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REINALDO MORABITO
MARCOS ARENALES

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Reinaldo Morabito

Departamento de Engenharia de Produção

Universidade Federal de São Carlos

13565-905 - São Carlos - S.P., Brazil

morabito@power.ufscar.br

Marcos Arenales

Departamento de Ciências de Computação e Estatística

Instituto de Ciências Matemáticas e de Computação

Universidade de São Paulo-campus de São Carlos

13560-970-São Carlos - S.P., Brazil

arenales@icmsc.sc.usp.br

Abstract

In this study we analyzed practical aspects of the application of a cutting stock model to a Brazilian company that manufactures furniture on a large scale with a high degree of standardization. The model is based on the classical approach of Gilmore and Gomory (1965), which combines a linear program and a column generation procedure. Besides the 2-stage and 3-stage guillotine cutting patterns, we also considered 1-group guillotine patterns that improve the productivity of the cutting equipment. Examples derived from the furniture company are used to illustrate some of the trade-offs involved, in particular, the trade-off between cutting simpler patterns and patterns that yield less waste material, but reduce the productivity of the cutting machine.

Key words: cutting and packing problems, furniture industry, guillotine cutting patterns, 1-group patterns.

Resumo

Neste estudo analisamos aspectos práticos de uma aplicação do modelo de corte de estoque a uma indústria brasileira de móveis de alto grau de padronização e larga escala de produção. O modelo é baseado na abordagem clássica de Gilmore e Gomory (1965), que combina um programa linear com um procedimento de geração de colunas. Além de padrões 2 e 3-estágios, consideramos também padrões mais simples, tipo *xadrez* (chamado por Gilmore e Gomory de *1-group*) que melhora a produtividade do equipamento de cortagem. Exemplos derivados da indústria de móveis estudada são usados para ilustrar o efeito do uso de padrões alternativos, em particular, a perda *versus* produtividade da serra.

Palavras-chaves: problemas de corte de estoque, problemas de corte e empacotamento, indústria de móveis, padrões guilhotinados, padrões 1-group

1. Introduction

In this study we analyzed the cutting stock problem of a Brazilian furniture company with highly standardized products manufactured on a large scale. The problem is how to cut a set of objects (rectangular plates of wood fiber), with known sizes and quantities, to produce a set of items (smaller rectangular plates) with specified sizes and demands. The objects are available in stock, or can be obtained from suppliers in a short time. The items are parts of the products, such as tables, chairs, cupboards, wardrobes, etc., manufactured by the company.

Given that the total amount of objects is, in general, sufficiently large to produce all ordered items of the planning horizon (typically of one or two weeks), the problem is to determine a set of feasible cutting patterns, that is, patterns that produce all ordered items. An immediate criteria to evaluate the solutions would be the waste of material, that is, solutions of lower waste should be preferable. Nevertheless, if the objects have different costs, the minimum waste solution can be less economical. This suggests the material cost as an alternative criteria. The minimum cost solution becomes particularly important for the furniture company for which the cost of objects (raw material) can represent more than one third of the total costs of some products. This solution can also be helpful for supporting stock management decisions.

Different practical aspects should be considered when solving this cutting problem, especially with respect to the cutting machine, which is a production bottleneck of the furniture company. There are non trivial trade-offs to be analyzed, for example, the trade-off between cutting simpler patterns (e.g., 1-group guillotine patterns) and patterns that yield less waste material (e.g., 2-stage and 3-stage patterns) but require longer processing times and, in this way, reduce the productivity of the equipment. Another example is the trade-off between the benefits of combining more products in a production run to reduce the waste material, and the corresponding difficulties in the control of the production, work in process, due dates, etc.

Although there are several papers in the literature dealing with the cutting stock problem in the furniture industry (e.g., Gilmore and Gomory, 1965, Foronda and Carino, 1991, Yanasse et al., 1991, Carnieri et al., 1994), we are not aware of previous work that specifically treated the issues above. For surveys of cutting and packing problems and their industrial applications, the readers may consult Brown (1971), Golden (1976), Hinxman (1980), Dyckhoff and Finke (1992), Dowsland and Dowsland (1992), Sweeney and Paternoster (1992), Bischoff and Waescher (1995), Dyckhoff et al. (1997) and the electronic databases of the Special Interest Group on Cutting and Packing (SICUP, 1998).

This paper is organized as follows: In section 2 we review the classical model of Gilmore and Gomory (1965) and how to solve it using the simplex algorithm combined with a column generation procedure. In section 3, together with the exact and non exact 2-stage and 3-stage guillotine cutting patterns, we consider the 1-group guillotine patterns which improve the productivity of the cutting machine. In section 4 we present details of the model implementation and analyze computational results of examples derived from the furniture company. We compare the minimum waste and minimum cost criteria and analyze the trade-offs between 2-stage and 3-stage patterns, and between 1-group and non 1-group patterns. Finally, in section 5 we present concluding remarks and discuss perspectives for future research, such as the importance of pattern sequencing and models integrating cutting stock and lot sizing problems in the furniture industry.

2. Mathematical formulation

Pieces of furniture such as tables, chairs, cupboards, wardrobes, etc, are produced from rectangular plates of wood fibers pressed together, known as hardboards. First, these plates are cut to produce items (smaller rectangles) which are parts of the products. Then, the items pass through several production stages (e.g., border and surface treatment, painting, gluing, etc), until they are assembled together to become the final products.

The hardboards are supplied in different standard sizes, such as (in meters): 1.830 × 2.130, 1.830 × 2.750, 2.130 × 2.440, 2.130 × 2.750, 2.130 × 3.050. Besides these sizes, the hardboard suppliers, who also have an interesting cutting stock problem in their production processes (see Morabito and Garcia, 1998), offer leftover hardboards of smaller sizes and with special discounts. Typical sizes of these offcuts are (in m): 1.220 × 2.440, 1.220 × 2.750, 1.220 × 3.050, 1.245 × 2.440, 1.700 x 2.100, etc. The standard sizes can be acquired in any quantity, whereas the offcuts are offered daily in different lots with varying prices.

In section 4 we analyze an example of a furniture company using some of these plate sizes and a typical weekly order list of 25 item types.

Consider the following notation for the mathematical formulation:

Objects:

N: number of plate types (hardboards);

$L_j \times W_j$: size of the plate type j, $j=1,...,N$;

D_j: available quantity of the plate type j, $j=1,...,N$;

C_j: unit cost of the plate type j, $j=1,...,N$ (proportional to the plate area).

Items:

m: number of item types (smaller rectangles);

$l_k \times w_k$: size of the item type k, $k=1,...,m$;

d_k: required quantity of item type k.

The mathematical model can be derived from the work of Gilmore and Gomory (1963) on 1-dimensional cutting stock problems. In that paper, the authors analyzed a machine balancing problem in which the capacity of the machines can be interpreted here as the plate availability D_j, $j=1,..,N$. To describe the model, we initially suppose that, for each plate type, all cutting patterns are defined (when solving the model, these patterns are considered only implicitly). Let n_j be the number of possible patterns for plate type j. For each pattern, we associate the following $m \times 1$ vectors:

$$a_1^j = \begin{pmatrix} \alpha_{11}^j \\ \alpha_{21}^j \\ \vdots \\ \alpha_{m1}^j \end{pmatrix}, \quad a_2^j = \begin{pmatrix} \alpha_{12}^j \\ \alpha_{22}^j \\ \vdots \\ \alpha_{m2}^j \end{pmatrix}, \quad \ldots, \quad a_{n_j}^j = \begin{pmatrix} \alpha_{1n_j}^j \\ \alpha_{2n_j}^j \\ \vdots \\ \alpha_{mn_j}^j \end{pmatrix}, \qquad j=1,\ldots,N.$$

where α_{ki}^j is the number of times that item type k appears in pattern i of plate type j. The decision variables are x_{ij}, the number of plates of type j cut by pattern i, $i=1,\ldots,n_j$, $j=1,\ldots,N$. Let $\boldsymbol{d} = (d_1, d_2, \ldots, d_m)$ be the demand vector. The cutting problem with minimum cost criteria can be formulated as:

$$\text{Minimize } f(x_{11}, x_{21}, \ldots) = \sum_{i=1}^{n_1} C_1 x_{i1} + \sum_{i=1}^{n_2} C_2 x_{i2} + \ldots + \sum_{i=1}^{n_N} C_N x_{iN} \tag{1}$$

$$\text{Subject to: } \quad \sum_{i=1}^{n_1} \boldsymbol{a}_i^1 x_{i1} + \sum_{i=1}^{n_2} \boldsymbol{a}_i^2 x_{i2} + \ldots + \sum_{i=1}^{n_N} \boldsymbol{a}_i^N x_{iN} = \boldsymbol{d} \tag{2}$$

$$\begin{array}{llll} \sum_{i=1}^{n_1} x_{i1} & & & \leq D_1 \\ & \sum_{i=1}^{n_2} x_{i2} & & \leq D_2 \\ & & \ddots & \vdots \\ & & \sum_{i=1}^{n_N} x_{iN} & \leq D_N \end{array} \tag{3}$$

$$x_{ij} \geq 0, \text{ integer}, \; i=1,\ldots,n_j, \; j=1,\ldots,N. \tag{4}$$

The objective function (1) represents the cost of the plates needed to produce all items. If the criteria is the minimum waste, then each C_j in $f(x_{11}, x_{21}, \ldots)$ should be replaced by $L_j W_j - \sum_{k=1}^{m} l_k w_k \alpha_{ki}^j$, that is, the waste of pattern i using plate type j. The constraint (2) guarantees that all d_k ordered items of type k are produced. The constraints (3) consider the availability of each plate type j, $j=1,..,N$, and the constraints (4) require the decision variables to be non negative integers. Note that if the number of possible patterns is very large (in the furniture company, this number can be hundreds of thousands, or even millions), the model (1)-(4) is much more difficult to be optimally solved.

Gilmore and Gomory (1961, 1965) applied the simplex method with a column generation procedure to solve the LP-relaxation of (1)-(4). This approach has been widely utilized to solve cutting problems in several industrial processes. The contributions to this method have been in the solution of the combinatorial optimization subproblem of generating a column in the linear program (see problem (7) below where only one plate is cut). To initiate the simplex method, the artificial variable technique can be applied and an initial basis is composed of the columns of the artificial variables associated with constraints (2), together with the slack variable columns associated with constraints (3).

The column generation derives from the simple observation that the columns of the model (1)-(4) can be iteratively generated, instead of being stored *a priori*. Let π be the simplex multiplier vector associated with constraint (2), and θ_j be the multiplier associated with the j^{th} constraint in (3), $j=1,\ldots,N$. The $(m+N)\times 1$ vector (π,θ), is the multiplier vector corresponding to the current basis. Then, the relative cost of variable x_{ij} is given by: $C_j - (\pi,\theta)\begin{pmatrix} \mathbf{a}_i^j \\ \mathbf{e}^j \end{pmatrix} = C_j - \pi \mathbf{a}_i^j - \theta_j$, where $\mathbf{e}^j$ is the $N\times 1$ vector with 1 in component j and 0 in the remaining components (see constraints (2) and (3)). Applying the Dantizg criteria to choose a column to enter the basis (i.e., the variable with the lowest relative cost), we obtain:

$$min_{(i,j)}\ \{ C_j - \pi \mathbf{a}_i^j - \theta_j \} \tag{5}$$

Given that the vectors $\mathbf{a}_i^j$ have the same *formation law* (since they correspond to cutting patterns), it follows that problem (5) consists of the solution to the subproblem:

$$\begin{aligned} &maximize\ \pi \mathbf{a}^j \\ &such\ that\ \mathbf{a}^j\ corresponds\ to\ a\ cutting\ pattern\ for\ plate\ L_j\times W_j \end{aligned} \tag{6}$$

Observe that problem (6) is equivalent to problem (5) since we have supposed that the cost of plate type j, C_j, is independent of how the pattern is cut. If the criteria is the minimum waste ($C_j \leftarrow L_j W_j - \sum_{k=1}^{m} l_k w_k \alpha_{ki}^j$), we should just redefine the coefficients of the objective function in (6) by: $\pi_k \leftarrow \pi_k - l_k w_k$. This is valid because the waste function is linearly defined in terms of the number of items of type k. In section 3 we discuss classes of cutting patterns for which additional costs of the cutting process should be considered and these costs do not vary linearly with the number of items in the pattern.

Problem (6) should be solved for each plate type $j=1,\ldots,N$. For each j, we obtain a cutting pattern represented by a vector $\mathbf{a}^j$. If

$$C_s - \pi \mathbf{a}^s - \theta_s = min_{j=1,\ldots,N}\ \{ C_j - \pi \mathbf{a}^j - \theta_j \} \geq 0, \tag{7}$$

then the current basis is optimal and the LP-relaxation is solved. Otherwise, $C_s - \pi a^s - \theta_s < 0$ and the column $\begin{pmatrix} a^s \\ e^s \end{pmatrix}$ enters the basis. The choice of the column to leave the basis follows the usual steps of the simplex method.

This procedure is particularly useful for the cutting problem of the furniture company, where the demands d_k are sufficiently large (typically of the order of hundreds or thousands; see the examples in section 4) and just a few items are produced in each pattern (at most tens of items), so that the value of the basic variables are large and its simple rounding produces satisfactory results. The integer 1-dimensional and 2-dimensional cutting stock problems have been considered in the literature (e.g., Haessler, 1980, Vance et al., 1994, Waescher and Gau, 1996, Riehme et al., 1996) and several heuristics have been developed to find an integer solution, instead of simple rounding.

3. Pattern generation

The variable vector a^j of problem (6) corresponds to a cutting pattern for the plate $L_j \times W_j$ and, therefore, is subject to a set of rules from the cutting process. Sometimes the mathematical formulation of the constraints that define a feasible cutting pattern is non trivial. In this section we briefly review how problem (6) can be solved for the cases of 2-stage and 3-stage guillotine cuttings, and we present another cutting rule called 1-group that, although yielding higher levels of waste, increases the cutting process productivity of the furniture company. For simplicity, the procedures are presented considering just one type of plate, say plates of size $L \times W$ and cost C. If we have different types of plates, the procedure should be repeated for each type.

3.1 2-stage cutting patterns

In the furniture company the cutting equipment is able to produce only guillotine cuts on the plates. A cut is of guillotine type if, when applied to a rectangle, it produces two new rectangles. Gilmore and Gomory (1965) presented a simple and effective method to solve problem (6) when the cutting pattern is obtained by guillotine cuts in, at most, two stages. In the first stage, parallel longitudinal guillotine cuts are produced on the plate, without moving the plate, to produce a set of strips. In the second stage, these strips are pushed, one by one, and the remaining parallel transversal guillotine cuts are made on each strip (figure 1). If there is no need for additional trimming (i.e., all items have the same width in each strip), the pattern is called exact 2-stage guillotine (figure 1a), otherwise, it is called non exact (figure 1b).

Figure 1 – 2-stage cutting pattern: (a) exact case, (b) non exact case

The method developed by Gilmore and Gomory (1965) to generate 2-stage patterns involves two phases. Approaches based on two phases are usual in the literature; see e.g. Farley (1983), Morabito and Arenales (1994), Hifi and Zissimopoulos (1996), Hifi (1997), Riehme et al. (1996), and Morabito and Garcia (1998). In the first phase, cutting patterns are determined for each strip of size (L,w_j), $j \in D_W$, where $D_W=\{j \mid w_j \neq w_i, i>j, i,j=1,...,m\}$ is the set of different widths. Then, the second phase decides how many times each strip should be used. Note that we need just one cutting pattern for each strip, the one that provides the best value for $\sum_{i=1}^{m} \pi_i \lambda_{ij}$, where λ_{ij} is the number of items of type i in a strip of type j. The two-phase procedure for the non exact case (figure 1b) is briefly presented below.

1st Phase: Let $W_j=\{i \mid w_i \leq w_j, i=1,...,m\}$ and V_j, $j \in D_W$, be defined as:

$$V_j = \max \sum_{i \in W_j} \pi_i \lambda_{ij} \tag{8}$$

$$\sum_{i \in W_j} l_i \lambda_{ij} \leq L \tag{9}$$

$$\lambda_{ij} \geq 0, \textit{ integer}, \ i=1,..,m \tag{10}$$

2nd Phase: Let μ_j be the number of strips of type j in the plate $j \in D_W$. Then,

$$\max \sum_{j \in D_w} V_j \mu_j \tag{11}$$

$$\sum_{j \in D_w} w_j \mu_j \leq W \tag{12}$$

$$\mu_j \geq 0, \textit{ integer}, \ j \in D_w \tag{13}$$

The procedure can be easily adapted to deal with the exact case (figure 1a) by simply redefining W_j in the first phase as $W_j=\{i \mid w_i=w_j,\ i=1,...,m\}$. The cutting pattern obtained with the solution of (6)-(8) and (9)-(11) is given by:

$$a = \begin{pmatrix} \alpha_1 = \sum_{j\in D_w} \lambda_{1j}\mu_j \\ \alpha_2 = \sum_{j\in D_w} \lambda_{2j}\mu_j \\ \vdots \\ \alpha_m = \sum_{j\in D_w} \lambda_{mj}\mu_j \end{pmatrix} \tag{14}$$

Let m_W be the number of elements of D_W. It should be noted that the m_W+1 knapsack problems in (8)-(10) and (11)-(13) produce the optimal pattern for the plate of size $L\times W$, considering the cuts of the first stage parallel to the plate length L (the optimality comes from the fact that the strips are independently cut. In section 3.3 we consider the 1-group pattern where the strips are not independent). The process should be repeated now considering the cuts of the first stage parallel to the plate width W. Note that this repetition requires the solution of other m_L+1 knapsack problems, where m_L is the number of elements in $D_W=\{j \mid l_j \neq l_i,\ i>j,\ i, j=1,...,m\}$). Cutting patterns should be determined for each plate type, and the pattern with the highest relative cost is chosen (see (7)).

3.2 3-stage cutting patterns

The cutting equipment of the furniture company is able to cut guillotine patterns in, at most, three stages (figure 2a).

Figure 2 – (a) 3-stage cutting pattern, (b) Heuristic third stage

Exact approaches to generate 3-stage cutting patterns can be found, for example, in Gilmore and Gomory (1965) and Beasley (1985). Gilmore and Gomory (1965) discussed an approach based on the application of the two-phase procedure (8)-(14) to each strip of size Lxw, $w\leq W$, where w is any linear combination of the widths w_1, w_2, ..., w_m (i.e., $w=\sum_{i=1}^{m} a_i w_i$, $a_i\geq 0$, integer). Beasley (1985) presented dynamic programming formulas with state spaces that depend on the size of the sets composed of all linear combinations of the lengths l_1, l_2, ..., l_m and the widths w_1, w_2, ..., w_m in LxW (*normal sets*). Obviously the inconveniences of these approaches are the computational requirements that, depending on the size of the problem, can be excessive. Next we present a simple heuristic that has been producing good results for the furniture company (Morabito, 1989) and practically involves the same computational effort as the 2-stage procedure.

Consider again the first phase of procedure (8)-(14) for the non exact case, where items of type $i \in W_j$ (i.e., i such that $w_i \leq w_j$) should be selected and arranged in a strip of size $L \times w_j$. Note that, for each item $i \in W_j$, we can arrange $\lfloor w_j/w_i \rfloor$ items along the width w_j of the strip (figure 2b), instead of only one item as in the original procedure. Thus, the heuristic simply replaces π_i in (8) by $\lfloor w_j/w_i \rfloor \pi_i$, and λ_{ij} in (14) by $\lfloor w_j/w_i \rfloor \lambda_{ij}$ (where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x). The remainder of the two-phase procedure stays as before.

3.3 1-group cutting patterns

Due to the particular characteristics of the cutting machine, the 1-group guillotine patterns are the ones that require the shortest processing times. These patterns belong to a special class of 2-stage patterns in which the second stage cuttings are performed simultaneously on the strips resulting from the first stage cuttings (Gilmore and Gomory, 1965). This implies that the second stage cuttings are produced together with the first stage cuttings, without moving the strips and, in this way, save processing time. Here we consider only the exact case where there is no need for additional trimming (see figure 3). Gilmore and Gomory (1965) also discussed p-group patterns, $p>1$, which are of less interest for the furniture company. Note that the 1-group patterns can be non homogeneous (a pattern is homogeneous if it contains only items of the same type), as illustrated in figure 3.

Figure 3 – 1-group cutting pattern

The 1-group patterns can be mathematically modeled as follows. Note that, although they are simpler than 2-stage patterns, an additional difficulty is introduced because the strips can not be independently cut anymore. This hypothesis of independence was necessary to validate Gilmore and Gomory's decomposition method. Let R_i be the ordered rectangle $l_i \times w_i$, $i=1,..,m$. Consider the $m \times m$ matrix $\Pi = (\pi_{ik})$, defined as:

$$\pi_{ik} = \begin{cases} \pi_r & \text{if } R_r = l_i \times w_k \\ 0 & \text{otherwise} \end{cases}$$

where π_r is the simplex multiplier associated with the r^{th}-contraint in (2). Note that $\pi_{ii}=\pi_i$, $i=1,...,m$. For illustration, let $R_1=1x2$, $R_2=1x3$ and $R_3=1x4$. Then, $\pi_{11}=\pi_1$, $\pi_{12}=\pi_2$ $(R_2=l_1 \times w_2)$ and $\pi_{13}=\pi_3$ $(R_3=l_1 \times w_3)$. In words, the non null elements of the i^{th}-line of matrix Π correspond to the items with the same length l_i, and the non null elements of column k correspond to the items with the same width w_k. The variables that define a 1-group pattern are:

λ_i = number of times that the length l_i is cut

μ_k = number of times that the width w_k is cut.

It should be noted that the product $\lambda_i \mu_k$ defines the number of items of size $l_i \times w_k$ in the pattern (see figure 3). Thus, the 1-group pattern with the lowest relative cost is obtained by solving the following integer quadratic program:

$$\text{Maximize } \sum_{i=1}^{m} \sum_{k=1}^{m} \pi_{ik}\, \lambda_i\, \mu_k \tag{15}$$

$$\text{Subject to: } \sum_{i=1}^{m} l_i\, \lambda_i \leq L \tag{16}$$

$$\sum_{k=1}^{m} w_k\, \mu_k \leq W \tag{17}$$

$$\lambda_i \geq 0,\ \mu_k \geq 0,\ \text{integer},\ i,\ k=1,...,m. \tag{18}$$

A heuristic solution to problem (15)-(18) can be obtained fixing μ_k, $k=1,...,m$ (or λ_i, $i=1,...,m$) and solving the resulting knapsack problem. In particular, a simple heuristic solution is given by (see section 4 for computational results):

$$\mu_j = \lfloor W / w_j \rfloor, \qquad \mu_k = 0, \quad k=1,...,m, \quad k \neq j, \text{ for any } j \in D_W. \tag{19}$$

In words, the strip $L \times w_j$ is repeated μ_j times. With vector μ fixed, we can determine vector λ, which defines the pattern of strip $L \times w_j$. Note that, because of the choice above, it follows that $\lambda_i \mu_k = 0$ for $k \neq j$ and, since $\pi_{ij} \neq 0$ if and only if $i \in W_j$ (items of same strip w_j), the objective function (14) can be rewritten as:

$$\lfloor W / w_j \rfloor \sum_{i=1}^{m} \pi_{ij} \lambda_i = \lfloor W / w_j \rfloor \sum_{i \in W_j} \pi_i \lambda_i \,. \tag{20}$$

Moreover, without loss of generality, we can fix $\lambda_p=0$ for all $p \notin W_j$. Therefore, the constraint in (16) becomes:

$$\sum_{i \in W_j} l_i \lambda_i \leq L \,. \tag{21}$$

In summary, by fixing μ according to (19), we obtain the problem in (8)-(10), that is:

$$V_j = maximize \sum_{i \in W_j} \pi_i \lambda_i \,, \quad s.t.: \sum_{i \in W_j} l_i \lambda_i \leq L, \quad \lambda_i \geq 0,\ integer,\ i=1,\dots,m$$

and the solution of (19) corresponds to the homogeneous solution of (11)-(13). Repeating this procedure for each $j \in D_W$ and selecting the best solution among them, we have a heuristic procedure to solve problem (15)-(18):

$$Max_{j \in D_W} \{ \lfloor W / w_j \rfloor V_j \}, \tag{22}$$

Note that, in this way, we are choosing the most valuable pattern among all patterns composed of strips of the same type (L,w_j), $j \in D_w$. Similarly, we can fix the vector λ:

$$\lambda_j = \lfloor L / l_j \rfloor, \qquad \lambda_i = 0,\ i=1,\dots,m,\ \ i \neq j, \text{ for any } j \in D_L. \tag{23}$$

and then,

$$Max_{j \in D_L} \{ \lfloor L / l_j \rfloor V_j \}, \tag{24}$$

where $V_j = maximize \sum_{k \in L_j} \pi_k \mu_k \,, \quad s.t.: \sum_{k \in L_j} w_k \mu_k \leq W,\ \mu_k \geq 0,\ integer,\ k=1,\dots,m$

$L_j = \{ k \text{ such that } w_k = w_j \}$.

3.4 Fixed pattern cost, item rotation and saw thickness

In the objective function (1) we have considered that the coefficient C_j of variable x_{ij} (number of plates of type j using pattern i) is independent of pattern i. Since the 1-group patterns are the ones that require the shortest processing times, a simple approximate way of describing the cost of cutting a more complex pattern is to increase the plate cost, for example, to add to C_j a fixed cost δ_i associated with the additional time needed to process pattern i. The coefficient of x_{ij} in the objective function (1) becomes:

$$C_{ij} = \begin{cases} C_j + \delta_i & \text{if } i \text{ is not a 1-group pattern} \\ C_j & \text{otherwise} \end{cases} \tag{25}$$

Then, in each simplex iteration, we search first for a 1-group pattern with negative relative cost to enter the basis (i.e., problem (6) is solved with the condition that $\boldsymbol{a}^j$ corresponds to a 1-group pattern). If condition (7) is verified, then we search for a non 1-group pattern (i.e., problem (6) is solved for exact and non exact 2-stage and 3-stage patterns). In this case, the optimality test (7) should be changed considering (25).

So far we have considered that the items are cut following a fixed orientation, that is, we have supposed that no items can be rotated in 90 degrees before being placed in the pattern. Nevertheless, in the furniture company, we can have part of the items with fixed orientation and part of them without. In this case, it is enough to consider in (8)-(10) each item type i without fixed orientation as two different types of sizes $l_i \times w_i$ and $w_i \times l_i$, respectively (note that the sets D_W and D_L should be redefined). Moreover, the quantity of items i (either $l_i \times w_i$ or $w_i \times l_i$) in the pattern should compose the i^{th}-component of vector $\boldsymbol{a}$ in (14). For example, assume that item type 1 has no fixed orientation and consider a new item type $m+1$, with $l_{m+1} = w_1$ and $w_{m+1} = l_1$. The coordinate α_1 in (14) should be calculated by:

$$\alpha_1 = \sum_{j \in D_w} (\lambda_{1j} + \lambda_{m+1,j}) \mu_j$$

We have also supposed so far that the saw thickness of the cutting machine can be disregarded. Without loss of generality, to consider the saw thickness we simply add it to the sizes of the stock plates and items when solving problems (8)-(10) and (11)-(13). For example, the new sizes of the plates and items of type i should be $(L+\sigma)\times(W+\sigma)$ and $(l_i+\sigma)\times(w_i+\sigma)$, where σ is the saw thickness (Gilmore and Gomory, 1965).

4. Model implementation and computational results

In this section we discuss some details of the computational implementation of the models of sections 2 and 3, and analyze the computational results obtained from examples of the furniture company. We compare the criteria of minimum waste and minimum cost, and analyze the trade-offs between 2-stage and 3-stage cutting patterns, and 1-group and non 1-group cutting patterns.

To solve the LP-relaxation of the model (1)-(4), we implemented the simplex algorithm with the column generation procedure described in section 3. For phase 1 of the simplex, we generate initial basic solutions as discussed in section 2. The linear systems involved in the simplex algorithm were solved applying the Gauss elimination method with partial pivoting (the sizes of these systems for the furniture company were of the order of tens). Finally, the knapsack problems (8)-(10) and (11)-(13) were solved by the lexicographic method described in Gilmore and Gomory (1963).

Gilmore and Gomory (1963) recommended the use of a stopping criteria (*cutoff*) to interrupt the simplex algorithm in cases where, after a few iterations, the improvements in the objective function are insignificant. In this work we implemented the criteria of interrupting the simplex method if the last 10 iterations did not reduce the value of the objective function in, at least, 0.1%.

The implementations were coded in Pascal language (Turbo-Pascal 5.5) and the computational results of the next tables were produced by a Pentium 100 microcomputer. A tolerance of $\pm 10^{-6}$ was considered for the floating point errors.

4.1 Minimum waste *versus* minimum cost

Consider the example of tables 1 and 2. The data for the stock plates are presented in table 1. In fact, the plate types 1, 2 and 3 were not available in the company stock at the time of the production planning, but could be supplied in a short time with discounts of 10% in comparison with the costs of the plate types 4 and 5 (note in the last column of table 1 that the plates have different costs per unit area). Table 2 presents the data for the ordered items. The items do not have a fixed orientation (i.e., they can rotate 90 degrees) and the saw thickness is σ=0.004 m.

Table 1- Data of the N=5 types of stock plates

Table 2 – Data of the m=25 types of ordered items

Table 3 compares the results obtained with the two criteria: the minimum waste and the minimum cost (see section 3.1), considering only non exact 2-stage cutting patterns (8)-(14). All values of the table correspond to the simple rounding of the optimal solution of the LP-relaxation of problem (1)-(4) (recall that the variables in (1)-(4) are integers). For example, the minimum waste and minimum cost values found by the simplex method were 501.3 m^2 (i.e., 2.32%) and \$22597.77, respectively. After the variable rounding, these values changed to 501.4 m^2 (2.32%) and \$22598.07 (table 3), that is, deviations of less than 10^{-3} with respect to the value of the optimal relaxed solution. The largest variation in the item demands is 0.3% (in fact, note in table 3 that the two solutions produce 48228 and 48230 items, respectively, which are very close to the original demanded value of 48227 items).

Table 3 – Comparison between the minimum waste and minimum cost solutions

Table 4 details the stock plates utilized in the minimum waste and minimum cost solutions. Observe that the solutions are quite different with respect to the utilization of plate types. For example, the minimum cost solution utilizes all plate types, in particular, all available plates of types 3 and 4 (compare tables 1 and 4). On the other hand, the minimum waste solution utilizes only plates of types 3, 4 and 5, in particular, all available plates of type 4. These solutions can be useful for supporting stock management decisions and reviewing the adopted inventory policies. In fact, after the implementation of the model in the company, the production manager reported that the model was also being used to help in the process of acquisition of stock plates.

Table 4 – Utilization of stock plates in the minimum waste and minimum cost solutions

It is worth noting that, for this example, any basis is composed of 30 columns (i.e., $m+N$ linearly independent columns), that is, at most 30 cutting patterns are utilized. The minimum waste solution uses only 26 patterns, whereas the minimum cost solution requires 27 patterns. The remaining basic columns correspond to the columns of the basic slack variables. For example, for the minimum cost solution, the other 3 columns correspond to the slack variables associated with the availability constraints of plate types 1, 2 and 5 (compare tables 1 and 4).

4.2 Trade-off between 2-stage and 3-stage cutting patterns

The third cutting stage reduces the productivity of the equipment due to the additional processing time. Considering that, on the one hand, the cutting equipment is a production bottleneck and, on the other hand, the cost of waste material is substantial in the furniture company, an interesting trade-off is that of choosing between patterns easy to cut, such as 2-stage, which saves processing time, and 3-stage patterns which save waste material.

To illustrate this trade-off, we solved the example of section 4.1 with the criteria of minimum waste for both the exact 2-stage and the 3-stage cutting patterns. Table 5 compares the results obtained. Similar to table 3, all values of the table correspond to the simple rounding of the optimal relaxed solution obtained by the simplex method. For example, the minimum waste values found for the two cases were 826.8 m^2 (3.78%) and 474.7 m^2 (2.20%), respectively. After the rounding of the variables, these values changed to 827.3 m^2 (3.78%) and 474.7 m^2 (2.20%) (see table 5), that is, deviations of less than 10^{-3} from the optimal relaxed solution. The largest variation in the item demand was 0.8%. (note in table 3 that the two solutions produce 48253 and 48227 items, respectively, which are very close to the original demanded value, 48227). Table 6 details the stock plates utilized in the exact 2-stage and 3-stage solutions.

Table 5 – Comparison between the exact 2-stage and 3-stage minimum waste solutions

Table 6 – Utilization of stock plates in the exact 2-stage and 3-stage solutions

As expected, the exact 2-stage solution (3.78%) results in higher waste than the non exact 2-stage solution (2.32%) (tables 3 and 5). On the other hand, the 3-stage solution (2.20%) yields lower waste. Estimating the processing times of 2-stage (exact and non exact) and 3-stage patterns, the production planner can select the best solution taking into account the benefits of reducing waste material and increasing productivity in the cutting process. Alternatively, he or she can add to the plate cost an estimated cost, proportional to the additional processing times of 3-stage cuttings with respect to 2-stage, and choose the minimum cost patterns, as in section 3.3. This idea is explored below.

4.3 Trade-off between 1-group and non 1-group cutting patterns

To illustrate the trade-off between 1-group and non 1-group patterns, let us consider another example derived from the furniture company. Plates of size 1.850 x 3.670 m should be used to produce all items of table 7. As in the example of tables 1 and 2, we assume that the items have no fixed orientation (i.e., they can rotate 90 degrees) and the saw thickness is

0.004 m.

Table 7 – Data of the m=15 types of ordered items

For ease of presentation, suppose that the unit cost of the plates is $1 and there is a sufficient number of such plates to produce all items. In order to avoid local perturbations due to the variable rounding, our analysis focuses on the relaxed solution of problem (1)-(4), which produces exactly 13227 items (table 7). Table 8 presents the results obtained by varying the fixed cost δ in (25) from $0.00 to $0.15 (i.e., from 0% to 15% of the plate value). For simplicity, these results were produced considering the non 1-group patterns as non exact 2-stage patterns.

Table 8 – Solutions obtained for different values of the fixed cost of non 1-group patterns

Note in table 8 that, as we increase the fixed cost (first column of table 8), the proportion of plates with non 1-group patterns decreases (last column), followed by an increase in the waste material (third and fourth columns).

In particular, if the fixed cost is $0.02, that is, a value equivalent to 2% of the plate value, the cost of the minimum waste solution (first row of table 8) rises from $348.71 to $355.22, that is, an increase of $6.51 (given that 93.4% of the patterns are non 1-group). On the other hand, the minimum cost solution is $354.68 (third row of table 8). The increase in the percentage waste is small (from 2.67% to 2.80%, or only 0.13%; compare the first and third rows) in comparison with the minimum waste solution. In this case, note that the proportion of plates using 1-group patterns reduces from 93.4% to 77.7%, but the total number of plates cut remains almost the same. If the fixed cost is greater or equal to $0.15 (last row of table 8), then it is better to cut all plates with 1-group patterns.

The data of table 8 can be very useful for a production manager who needs to determine the best set of cutting patterns taking into account the trade-off between the economy of material (patterns with lower levels of waste) and the benefits of increasing process productivity (simpler patterns such as 1-group) and, in this way, reducing overtime, avoiding additional shifts, satisfying due dates, etc. Figure 4 illustrates a trade-off curve for the example above (fourth and sixth columns of table 8). Note that as the proportion of non 1-group patterns decreases (and hence, the productivity increases), the waste material increases.

Figure 4 – Trade-off curve between waste material and proportion of non 1-group patterns

5. Concluding remarks and perspectives for future research

In this paper we analyzed the application of Gilmore and Gomory's approach for the cutting stock problem of a Brazilian furniture company. Different rules for generating a cutting pattern were studied under alternative objective functions. A trade-off curve was depicted showing the variation of the waste material as the proportion of 1-group cutting patterns is increased to improve the productivity of the cutting machine. Besides being helpful for the production planning of the cutting process, the model can also be useful for deriving an inventory policy for the stock plates.

The cutting stock problem is part of the production planning of the company and the solution of the model (1)-(4) produces a set of cutting patterns which can not always be cut in any sequence, since the items obtained must be grouped for the production of the products and there may be limitations for the work in process and different product due dates. The pattern sequencing has been studied in the literature with different objectives, such as minimizing the order spread and the number of open products (a product is open if its demand was only partially produced) during the production of the cutting patterns (Madsen, 1988, Yuen, 1995, Yuen and Richardson, 1995, Yanasse, 1997). To be applied to the furniture industry, these approaches should be adapted to include additional constraints, such as product due dates and work in process limitations.

In practice, due to the difficulties of an integrated analysis for the cutting and sequencing problems, the production planning of the company defines production lots such that the due dates and the work in process limitations can be trivially satisfied after solving the cutting problem. However, this procedure increases the waste material, since only part of the products (consequently, their corresponding items) is considered each time by the cutting problem and, in this way, the number of candidate cutting patterns can be dramatically decreased. An approach integrating the cutting stock and the lot sizing problems, namely the coordination problem (Drexl and Kimms, 1997), is a topic for our future research.

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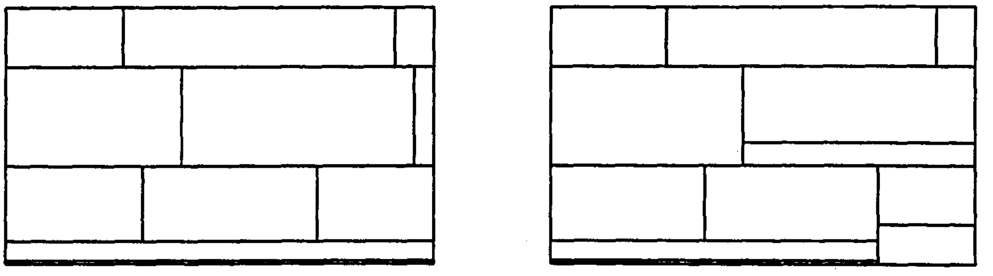
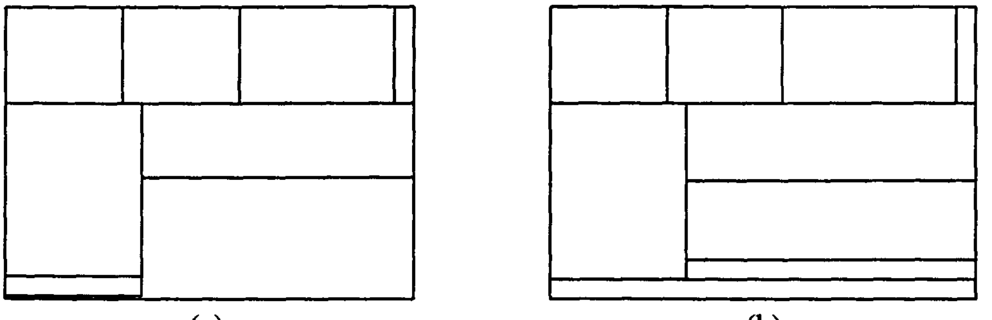


(a) (b)

Figure 1 – 2-stage cutting pattern: (a) exact case, (b) non exact case



(a) (b)

Figure 2 – (a) 3-stage cutting pattern, (b) Heuristic third stage

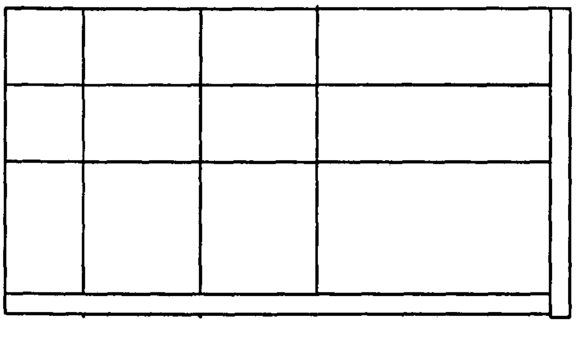


Figure 3 – 1-group cutting pattern

Table 1 – Data of the N=5 types of stock plates

Type j	L_j *(m)*	W_j *(m)*	D_j	C_j *($)*	C_j/L_jW_j *($/m²)*
1	1.220	2.750	975	3.3550	1.00
2	1.220	3.050	2872	3.7210	1.00
3	1.700	2.100	2305	3.5700	1.00
4	1.830	2.750	391	5.5917	1.11
5	2.130	2.440	3452	5.7747	1.11
Total			9995		

Table 2 – Data of the m=25 types of ordered items

Type k	l_k *(m)*	w_k *(m)*	d_k
1	0.454	2.130	4321
2	0.454	2.060	713
3	0.256	1.425	735
4	0.390	1.425	735
5	0.454	1.342	1447
6	0.454	0.636	1034
7	0.484	1.352	1050
8	0.666	1.440	2030
9	0.345	0.610	1384
10	0.351	1.187	4410
11	0.405	0.698	2940
12	0.341	0.780	5066
13	0.395	1.585	735
14	0.415	1.675	735
15	0.384	0.551	368
16	0.454	1.105	803
17	0.454	0.778	751
18	0.338	0.431	1280
19	0.454	1.578	1156
20	0.454	0.674	550
21	0.680	0.803	630
22	0.322	0.485	7308
23	0.322	1.445	5040
24	0.328	0.670	2376
25	0.205	1.368	630
Total			48227

Table 3 – Comparison between the minimum waste and minimum cost solutions

	Minimum waste solution	*Minimum cost solution*
Total waste (m^2)	501.4 (2.32%)	1100.6 (4.96%)
Total cost ($)	23652.84	22598.07
Number of plates cut	4413	5836
Area of plates cut (m^2)	21572.3	22173.3
Number of items produced	48228	48230
Area of items produced (m^2)	21071.0	21072.7
Number of patterns used	26	27
Number of iterations	218	184
Total runtime (min)	0.5	0.4

Table 4 – Utilization of stock plates in the minimum waste and minimum cost solutions

	Minimum waste solution		*Minimum cost solution*	
Type j	*Utilization*	*Cost ($)*	*Utilization*	*Cost ($)*
1	0	0	640	2147.20
2	0	0	2143	7974.10
3	798	2848.86	2305	8228.85
4	391	2186.35	391	2186.35
5	3224	18617.63	357	2061.57
Total	4413	23652.84	5836	22598.07

Table 5 – Comparison between the exact 2-stage and 3-stage minimum waste solutions

	Exact 2-stage solution	*3-stage solution*
Total waste (m^2)	827.3 (3.78%)	474.7 (2.20%)
Number of plates cut	4599	4379
Area of the plates cut (m^2)	21911.0	21545.8
Number of items produced	48253	48227
Area of the items produced (m^2)	21083.7	21071.2
Number of patterns used	26	26
Number of iterations	127	168
Total runtime (min)	0.1	0.6

Table 6 – Utilization of stock plates in the exact 2-stage and 3-stage solutions

Type j	*Exact 2-stage solution*	*3-stage solution*
1	53	0
2	314	165
3	839	556
4	391	391
5	3001	3267
Total	4599	4379

Table 7 – Data of the m=15 types of ordered items

Type k	l_k *(m)*	w_k *(m)*	d_k
1	0.274	0.609	630
2	0.274	0.380	1260
3	0.330	0.425	630
4	0.361	0.650	630
5	0.270	0.348	315
6	0.270	0.705	893
7	0.328	0.718	2520
8	0.300	0.705	90
9	0.330	0.465	4410
10	0.330	0.480	315
11	0.330	0.465	630
12	0.250	1.956	112
13	0.302	0.674	118
14	0.270	0.674	181
15	0.270	0.636	493
Total			13227

Table 8 – Solutions obtained for different values of the fixed cost of non 1-group patterns

Fixed cost ($)	*Total minimum cost ($)*	*Waste material (m^2)*	*Percentage Waste (%)*	*Number of plates cut*	*Proportion of non 1-group patterns (%)*
0.00	348.71	63.3	2.67	348.7	93.4
0.01	351.73	64.0	2.70	348.8	83.7
0.02	354.68	66.3	2.80	349.1	77.7
0.03	357.19	71.3	3.00	349.9	69.4
0.05	361.30	96.2	4.01	353.6	43.4
0.10	363.64	153.8	6.26	362.0	4.4
≥0.15	364.29	169.1	6.84	364.3	0.0

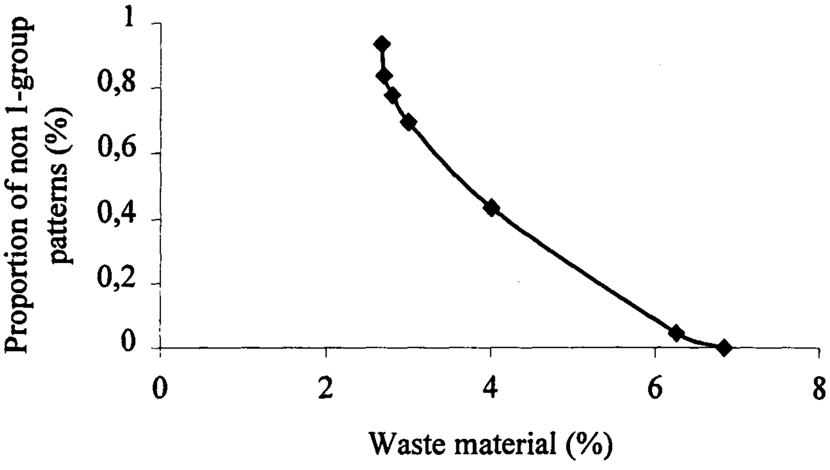


Figure 4 – Trade-off curve between waste material and proportion of non 1-group patterns

NOTAS DO ICMC

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041/98 SANTOS-MEZA, E.; SANTOS, M.O.; ARENALES, M.N. – Lot sizing and scheduling in na automated foundry.

040/98 FELTRIM, V.D.; FORTES, R.P.M. – Uma modelagem do domínio de engenharia reversa de software utilizando o método OOHDM.

039/98 FERREIRA, V.G.; MELLO, O .D.; OLIVEIRA, J.N.; FORTUNA, A . O . - Tópicos teóricos e computacionais em escoamentos de fluidos.

038/98 CAVICHIA, M.C.; ARENALES, M.N. - Piecewise linear programming via interior points.

037/98 REZENDE, S. O. ; PUGLIESI, J.B. - Aquisição de conhecimento explícito ou manual - versão 1.0

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