UNIVERSIDADE DE SÃO PAULO

Instituto de Ciências Matemáticas e de Computação

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ISSN 0103-2577

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Série Computação

São Carlos – SP
Out./2007

Extensões de desigualdades válidas para problemas de corte de estoque unidimensionais

Resumo: O problema de corte de estoque consiste em cortar objetos grandes para produzir uma quantidade requisitada de peças menores, de modo que uma certa função objetivo seja otimizada. Um modelo de otimização linear tem sido amplamente utilizado na solução desse problema desde os anos 60 e incorpora parte da estrutura combinatória inerente ao problema nas colunas da matriz de restrições. O método simplex com geração de colunas é empregado na resolução desse modelo e, a cada iteração, um subproblema é resolvido para determinar uma nova coluna. Embora a técnica de geração de colunas seja amplamente utilizada para resolver problemas de otimização combinatória, é bem conhecido que esta técnica apresenta baixa convergência próximo da otimalidade, fazendo muitas iterações que pouco melhoram a função objetivo. Recentemente foi proposta uma abordagem que utiliza uma família de cortes duais, isto é, adiciona novas colunas ao problema primal, e reduz significativamente o número de subproblemas (neste caso, o problema da mochila) a serem resolvidos. Neste artigo, interpretamos esses cortes duais como mudança de variáveis e também propomos uma nova família de cortes duais.

Palavras-chaves: problemas de corte, geração de colunas, desigualdades válidas

Some Extensions of Dual Cuts to One-dimensional Cutting Stock Problems

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Abstract. Cutting stock problems consist of cutting available large objects to produce a quantity of smaller ordered items, in such a way as to optimise a given objective function. Linear optimisation, with a very large number of variables, has been widely used to model these problems since the 1960s. It embeds the essential part of the problem's combinatorial nature (the cutting patterns) in the model's columns. These models are well solved by the simplex method using the column generation technique, where a column generator sub-problem, which is in general a combinatorial optimisation problem, has to be solved in each iteration to find the entering column. Although the column generation technique is widely used to solve a number of combinatorial optimisation problems, it is well known that it has poor convergence, which requires solving many pricing problems. Recently, a new approach to the one-dimensional cutting stock problem was published, using a family of dual cuts (i.e., adding new columns to the primal problem) that significantly reduces the number of sub-problems (in this case, knapsack problems) to be solved. In this paper we reinterpret these dual cuts as elementary column operations (variable exchanges) and also propose an extended family of dual cuts.

Keywords. Cutting Stock Problems, Column Generation, Valid Inequalities

1. Introduction

There has been a renewed interest in the column generation technique due to successful experiments in solving combinatorial optimisation problems by the branch-and-price method. In these experiments, the problem in each node of the branch-and-bound method is solved by the column generation technique. The roots of the branch-and-price method are in the Dantzig-Wolfe decomposition, which allows reformulating a given combinatorial optimisation problem as one with a large number of variables. Typically, essential combinatorial characteristics of the overall problem are embedded in the reformulated problem. New rules for branching, which are problem-dependent, have to be devised in order to preserve the basic structure of the problem and use the column generation technique (Barnhart et al., 1998, Vanderbeck, 1999, 2000, Desaulniers et al. 2005, Ben Amor and Valerio de Carvalho, 2005). There are also many problems that are straightforwardly modelled with a large number of variables, such as cutting stock problems, bin-packing problems, covering problems, vehicle routing problems, etc. To

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solve either exactly or heuristically these linear integer optimisation problems, a number of linear relaxation problems with a very large number of variables have to be solved. Furthermore, it is well known that the column generation technique presents poor convergence when approaching optimality. A kind of *tail* arises (see Fig. 1), where many of the final iterations (sometimes the last 50% of them) produce only small decreases in the objective function value. Fig. 1 is the outcome of a one-dimensional cutting stock problem, where a larger object of given length is cut into 45 items whose lengths and ordered quantities are given as well. In iteration number 212 the objective function value (the total number of objects cut) equals 1838 and in iteration 298 the objective function value is 1836. In other words, the objective function value decreases only 2 units or 0.1% after 86 iterations. This means that 86 knapsack problems (the column generator sub-problem, or pricing problem) had to be solved to improve the objective function value by only 0.1%.

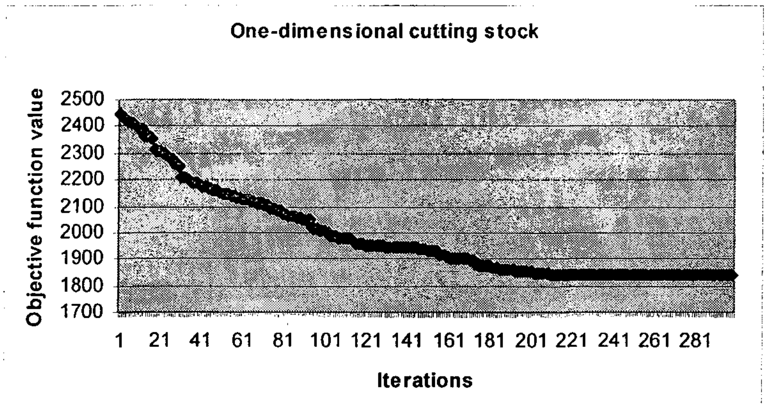

Figure 1. The tail phenomenon.

In order to accelerate and stabilize the column generation technique some strategies have been devised. For example, some researchers put effort into controlling oscillation in dual variables (Marsten, 1975, du Merle *et al.,* 1999, Ben Amor and Desrosiers, 2006).

Another strategy consists of reducing dual space by adding new constraints (called dual cuts), which involves adding new columns to the primal problem. Valério de Carvalho (2005) proposed a family of dual cuts to the one-dimensional cutting stock problem, when the objective function is the total number of objects cut. These dual cuts are not valid inequalities in the sense that arises in integer and combinatorial optimisation. These cuts are valid in the sense that if they are active at an optimal solution (i.e., basic columns in terms of the primal problem) there will be another optimal basis of original columns that are easy to build without needing to solve extra knapsack problems. In this paper we follow this strategy and propose extended families of dual cuts, for the problem studied by Valerio de Carvalho and extend them to the case where the objective is minimizing total waste.

2. Cutting stock problem modelling

Consider the one-dimensional cutting stock problem, where a number of objects in stock, each one with length L, are cut into m types of items, and an item of type i has a length ℓ_i, i=1,...,m. The problem consists of cutting the quantity b_i of type item i, i=1,...,m in such a way as to minimise the total number of objects cut.

In order to model this problem consider all possible cutting patterns, that is, all possible ways of cutting the objects, **say** n. For a cutting pattern j there is an associated m-vector of nonnegative integers denoted as $\mathbf{a}_j = (a_{1j} \ldots a_{mj})^T$, where a_{ij} is the number of items type i in the cutting pattern j. Let x_j be the number of objects cut according to cutting pattern j. Therefore, the following integer linear optimisation problem models the cutting stock problem:

$$\begin{aligned} &\text{Minimise } f(\mathbf{x}) = c_1x_1 + c_2x_2 + \ldots + c_nx_n \\ &\text{subject to: } \mathbf{a}_1x_1 + \mathbf{a}_2x_2 + \ldots + \mathbf{a}_nx_n = \mathbf{b} \\ &\qquad\qquad x_j \geq 0, j=1,\ldots,n, \end{aligned} \tag{1}$$

where c_j = 1, j=1,...,n. Therefore, the function $f(\mathbf{x})$ means the total number of objects cut.

Although the variables in model (1) should be integers, we consider a relaxation that generally gives a very good approximate solution for practical instances that present a large demand for items. For heuristic or exact methods to obtain an integer solution from the relaxed problem see, e.g., Gau and Wascher (1996), Ben Amor and Valerio de Carvalho (2005), Valerio de Carvalho (1999), Belov and Scheithauer (2002), Vanderbeck (2005). Typically, these methods solve several problems like (1) using column generation technique and, therefore the step of solving problem (1) more efficiently is a fundamental step in speeding up a number of other methods.

3. The column generation technique and dual space

Although model (1) has a very large number of variables, only a few of them are indeed absolutely necessary for its optimal solution. In other words, only a few cutting patterns should need be generated. We should have in mind that cutting pattern generation is the most time consuming routine of the whole column generation method, and any effort to generate correct cutting patterns is welcome. In terms of dual space, the column generation method works at finding unfeasible dual solutions that fulfil a sub set of dual constraints and determines violated dual constraints, which are fulfilled in the next iteration. Therefore, dual space is iteratively built until the optimal solution is obtained. The dual problem of (1) is given by:

$$\begin{aligned} &\text{Maximise } g(\boldsymbol{\pi}) = \boldsymbol{\pi}^T\mathbf{b} \\ &\text{subject to: } \boldsymbol{\pi}^T\mathbf{a}_j \leq c_j, j=1,\ldots,n. \end{aligned} \tag{2}$$

If we add new (illegitimate) constraints to (2) in such a way that we are able to rescue an optimal solution, then we may avoid a number of solutions that would have to generate cutting patterns. Consider *the constrained dual problem*:

$$\begin{aligned} &\text{Maximise } g(\boldsymbol{\pi}) = \boldsymbol{\pi}^T\mathbf{b} \\ &\text{subject to: } \boldsymbol{\pi}^T\mathbf{a}_j \leq c_j, j=1,\ldots,n, \\ &\qquad\qquad \boldsymbol{\pi}^T\mathbf{d}_k \leq d_k, k=1,\ldots,p, \end{aligned} \tag{3}$$

where $\pi^T \mathbf{d}_k \leq d_k$, $k=1,\ldots,p$ are called dual cuts. These new constraints correspond to new columns in primal problem (1), denoted as *the relaxed primal problem*:

$$\begin{aligned} &\text{Minimise } f(\mathbf{x}) = c_1x_1 + c_2x_2 + \ldots + c_nx_n + d_1y_1 + d_2y_2 + \ldots + d_py_p \\ &\text{subject to: } \mathbf{a}_1x_1 + \mathbf{a}_2x_2 + \ldots + \mathbf{a}_nx_n + \mathbf{d}_1y_1 + \mathbf{d}_2y_2 + \ldots + \mathbf{d}_py_p = \mathbf{b} \\ &\qquad x_j \geq 0, j=1,\ldots,n,\ y_k \geq 0,\ k=1,\ldots,p. \end{aligned} \tag{4}$$

The dual cuts are said to be *valid* if one is able to rescue an optimal solution of problem (1) from the optimal solution of problem (4).

It is worth noting that the first valid dual cut was suggested by Gilmore and Gomory (1961), who showed how to obtain an optimal solution to (1) after determining an over-fulfilled order optimal solution for the following problem.

$$\begin{aligned} &\text{Minimise } f(\mathbf{x}) = c_1x_1 + c_2x_2 + \ldots + c_nx_n \\ &\text{subject to: } \mathbf{a}_1x_1 + \mathbf{a}_2x_2 + \ldots + \mathbf{a}_nx_n \geq \mathbf{b} \\ &\qquad x_j \geq 0, j=1,\ldots,n. \end{aligned} \tag{5}$$

In terms of (4), $\mathbf{d}_k = (0 \ \ldots \ -1 \ \ldots \ 0)^T$ where the element -1 is in k^{th} position, and $d_k=0$, $k=1,\ldots,m$. In terms of dual space it means that $\pi_k \geq 0$, $k=1,\ldots,m$, that is, when solving problem (1), one can just check if $\pi_k<0$ and enter $\mathbf{d}_k = (0 \ \ldots \ -1 \ \ldots \ 0)^T$ into the next basis without solving the knapsack problem, making the pricing problem trivial. We denote these cuts as *G&G* dual cuts, which are a particular case of Valério de Carvalho's family of dual cuts.

4. Dual cuts: The Valério de Carvalho's Family

Valério de Carvalho (2005) presented a family of valid dual cuts as follows:

Proposition 1 (Valério de Carvalho, 2005). *Let $S \subseteq \{1,\ldots,m\}$ be any set of item types such that:*

$$\sum_{s \in S} \ell_s \leq \ell_i \tag{6}$$

for a given item type i. Then, a valid dual cut is given by:

$$-\pi_i + \sum_{s \in S} \pi_s \leq 0. \tag{7}$$

We call the dual cut in (7) a *V-cut*. Note that if $S = \varnothing$, then the V-cut is exactly the G&G-cut. Therefore, if for a multiplier simplex in a given iteration it happens that $-\pi_i + \sum_{s \in S} \pi_s > 0$ then the following column enters into the basis: (0… −1 … +1 …+1 … 0), where −1 is in the i^{th} position and each +1 is in the s^{th} position, $s \in S$. Of course, generating all these sets S according to (6) beforehand could be a non-polynomial task, but computational experiments show that one may limit the sets S to have low cardinalities, $|S|=1$ or 2, to reduce significantly the number of pricing sub problems solved. For instance, consider that $\ell_1 \geq \ell_2 \cdots \geq \ell_m$ then $-\pi_i + \pi_{i+1} \leq 0$, $i=1,\ldots,m-1$ are valid cuts according to (7) where $S = \{i+1\}$ for each i.

Valério de Carvalho (2005) proved Proposition 1 by considering a number of V-cuts active in an optimal solution, and showed constructively how to obtain new cutting patterns without solving new pricing problems by substituting the V-cuts in the optimal solution, that is, driving y_k to zero.

We show in the next section how to build this family of cuts by simply changing variables in model (1), which allows us to extend it to deeper dual cuts.

5. Dual cuts: An Extended Family

Proposition 1 can be extended as follows:

Proposition 2. *Let $S \subseteq \{1,...,m\}$ be any set of item types such that:*

$$\sum_{s \in S} \beta_s \ell_s \leq \ell_i \tag{8}$$

where $\beta_s \geq 0$ and integer, for a given item type i. Then, a valid dual cut is given by:

$$-\pi_i + \sum_{s \in S} \beta_s \pi_s \leq 0. \tag{9}$$

In order to build the dual cuts in Proposition 2 consider, for example, that $\ell_1 \geq \beta_2 \ell_2$ (in what follows, one may use $\ell_i \geq \beta_{s_1} \ell_{s_1} + \beta_{s_2} \ell_{s_2} + \cdots + \beta_{s_r} \ell_{s_r}$ and the same reasons given in this simpler case).

Assume that $\mathbf{a}_1$ is a column of the matrix **A**, that is, an m-vector that corresponds to a cutting pattern: $\mathbf{a}_1 = (a_{11}\ a_{21}\ \ldots\ a_{m1})^T$, with $a_{11} > 0$. Therefore, another possible column of **A**, say $\mathbf{a}_2$, is given by: $\mathbf{a}_2 = (0\ \beta_2 a_{11} + a_{21}\ \ldots\ a_{m1})^T$. This second cutting pattern is obtained from the first one, by simply substituting all items of type 1 with β_2 items of type 2, which is acceptable since $\ell_1 \geq \beta_2 \ell_2$.

Therefore, $\frac{1}{a_{11}}(\mathbf{a}_2 - \mathbf{a}_1) = \mathbf{d}_1 = (-1\ \beta_2\ 0\ \ldots\ 0)^T$, that is, $\mathbf{a}_2 - \mathbf{a}_1$ is a multiple of $\mathbf{d}_1$, which corresponds to the left hand side in (9) where i=1 and S={2} (note that if β_2=1 then it is a V-cut, and if β_2=0 then it is a G&G-cut). Since $c_1 = c_2 = 1$, the right hand side in (9) follows. We may formalise this procedure, by introducing the following matrix notation.

Let **E** be the $n \times n$ elementary matrix that represents the elementary operation above:

$$\mathbf{E} = \begin{bmatrix} 1 & -\frac{1}{a_{11}} & \cdots & 0 \\ 0 & \frac{1}{a_{11}} & & \\ \vdots & & \ddots & \vdots \\ 0 & & & 1 \end{bmatrix} \tag{10}$$

Therefore, the original problem (1) can be equivalently rewritten as:

$$\mathbf{AEE^{-1}x = b \Leftrightarrow A'x' = b},$$

where **A'** has the same columns as **A**, except for the second one, which is $\mathbf{d}_1$ given above, and $\mathbf{x}' = \mathbf{E}^{-1}\mathbf{x}$. Substituting $\mathbf{x} = \mathbf{E}\mathbf{x}'$ in the objective function, it follows that the coefficient of the second variable is zero, that is, d_1=0 in the relaxed primal problem (4) (here, the second variable x'_2, or y_1 in problem (4)).

Therefore, after solving the problem for **x**', the **x**-solution can be rescued by: $x_1 = x_1' - \frac{1}{a_{11}} x_2'$, $x_2 = \frac{1}{a_{11}} x_2'$ and $x_j = x_j'$, j=3…n. The column $\mathbf{a}_2$ of variable x_2 is built from column $\mathbf{a}_1$ as given above, without solving an extra knapsack problem.

However, when one solves the problem with the new variables **x**', i.e., column $\mathbf{d}_1$ added (as the second column in the case above) one does not know beforehand the column $\mathbf{a}_1$. Therefore, if $x_2' > 0$ (i.e., y_1>0) in the optimal solution (i.e. cut $\mathbf{d}_1$ is basic) then one needs to look for a cutting pattern column in the optimal basis, say $\mathbf{a}_1 = (a_{11} \ldots a_{m1})$, such that a_{11}>0. There will always be such a column, since b_1>0 and the coefficient of $x_2' > 0$ is –1 in the first equation. Although we have used in the arguments above item type 1 (the largest one) if other dual cuts are taken, it is enough to order them lexicographically and follow this order (see Valério de Carvalho, 2005).

Moreover, in order to preserve **x≥0** we should have $x_1 = \left(x_1' - \frac{1}{a_{11}} x_2' \right) \geq 0$, or $x_2' \leq a_{11} x_1'$. If there is no such a column, that is, $x_2' > a_{1j} x_j'$ for all j, then choose any column with a_{1j}>0 and x_j'>0. For simplicities sake, suppose that a_{11}>0 and x_1'>0, despite $x_2' > a_{11} x_1'$. Equation (4), but using **x**'-variable, is written by: $\mathbf{a}_1 x_1' + \mathbf{d}_1 x_2' + \ldots = \mathbf{b}$. Now, noting that $\mathbf{a}_2 = \mathbf{a}_1 + a_{11} \mathbf{d}_1$ and substituting $\mathbf{a}_1 = \mathbf{a}_2 - a_{11} \mathbf{d}_1$ in equation (4) we can rewrite it as: $\mathbf{a}_2 x_1' + \mathbf{d}_1 (x_2' - a_{11} x_1') + \ldots = \mathbf{b}$, which means that if we substitute the cutting pattern in column $\mathbf{a}_1$ by the cutting pattern in column $\mathbf{a}_2$, and the new feasible basic solution is given by $x_1' \leftarrow x_1'$ (now the first column is given by the cutting pattern in column $\mathbf{a}_2$), $x_2' \leftarrow x_2' - a_{11} x_1'$. Note that this is again an elementary column operation of adding column $\mathbf{d}_1$ times a_{11} to column $\mathbf{a}_1$. Since the coefficient of x_2' is zero in the objective function, the objective function value remains the same, but with a reduced frequency x_2' of the dual cut. This procedure can be repeated until finding a column j such that $x_2' \leq a_{1j} x_j'$.

In order to illustrate the effect of the dual cuts in dual space, consider ℓ_1=10 and ℓ_2=4. A V-cut according to (7) is given by: $-\pi_1 + \pi_2 \leq 0$, and another one according to (9) could be: $-\pi_1 + 2\pi_2 \leq 0$ (since $\ell_1 \geq 2\ell_2$). When $|S|$=1, that is, just one item type in S, say S={j}, one may define β_j as: $\beta_j = \left\lfloor \frac{\ell_i}{\ell_j} \right\rfloor$, where $\lfloor x \rfloor$ is the biggest integer number less than or equal to x. Fig. 2 depicts the dual space reduced due to a G&G-cut, halved by a V-cut and halved once more by an extended cut.

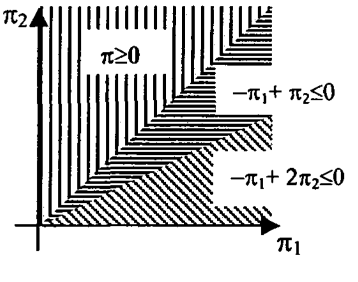

Figure 2. Effects of dual cuts on the dual solution space

6 Extending Dual Cut Families to the total waste objective function

It is also possible to extend the previous dual cuts when we deal with problem (1), but considering the objective function as the total waste as:

$$c_j = L - (a_{1j}\ell_1 + a_{2j}\ell_2 + \ldots + a_{mj}\ell_m),\ j = 1,\ldots,n.$$

Proceeding as before, let $\mathbf{a}_1$ be a column of matrix **A**, that is, an m-vector to a cutting pattern: $\mathbf{a}_1 = (a_{11}\ a_{21}\ \ldots\ a_{m1})^T$, with $a_{11}>0$, and if $\ell_1 \geq \beta_2\ell_2$ then $\mathbf{a}_2 = (0\ \beta_2 a_{11}+a_{21}\ \ldots\ a_{m1})^T$ is another valid cutting pattern. Exactly as is done in section 5, we consider $\mathbf{d}_1 = \frac{1}{a_{11}}(\mathbf{a}_2 - \mathbf{a}_1)$ $= (-1\ \beta_2\ 0\ \ldots\ 0)^T$, that means a variable changing in model (1). The coefficient in the objective function of $\mathbf{d}_1$ column is now given by:

$$\frac{1}{a_{11}}(\mathbf{c}_2 - \mathbf{c}_1) =$$

$$\frac{1}{a_{11}}\{[L - ((\beta_2 a_{11} + a_{21})\ell_2 + \ldots + a_{m1}\ell_m)] - [L - (a_{11}\ell_1 + a_{21}\ell_2 + \ldots + a_{m1}\ell_m)]\} = (\ell_1 - \beta_2\ell_2)$$

Therefore, following the same steps, we can write Proposition 3.

Proposition 3. *Let* $S \subseteq \{1,\ldots,m\}$ *be any set of item types such that*:

$$\sum_{s\in S}\beta_s\ell_s \leq \ell_i \qquad (11)$$

where $\beta_s \geq 0$ *and integer, for a given item type i. Then, a valid dual cut is given by:*

$$-\pi_i + \sum_{s\in S}\beta_s\pi_s \leq \ell_i - \sum_{s\in S}\beta_s\ell_s\ . \qquad (12)$$

The interpretation relating to the variables remains the same as before, as well as the recovery of the primal solution from a solution that contains dual cuts.

In order to illustrate the effect of dual cuts in (12), consider $\ell_1=10$ and $\ell_2=4$. Two dual cuts according to (12) can be given by: $-\pi_1+\pi_2 \leq 6$, if $\beta_2=1$, and $-\pi_1+2\pi_2 \leq 2$, if $\beta_2=2$ (since $\ell_1 \geq 2\ell_2$). Fig. 13 depicts the dual space reduced due to a G&G-cut, and the extended cuts in (12).

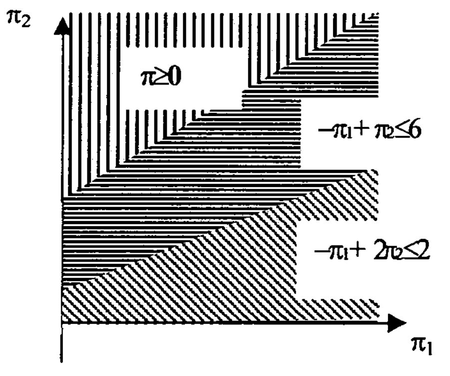

Figure 3. Effects of dual cuts on the dual solution space when waste is the objective

We can notice that regardless of the objective function (the waste or the number of objects) the dual cuts are still only dependent on the length of the items, and can therefore be generated beforehand.

7 Conclusions

In this article we reviewed a family of dual cuts proposed by Valerio de Carvalho (2005) in Proposition 1, who computationally showed their efficiency in terms of reducing the number of times that the pricing sub-problems were solved, as well as in terms of reducing run time when solving one-dimensional cutting stock problem. We gave another way of understanding Valerio de Carvalho's dual cuts (V-cuts) based on simple variable changing, that is, elementary column operations. This new interpretation of the V-cuts made it possible to easily extend them to other deeper dual cuts in Proposition 2, which lead to a further reduced dual space. Moreover, using the same strategy of changing variables, we also extended the dual cut families when the objective was changed from minimising the total number of objects cut (considered by Valerio de Carvalho) to minimising waste, which is used quite often in the cutting and packing area.

Acknowledgment

This research was sponsored by CNPq and FAPESP.

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