.47075	0.52095 (10.66%)	0.0071	0.47100 (0.053%)
$R(t_0)$	2000	0.06972	0.18043 (158.8%)	0.1596	0.06873 (1.42%)

Table 3 - Laplace approximations for $E(R(t_0)|data)$ (percentage errors in parentheses)

6.2 An Example Considering a Weibull Distribution

Consider a type II censored data set introduced by Lawless, 1982, p. 152 consisting of $n = 40$ units where the experiment terminated when it was observed $d = 28$ failures (see table 4).

0.0507	0.0579	0.0784	0.0954	0.1376	0.2249
0.2362	0.2481	0.2501	0.2811	0.3027	0.3091
0.4295	0.5379	0.5621	0.5781	0.7811	0.8228
0.9455	0.9871	1.0060	1.0335	1.0377	1.0471
1.0876	1.2473	1.2776	1.3445		

Table 4 - Observed lifetimes (in hours)

Assuming the Weibull distribution with density (5), the maximum likelihood estimators for the parameters α and β are given by $\hat{\alpha} = 1.1692$ and $\hat{\beta} = 1.0984$. Considering $\beta = 1.0984$ known, we have in table 5, exact and approximate confidence intervals for $R(t_0)$, for different values of t_0. For the approximate confidence intervals of $R(t_0)$, we used the normal limiting distribution (10) for the maximum likelihood estimator $\hat{R}(t_0)$. For the exact confidence intervals, observe that $2\, T \big/ \alpha^{\beta}$ where β is known and $T = \sum_{i=1}^{d} t_{(i)} + (n-d)t_{(d)}^{\beta}$, has a chi-square distribution with 2d degrees of freedom. Since $T = 33.2455$ and $\beta = 1.0984$, we find a 95% confidence interval for α given by 0.8581 ; 1.6977) from which, we get 95% confidence intervals for $R(t_0) = exp\left\{-\left(\frac{t_0}{\alpha}\right)^{\beta}\right\}$.

In table 5, we also have approximate 95% confidence intervals for $R(t_0)$ considering the parametrization $\phi_{GJ}(\lambda)$ and using the normal limiting distribution (15). We observe very accurate confidence intervals considering the parametrization $\phi_{GJ}(\lambda)$.

t_0	Using Exact Distribution For $2T/\alpha^\beta$	Using Asymptotical Normality For $\widehat{R}(t_0)$	λ Given in (14)	Using Asymptotical Normality For $\widehat{\phi}_{GJ}(\lambda)$
0.001	(0.9994;0.9997)	(0.9994;0.9997)	-0.3330	(0.9994;0.9997)
0.010	(0.9925;0.9964)	(0.9927;0.9966)	-0.3298	(0.9924;0.9964)
0.100	(0.9099;0.9564)	(0.9118;0.9583)	-0.2903	(0.9092;0.9558)
0.400	(0.6489;0.8151)	(0.6512;0.8188)	-0.1612	(0.6466;0.8125)
1.000	(0.3063;0.5717)	(0.2964;0.5651)	0.0198	(0.3037;0.5662)
2.000	(0.0794;0.3020)	(0.0547;0.2748)	0.1440	(0.0778;0.2960)
2.500	(0.0393;0.2166)	(0.0146;0.1850)	0.1606	(0.0381;0.2114)
3.000	(0.0192;0.1543)	(-0.0026;0.1224)	0.1627	(0.0184;0.1501)
5.000	(0.0009;0.0378)	(-0.0059;0.0204)	0.1269	(0.0009;0.0365)

Table 5 - 95% Confidence intervals for $R(t_0)$ assuming $\beta = 1.0894$ known

We also could consider the reparametrization $\phi_{GJ}(\lambda)$ assuming β unknown. We observe in figure 5, a markedly curved form for the t-plot (see (18)) in the parametrization $R(t_0)$ with $t_0 = 2.5$, indicating nonnormality of the profile likelihood function $L_{\widehat{\beta}}(R)$ (see (16)).

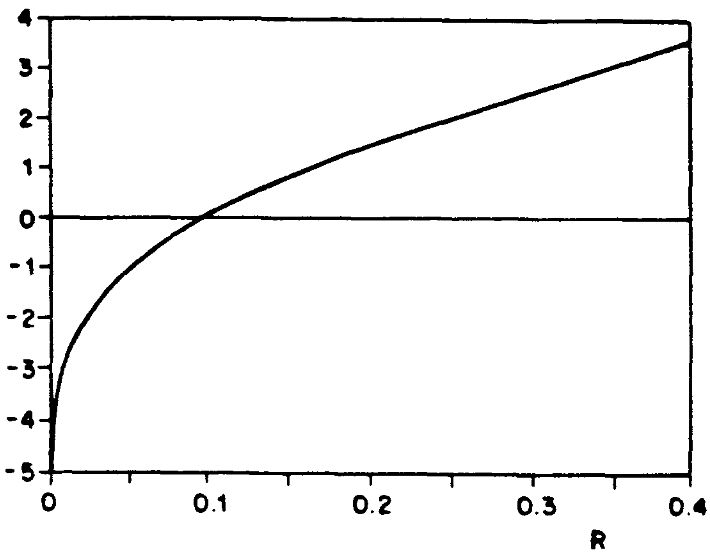

Figure 5 - t-plot for the profile likelihood of $R(2.5)$

To find an appropriate value for λ in the parametrization $\phi_{GJ}(\lambda)$, we observe that $\lambda = 0.1606$ gives third derivative of the logarithm of the profile likelihood at its maximum $\widehat{\phi}_{GJ}$ close to zero (see (4)). Therefore, we consider the reparametrization $\phi_{GJ}(0.1606)$, where we observe in figure 6, close linearity for the t-plot (see (18)) indicating good "normality" of the profile likelihood for $\phi_{GJ}(0.1606)$.

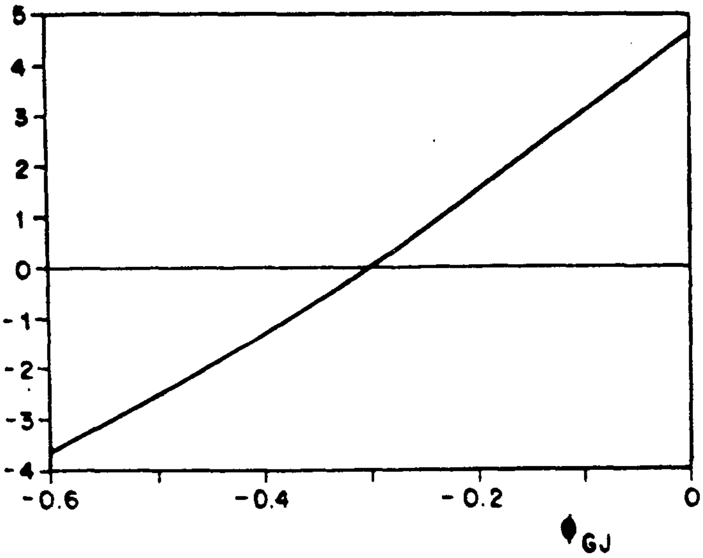

Figure 6 - t-plot for the profile likelihood of $\phi_{GJ}(0.1606)$

7 Overall Conclusions

The use of reparametrization $\phi_{GJ}(\lambda)$ could be of great practical interest, since we get very simple and accurate approximate inference results for the reliability function $R(t_0)$ at time t_0. The proposed method introduced in this paper, to find the appropriate value of λ in the parametrization $\phi_{GJ}(\lambda)$, also could be extended to other invertible parametric family of transformations for proportions (see for example, Atkinson, 1985). One of these transformations given by $\phi_{A0}(\lambda) = \left[R^\lambda - (1-R)^\lambda\right] / \left[R^\lambda + (1-R)^\lambda\right]$, where $R = R(t_0)$, was introduced by Aranda-Ordaz (1981). We also could consider other lifetime distributions to get similar results.

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