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ISSN - 0103-2577

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N O T A S D O I C M S C
Série Estatística

São Carlos
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# Some Useful Reparametrizations for the Reliability Function Considering the Exponential Distribution and Censored Lifetime Data

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**Abstract**

In this paper, we present some useful formulas to get appropriate reparametrization for the reliability function at time $t$ considering censored lifetime data and an exponential distribution. With the obtained reparametrizations, we get very accurate approximate inference results based on the usual normal limiting distribution for the maximum likelihood estimator of the reliability function.

**Key words:** Reliability function, reparametrization, exponential distribution, censored data.

# 1 Introduction

Usually, statisticians consider the asymptotical normality of the maximum likelihood estimators to get inferences for the reliability function at time $t$ assuming different parametric models and censored observations (see for example, Lawless, 1982). These asymptotical results could be very poor, considering small or moderate sample sizes. One alternative to improve the approximate inference results, is to consider an appropriate reparametrization to get good "normality" of the likelihood function (see for example, Anscombe, 1964; or Sprott, 1973, 1980).

Good parametrization also is very important to get accurate Bayesian inferences considering numerical or approximation methods for posterior moments or posterior densities of interest (see for example, Achcar and Smith, 1990; Kass and Slate, 1992; or Hills and Smith, 1993).

Assuming an exponential distribution for the lifetimes in a reliability experiment, we use some well known parametric families of transformations for proportions (see for example, Guerrero and Johnson, 1982; or Aranda-Ordaz, 1981) to get good "normality" for the likelihood function. To get an appropriate data dependent reparametrization for the reliability function at time $t$, we explore a measure to nonnormality of likelihood functions given by the standardized form of the third derivative of the logarithm of the likelihood function (see Sprott, 1973; or Kass and Slate, 1992).

With the proposed reparametrization, we get in a very simple way, accurate inference results for the reliability function at time $t$. We can check the adequability of the obtained reparametrizations by considering plots or some diagnostic measures for nonnormality of likelihood functions (see for example, Kass and Slate, 1992; or Hills and Smith, 1993).

We illustrate the proposed methodology, considering an example with type II censored data, where it is possible to compare obtained approximate confidence intervals with exact results.

# 2 Some Reparametrizations for the Reliability Function at Time $t$

One way to improve the "normality" of the maximum likelihood estimator for the reliability function at time $t$, $R(t) = P(T > t)$, is to consider different parametrizations or transformations of $R(t)$. For example, we could explore the usual logit reparametrization $\phi_L = \ln[R/(1-R)]$, where $R = R(t)$. Some parametric families of transformations for proportions (see for example, Atkinson, 1985) also could be explored to improve the "normality" of the likelihood of $R(t)$. One of these transformations which yields the logit as $\lambda$ approaches zero, is the folded power transformation (see Mosteller and Tukey, 1977, p.92) given by

$$\phi_F(\lambda) = R^\lambda - (1-R)^\lambda \tag{1}$$

where $0 \leq R \leq 1$. One disadvantage of the folded power transformation, is that except for a few special values of $\lambda$, such as zero and one, analytical inversion of transformation (1) is not possible.

To obtain an invertible family of transformations which includes the logit, Guerrero and Johnson (1982), suggested the transformation

$$\phi^*_{GJ}(\lambda) = \left\{ \left( \frac{R}{1-R} \right)^\lambda - 1 \right\} \Big/ \lambda \, . \tag{2}$$

For a given $\lambda$, we can consider a modified form of Guerrero and Johnson transformation given by

$$\phi_{GJ}(\lambda) = \left( \frac{R}{1-R} \right)^\lambda - 1 \, , \tag{3}$$

which should not produce different results as considering (2).

The advantage of the transformation (3) is that it is readily inverted. With $\phi_{GJ} = \phi_{GJ}(\lambda)$, we obtain,

$$R = \frac{(\phi_{GJ}+1)^{1/\lambda}}{1 + (\phi_{GJ}+1)^{1/\lambda}} \, . \tag{4}$$

Another transformation for proportions is suggested by Aranda-Ordaz (1981),

$$\phi_{A_0}(\lambda) = \frac{R^\lambda - (1-R)^\lambda}{R^\lambda + (1-R)^\lambda} . \tag{5}$$

The inverse of transformation (5) is given by

$$\left(\frac{R}{1-R}\right)^\lambda = \left(\frac{1+\phi_{A_0}}{1-\phi_{A_0}}\right) . \tag{6}$$

We observe that both transformations $\phi^*_{GJ}(\lambda)$ and $\phi_{A_0}(\lambda)$ give the logit transformation for $\lambda = 0$.

To consider one of the reparametrizations (3) or (5), we should have an appropriate value of $\lambda$ that gives good "normality" of the likelihood function for $\phi(\lambda)$. One way to find this value, is to choose $\lambda$ in (3) or (5) that gives third derivatives of the logarithm of the likelihood function $\ell(\phi(\lambda))$ at the maximum likelihood estimator $\widehat{\phi}(\lambda)$ in a standardized form,

$$STD(\widehat{\phi}(\lambda)) = \left|\ell'''(\widehat{\phi}(\lambda))\left(-\ell''(\widehat{\phi}(\lambda))\right)^{-3/2}\right| \tag{7}$$

close to zero (see for example, Sprott, 1973; or Kass and Slate, 1992).

# 3 Asymptotical Inferences for the Reliability Function at Time $t$ Considering an Exponential Distribution and Censored Data

Suppose there is a random sample of $n$ units with lifetimes $T_1, T_2, \ldots, T_n$, but that associated to each unit is also a fixed censoring time $L_i > 0$ (type $I$ censored data). We observe $T_i$ only if $T_i \leq L_i$ and the data consists of pairs $(t_i, \delta_i)$, $i = 1, \ldots, n$, where $t_i = \min(T_i, L_i)$ and $\delta_i = 1$ if $t_i = T_i$ or $\delta_i = 0$ if $t_i = L_i$.

Considering the exponential model,

$$f(t;\theta) = \frac{1}{\theta}\, e^{-t/\theta}\,, \tag{8}$$

and type $I$ censored data, the likelihood function for $\theta$ is given by

$$L(\theta) = \frac{1}{\theta^d} \exp\left\{-\frac{T}{\theta}\right\} \tag{9}$$

where $d = \sum_{i=1}^{n} \delta_i$ is the observed number of lifetimes, and $T = \sum_{i=1}^{n} t_i = \sum_{i \in D} T_i + \sum_{i \in C} L_i$ is the total observed lifetime for the $n$ units, where $D$ and $C$ denote the sets of units for whom lifetimes are observed and censored, respectively (see Lawless, 1982, p.105).

With type II censored data, the form of the likelihood function is identical, but $d$ is fixed and $T = \sum_{i=1}^{d} t_{(i)} + (n-d)t_{(d)}$, where $t_{(1)}, t_{(2)}, \ldots, t_{(d)}$ are the first $d$ ordered observations of a random sample of size $n$ from the exponential distribution (8).

From (9), we get the likelihood function for the reliability at time $t$ given by

$$L(R) = \frac{R^{T/t}}{t^d}\,(-\ln R)^d \tag{10}$$

where $0 \leq R \leq 1$.

The maximum likelihood estimator for the reliability function at time $t$, $\hat{R}(t) = \exp\{-dt/T\}$, has an asymptotic normal distribution based on Fisher observed information, given by

$$\hat{R}(t) \overset{a}{\sim} N\left\{e^{-t/\theta};\ \frac{dt^2 e^{-2dt/T}}{T^2}\right\} \tag{11}$$

which could be used to construct hypothesis tests or confidence intervals for $R(t)$. In practical work, this normal approximation can be very poor for small or moderate sample sizes.

Considering the Guerrero and Johnson transformation (3), the logarithm of the likelihood function for $\phi_{GJ}(\lambda)$ is given by

$$\ell(\phi_{GJ}) = -d\ln t + d\ln\left\{\ln\left[1 + (\phi_{GJ}+1)^{-1/\lambda}\right]\right\} - \frac{T}{t}\ln\left[1 + (\phi_{GJ}+1)^{-1/\lambda}\right]\,. \tag{12}$$

At the maximum likelihood estimator $\hat{\phi}_{GJ} = \left(e^{dt/T} - 1\right)^{-\lambda} - 1$, the second and third derivatives of $\ell(\phi_{GJ})$ (see appendix 1) are given by

$$\ell''(\hat{\phi}_{GJ}) = -\frac{T^2}{d\lambda^2 t^2} e^{-2dt/T} \left(e^{dt/T} - 1\right)^{2(\lambda+1)} \tag{13}$$

and

$$\ell'''(\hat{\phi}_{GJ}) = \frac{T^2(e^{dt} - 1)^{3(\lambda+1)}}{d\lambda^2 t^2 e^{2dt/T}} \left\{ \frac{3\left(1 + \lambda^{-1} e^{-dt/T}\right)}{e^{dt/T} - 1} - \frac{2T}{d\lambda t e^{dt/T}} \right\} .$$

Thus, the standardized form of the third derivative of $\ell(\phi_{GJ})$ at the maximum likelihood estimator $\hat{\phi}_{GJ}$ (see (7)) is given by

$$STD(\hat{\phi}_{GJ}) = \left| \frac{\lambda t d^{1/2}}{T e^{-dt/T}} \left( \frac{3\left(1 + \lambda^{-1} e^{-dt/T}\right)}{e^{dt/T} - 1} - \frac{2T}{d\lambda t e^{dt/T}} \right) \right| \tag{14}$$

Therefore, we find the appropriate value of $\lambda$ such that $STD(\hat{\phi}_{GJ}) = 0$, given by

$$\lambda = \left(\frac{2T}{3dt} + 1\right) \left(1 - e^{-dt/T}\right) - 1 . \tag{15}$$

With $\lambda$ given by (15), we can consider the asymptotic normality of $\hat{\phi}_{GJ}(\lambda)$,

$$\hat{\phi}_{GJ}(\lambda) \overset{a}{\sim} N \left\{ \phi_{GJ}(\lambda);\ \frac{d\lambda^2 t^2 e^{2dt/T}}{T^2 \left(e^{dt/T} - 1\right)^{2(\lambda+1)}} \right\} \tag{16}$$

to get better inferences, especially for small or moderate sample sizes.

Considering the Aranda-Ordaz transformation (5), the logarithm of the likelihood function for $\phi_{A_0}(\lambda)$ is given by

$$\ell(\phi_{A_0}) = -d \ln t + d \ln \left\{ \ln \left[ 1 + \left( \frac{1 - \phi_{A_0}}{1 + \phi_{A_0}} \right)^{1/\lambda} \right] \right\} - \frac{T}{t} \ln \left\{ 1 + \left( \frac{1 - \phi_{A_0}}{1 + \phi_{A_0}} \right)^{1/\lambda} \right\} \tag{17}$$

At the maximum likelihood estimator

$$\widehat{\phi}_{A_0} = \left[1 - \left(e^{dt/T} - 1\right)^{\lambda}\right] \Big/ \left[1 + \left(e^{dt/T} - 1\right)^{\lambda}\right] ,$$

the second and third derivatives of $\ell(\phi_{A_0})$ (see appendix 2) are given by

$$\ell''(\widehat{\phi}_{A_0}) = -\frac{4T^2 e^{-2dt/T}\left(e^{dt/T} - 1\right)^2}{dt^2\lambda^2\left(1 - \widehat{\phi}^2_{A_0}\right)^2} \tag{18}$$

and

$$\ell'''(\widehat{\phi}_{A_0}) = \frac{8T^2 e^{-2dt/T}\left(e^{dt/T} - 1\right)^2}{dt^2\lambda^2\left(1 - \widehat{\phi}^2_{A_0}\right)^3}\left\{\frac{3e^{-dt/T}}{\lambda} - 3\widehat{\phi}_{A_0} - \frac{2T}{dt\lambda} + \frac{2Te^{-dt/T}}{dt\lambda}\right\} . \tag{19}$$

The standardized form of the third derivative of $\ell(\phi_{A_0})$ at $\widehat{\phi}_{A_0}$ is given by

$$STD(\widehat{\phi}_{A_0}) = \left|\frac{d^{1/2}t\lambda e^{dt/T}}{T\left(e^{dt/T} - 1\right)}\left(\frac{3e^{-dt/T}}{\lambda} - 3\widehat{\phi}_{A_0} - \frac{2T}{dt\lambda} + \frac{2Te^{-dt/T}}{dt\lambda}\right)\right| . \tag{20}$$

Using an iterative procedure (e.g., Newton-Raphson method), we find $\lambda$ such that,

$$g(\lambda) = \frac{3e^{-dt/T}}{\lambda} + \frac{2Te^{-dt/T}}{dt\lambda} - \frac{2T}{dt\lambda} - 3\left\{\frac{1 - \left(e^{dt/T} - 1\right)^{\lambda}}{1 + \left(e^{dt/T} - 1\right)^{\lambda}}\right\} = 0 . \tag{21}$$

With $\lambda$ given by (21), we also can consider the usual asymptotic normality of the maximum likelihood estimator $\widehat{\phi}_{A_0}(\lambda)$ based on the Fisher observed information to get better inferences.

# 4 An Example

Consider a type II censoring data set consisting of $n = 12$ units where the experiment terminated when it was observed $d = 8$ failures (data set introduced by Lawless, 1982, p.103). The observed lifetimes (in hours) are given by 31, 58, 157, 185, 300, 470, 497 and 673, where $T = \sum_{i=1}^{8} t_{(i)} + 4t_{(8)} = 5063$. Assuming the exponential distribution with density (8), the maximum likelihood estimator for the reliability function at time $t = 5$ is given by $\widehat{R}(5) = 0.9921$. From the normal limiting distribution (11) for $\widehat{R}(5)$, we find a 95% confidence interval for $R(5)$ given by (0.9867; 0.9976).

It is interesting to observe that $2T/\theta$ has an exact chi-square distribution with $2d$ degrees of freedom. An exact 95% confidence interval for $\theta$ is given by (351.6; 1465.4) which corresponds to a 95% confidence interval for $R(5)$ given by (0.9859; 0.9966).

Considering the Guerrero and Johnson transformation (3), we could improve the "normality" of the likelihood function considering an appropriate value for $\lambda$ in $\phi_{GJ}(\lambda)$. With $t = 5$, we find from (15), $\lambda = -0.3281$.

From the normal limiting distribution (16) for the maximum likelihood estimator $\widehat{\phi}_{GJ}(-0.3281) = -0.7955$, we find an appropriate 95% confidence interval for $\phi_{GJ}(-0.3281)$ given by (-0.8422; -0.7487), which corresponds to a better 95% confidence interval for $R(5)$ given by (0.9854; 0.9964).

We also could check the "normality" of the likelihood function in the parametrization $\phi_{GJ}(-0.3281)$ considering the $t$-plot (see Hills and Smith, 1993) $T(\phi_{GJ})$ against some values of $\phi_{GJ}$, where

$$T(\phi_{GJ}) = \text{sgn}(\phi_{GJ} - \widehat{\phi}_{GJ})\left\{-2\ell(\phi_{GJ}) + 2\ell(\widehat{\phi}_{GJ})\right\}^{1/2} \tag{22}$$

and $\widehat{\phi}_{GJ}$ is the maximum likelihood estimator of $\phi_{GJ}$.

Since we observe a straight line (see figure 1), we conclude by the "normality" of the likelihood function for $\phi_{GJ}(-0.3281)$. In the original parametrization $R(5)$, the plot of $T(R(5))$ against $R(5)$ is markedly curved (see figure 2), which indicates the nonnormality of the likelihood function for $R(5)$.

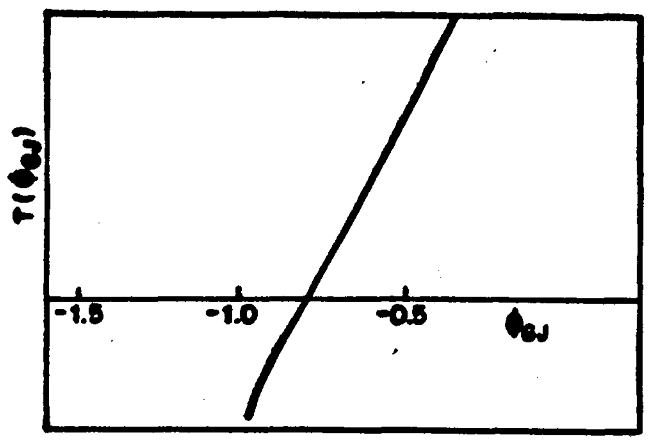


Figure 1: $t$-plot for $\phi_{GJ}(-0.3281)$

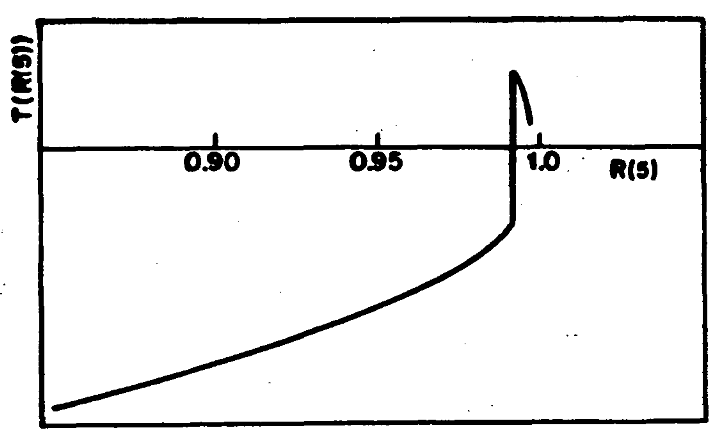


Figure 2: $t$-plot for $R(5)$

In the same way, we can improve the "normality" of the likelihood function considering an appropriate value for $\lambda$ in the Aranda-Ordaz transformation $\phi_{A_0}(\lambda)$ given in (5). With $t = 5$, and using Newton-Raphson procedure, we find from (20), $\lambda = 0.4246$.

Using the normal limiting distribution for the maximum likelihood estimator $\hat{\phi}_{A_0}(0.4246) = 0.7726$, we find an approximate 95% confidence interval for $\phi_{A_0}(0.4246)$ given by (0.7131; 0.8322), which corresponds to a 95% confidence interval for $R(5)$ given by (0.9854; 0.9964).

In table 1, we have exact and approximate 95% confidence intervals for $R(t)$ with $t = 30, 500$ and 2000, considering the parametrizations $R(t)$, $\phi_{GJ}(\lambda)$ and $\phi_{A_0}(\lambda)$, respectively. We observe good inference results considering the parametrizations $\phi_{GJ}(\lambda)$ and $\phi_{A_0}(\lambda)$ with the appropriate values for $\lambda$.

In figures 3,4 and 5, we have plots for the likelihood functions in the different parametrizations. We observe good "normality" for the likelihood functions considering the parametrizations $\phi_{GJ}(\lambda)$ and $\phi_{A_0}(\lambda)$.

Table 1: 95% Confidence Intervals for $R(t)$

| $t$ | Using Exact Distribution for $2T/\theta$ | Asymptotical Normality for $\hat{R}(t)$ | $\lambda$ Given by (15) | Asymptotical Normality for $\hat{\phi}_{GJ}(\lambda)$ | $\lambda$ Given by (21) | Asymptotical Normality for $\hat{\phi}_{A_0}(\lambda)$ |
|---|---|---|---|---|---|---|
| 30 | (0.9182;0.9797) | (0.9224;0.9850) | -0.3026 | (0.9155;0.9786) | 0.4844 | (0.9152;0.9788) |
| 500 | (0.2412;0.7109) | (0.2054;0.7023) | 0.0071 | (0.2332;0.6930) | 0.2761 | (0.2319;0.6942) |
| 2000 | (0.0034;0.2554) | (-0.5050;0.1353) | 0.1596 | (0.0026;0.2373) | 0.3339 | (0.0021;0.2477) |

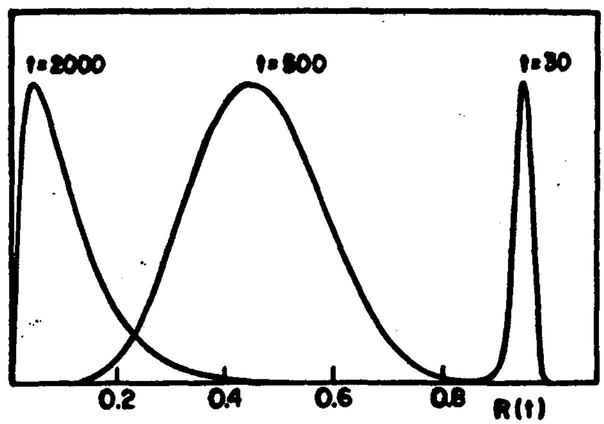


Figure 3: Likelihood Function for $R(t)$

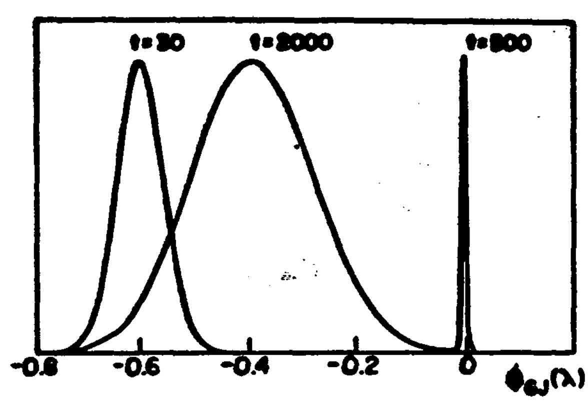


Figure 4: Likelihood Function for $\phi_{GJ}(\lambda)$

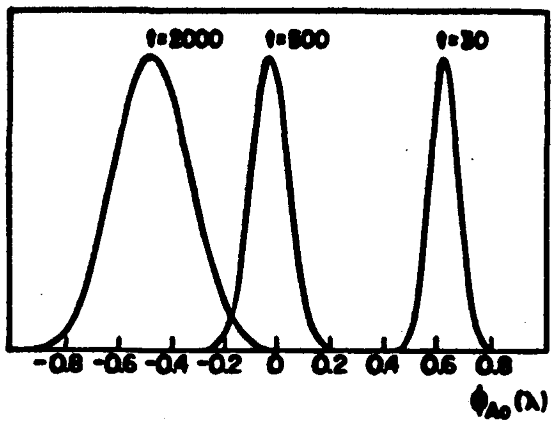


Figure 5: Likelihood Function for $\phi_{A_0}(\lambda)$

# 5 Overall Conclusions

The use of the reparametrizations $\phi_{GJ}$ and $\phi_{A_0}$ could be of great practical interest, since we get very simple and accurate inference results for the reliability function at time $t$ based on the asymptotical normality of the maximum likelihood estimators. We also could consider other lifetime distributions to get similar results. If the proposed model has two or more parameters, we could consider the "profile" likelihood for the reliability function at time $t$, the obtain appropriate values for $\lambda$ in $\phi_{GJ}(\lambda)$ and $\phi_{A_0}(\lambda)$ using (7). In the same way, we could consider the reparametrizations $\phi_{GJ}(\lambda)$ and $\phi_{A_0}(\lambda)$ to get better inferences for "$K$-out-of-$m$" system reliability (see Martz and Waller, 1982).

# Appendix

## (A.1) Derivatives of the Log-Likelihood Function for $\phi_{GJ}(\lambda)$ Locally at the Maximum Likelihood Estimator $\widehat{\Phi}_{GJ}(\lambda)$

The logarithm of the likelihood function for $\phi_{GJ} = \phi_{GJ}(\lambda)$ (see (12)) is given by

$$\ell(\phi_{GJ}) = -d \ln t + d \ln B(\phi_{GJ}) - \frac{T}{t} B(\phi_{GJ}) \tag{A.1}$$

where $B(\phi_{GJ}) = \ln[1 + A(\phi_{GJ})]$, and $A(\phi_{GJ}) = (\phi_{GJ} + 1)^{-1/\lambda}$.

The first three derivatives of $\ell(\phi_{GJ})$ are given by

$$\begin{aligned} \ell'(\phi_{GJ}) &= d\, \tfrac{B'(\phi_{GJ})}{B(\phi_{GJ})} - \tfrac{T}{t}\, B'(\phi_{GJ}) \\ \ell''(\phi_{GJ}) &= d\, \tfrac{B''(\phi_{GJ})}{B(\phi_{GJ})} - d \left(\tfrac{B'(\phi_{GJ})}{B(\phi_{GJ})}\right)^2 - \tfrac{T}{t}\, B''(\phi_{GJ}) \end{aligned} \tag{A.2}$$

and

$$\ell'''(\phi_{GJ}) = d\, \frac{B'''(\phi_{GJ})}{B(\phi_{GJ})} - 3d\, \frac{B'(\phi_{GJ})B''(\phi_{GJ})}{B^2(\phi_{GJ})} + 2d \left(\frac{B'(\phi_{GJ})}{B(\phi_{GJ})}\right)^3 - \frac{T}{t}\, B'''(\phi_{GJ})$$

where $B'(\phi_{GJ}) = A'(\phi_{GJ}) / (1 + A(\phi_{GJ}))$,

$$B''(\phi_{GJ}) = \tfrac{A''(\phi_{GJ})}{1+A(\phi_{GJ})} - \left(\tfrac{A'(\phi_{GJ})}{1+A(\phi_{GJ})}\right)^2 ,$$

$$B'''(\phi_{GJ}) = \tfrac{A'''(\phi_{GJ})}{1+A(\phi_{GJ})} - \tfrac{3A'(\phi_{GJ})A''(\phi_{GJ})}{(1+A(\phi_{GJ}))^2} + 2\left(\tfrac{A'(\phi_{GJ})}{1+A(\phi_{GJ})}\right)^3 ,$$

$$A'(\phi_{GJ}) = -\tfrac{1}{\lambda}(\phi_{GJ} + 1)^{-1/\lambda - 1} ,$$

$$A''(\phi_{GJ}) = \tfrac{1}{\lambda}\left(\tfrac{1}{\lambda} + 1\right)(\phi_{GJ} + 1)^{-1/\lambda - 2} ,$$

and

$$A'''(\phi_{GJ}) = -\frac{1}{\lambda}\left(\frac{1}{\lambda}+1\right)\left(\frac{1}{\lambda}+2\right)(\phi_{GJ}+1)^{-1/\lambda-3} .$$

At the maximum likelihood estimator $\widehat{\phi}_{GJ} = \left(e^{dt/T}-1\right)^{-\lambda} - 1$, we have,

$$A(\widehat{\phi}_{GJ}) = e^{dt/T} - 1 ,$$

$$A'(\widehat{\phi}_{GJ}) = -\tfrac{1}{\lambda}\left(e^{dt/T}-1\right)^{\lambda+1} ,$$

$$A''(\widehat{\phi}_{GJ}) = \tfrac{1}{\lambda}\left(\tfrac{1}{\lambda}+1\right)\left(e^{dt/T}-1\right)^{2\lambda+1} , \qquad \text{(A.3)}$$

$$A'''(\widehat{\phi}_{GJ}) = -\tfrac{1}{\lambda}\left(\tfrac{1}{\lambda}+1\right)\left(\tfrac{1}{\lambda}+2\right)\left(e^{dt/T}-1\right)^{3\lambda+1} ,$$

$$B(\widehat{\phi}_{GJ}) = \tfrac{dt}{T} ,$$

$$B'(\widehat{\phi}_{GJ}) = -\tfrac{1}{\lambda}\, e^{-dt/T}\left(e^{dt/T}-1\right)^{\lambda+1} ,$$

$$B''(\widehat{\phi}_{GJ}) = \tfrac{1}{\lambda}\, e^{-dt/T}\left(e^{dt/T}-1\right)^{2\lambda+1} + \tfrac{1}{\lambda^2}\, e^{-2dt/T}\left(e^{dt/T}-1\right)^{2\lambda+1}$$

and

$$B'''(\widehat{\phi}_{GJ}) = -\tfrac{1}{\lambda}\left(\tfrac{1}{\lambda}+1\right)\left(\tfrac{1}{\lambda}+2\right) e^{-dt/T}\left(e^{dt/T}-1\right)^{3\lambda+1} +$$

$$+ \tfrac{3}{\lambda^2}\left(\tfrac{1}{\lambda}+1\right) e^{-2dt/T}\left(e^{dt/T}-1\right)^{3\lambda+2} - \tfrac{2}{\lambda^3}\, e^{-3dt/T}\left(e^{dt/T}-1\right)^{3\lambda+3} .$$

Thus

$$\ell''(\widehat{\phi}_{GJ}) = -\tfrac{T^2}{d\lambda^2 t^2}\, e^{-2dt/T}\left(e^{dt/T}-1\right)^{2(\lambda+1)}$$

and

$$\ell'''(\widehat{\phi}_{GJ}) = \frac{T^2\left(e^{dt/T}-1\right)^{3(\lambda+1)}}{d\lambda^2 t^2\, e^{2dt/T}}\left\{\frac{3\left(1+\lambda^{-1}e^{-dt/T}\right)}{e^{dt/T}-1} - \frac{2T}{d\lambda t e^{dt/T}}\right\} . \qquad \text{(A.4)}$$

## (A.2) Derivatives of the Log-Likelihood Function for $\phi_{A_0}(\lambda)$ Locally at the Maximum Likelihood Estimator $\widehat{\Phi}_{A_0}(\lambda)$

The logarithm of the likelihood function for $\phi_{A_0} = \phi_{A_0}(\lambda)$ (see (17)) is given by

$$\ell(\phi_{A_0}) = -d\ln t + d\ln D(\phi_{A_0}) - \frac{T}{t}D(\phi_{A_0}) \qquad \text{(A.5)}$$

where $D(\phi_{A_0}) = \ln(1 + C(\phi_{A_0}))$, and $C(\phi_{A_0}) = \left(\frac{1-\phi_{A_0}}{1+\phi_{A_0}}\right)^{1/\lambda}$.

The first three derivatives of $\ell(\phi_{A_0})$ are given by

$$\ell'(\phi_{A_0}) = d\frac{D'(\phi_{A_0})}{D(\phi_{A_0})} - \frac{T}{t}D'(\phi_{A_0})$$

$$\ell''(\phi_{A_0}) = d\frac{D''(\phi_{A_0})}{D(\phi_{A_0})} - d\left(\frac{D'(\phi_{A_0})}{D(\phi_{A_0})}\right)^2 - \frac{T}{t}D''(\phi_{A_0})$$

and

$$\ell'''(\phi_{A_0}) = d\frac{D'''(\phi_{A_0})}{D(\phi_{A_0})} - 3d\frac{D'(\phi_{A_0})D''(\phi_{A_0})}{D^2(\phi_{A_0})} + 2d\left(\frac{D'(\phi_{A_0})}{D(\phi_{A_0})}\right)^3 - \frac{T}{t}D'''(\phi_{A_0}) \qquad \text{(A.6)}$$

where $D'(\phi_{A_0}) = C'(\phi_{A_0})/(1 + C(\phi_{A_0}))$ ,

$$D''(\phi_{A_0}) = \frac{C''(\phi_{A_0})}{1+C(\phi_{A_0})} - \left(\frac{C'(\phi_{A_0})}{1+C(\phi_{A_0})}\right)^2 ,$$

$$D'''(\phi_{A_0}) = \frac{C'''(\phi_{A_0})}{1+C(\phi_{A_0})} - \frac{3C'(\phi_{A_0})C''(\phi_{A_0})}{\left(1+C(\phi_{A_0})\right)^2} + 2\left(\frac{C'(\phi_{A_0})}{1+C(\phi_{A_0})}\right)^3 ,$$

$$C'(\phi_{A_0}) = -\frac{2C(\phi_{A_0})}{\lambda\left(1-\phi_{A_0}^2\right)} , \quad C''(\phi_{A_0}) = \frac{4C(\phi_{A_0})}{\lambda\left(1-\phi_{A_0}^2\right)^2}\left(\frac{1}{\lambda} - \phi_{A_0}\right)$$

and

$$C'''(\phi_{A_0}) = \frac{4C(\phi_{A_0})}{\lambda\left(1-\phi_{A_0}^2\right)^2}\left\{\frac{2}{\left(1-\phi_{A_0}^2\right)}\left(2\phi_{A_0} - \frac{1}{\lambda}\right)\left(\frac{1}{\lambda} - \phi_{A_0}\right) - 1\right\} .$$

At the maximum likelihood estimator

$$\widehat{\phi}_{A_0} = \left[1 - \left(e^{dt/T} - 1\right)^{\lambda}\right] \Big/ \left[1 + \left(e^{dt/T} - 1\right)^{\lambda}\right] \ ,$$

we have:

$$C(\widehat{\phi}_{A_0}) = e^{dt/T} - 1 \ ,$$

$$C'(\widehat{\phi}_{A_0}) = -\frac{2\left(e^{dt/T}-1\right)}{\lambda\left(1-\widehat{\phi}^2_{A_0}\right)} \ , \tag{A.7}$$

$$C''(\widehat{\phi}_{A_0}) = \frac{4\left(e^{dt/T}-1\right)}{\lambda\left(1-\widehat{\phi}^2_{A_0}\right)^2}\left(\frac{1}{\lambda} - \widehat{\phi}_{A_0}\right) \ ,$$

$$C'''(\widehat{\phi}_{A_0}) = \frac{4\left(e^{dt/T}-1\right)}{\lambda\left(1-\widehat{\phi}^2_{A_0}\right)^2}\left\{\frac{2}{1-\widehat{\phi}^2_{A_0}}\left(2\widehat{\phi}_{A_0} - \frac{1}{\lambda}\right)\left(\frac{1}{\lambda} - \widehat{\phi}_{A_0}\right) - 1\right\} \ ,$$

$$D(\widehat{\phi}_{A_0}) = dt/T \ , \quad D'(\widehat{\phi}_{A_0}) = -\frac{2e^{-dt/T}\left(e^{dt/T}-1\right)}{\lambda\left(1-\widehat{\phi}^2_{A_0}\right)} \ ,$$

$$D''(\widehat{\phi}_{A_0}) = \frac{4e^{-dt/T}\left(e^{dt/T}-1\right)}{\lambda\left(1-\widehat{\phi}^2_{A_0}\right)^2}\left(\frac{e^{-dt/T}}{\lambda} - \widehat{\phi}_{A_0}\right) \ ,$$

and

$$D'''(\widehat{\phi}_{A_0}) = \frac{8e^{-dt/T}\left(e^{dt/T}-1\right)}{\lambda\left(1-\widehat{\phi}^2_{A_0}\right)^3}\left\{-2\widehat{\phi}^2_{A_0} + \frac{e^{-dt/T}}{\lambda^2} + \frac{3\widehat{\phi}_{A_0}e^{-dt/T}}{\lambda} - \right.$$

$$\left. - \frac{2e^{-2dt/T}}{\lambda^2}\right\} - \frac{4e^{-dt/T}\left(e^{dt/T}-1\right)}{\lambda\left(1-\widehat{\phi}^2_{A_0}\right)^2} \ .$$

Thus

$$\ell''(\hat{\phi}_{A_0}) = \frac{-4T^2 e^{-2dt/T}\left(e^{dt/T}-1\right)^2}{dt^2\lambda^2\left(1-\hat{\phi}^2_{A_0}\right)^2}$$

and

$$\ell'''(\hat{\phi}_{A_0}) = \frac{8T^2 e^{-2dt/T}\left(e^{dt/T}-1\right)^2}{dt^2\lambda^2\left(1-\hat{\phi}^2_{A_0}\right)^3}\left\{\frac{3e^{-dt/T}}{\lambda} - 3\hat{\phi}_{A_0} - \right. \tag{A.8}$$

$$\left. - \frac{2T}{dt\lambda} + \frac{2T e^{-dt/T}}{dt\lambda}\right\} .$$

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Observação: A partir do nº 134/93, a publicação "Notas do ICMSC" subdividiu-se em três séries: COMPUTAÇÃO, ESTATÍSTICA E MATEMÁTICA.