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Abstract

This work is concerned with the construction and analysis of convergence of high order product integration methods for Volterra integral equations with weakly singular kernels. Some numerical examples are presented which illustrate the theoretical estimates obtained.

Resumo

Este trabalho trata da construção e análise de convergência de métodos do tipo integração produto para equações integrais de Volterra com núcleo singular. São apresentados alguns exemplos numéricos que confirmam os resultados teóricos obtidos.

Keywords: Volterra Integral Equations, Product Integration Methods, Convergence.

Palavras-chave: Equações Integrais de Volterra, Métodos do tipo Integração Produto, Convergência.

1 Introduction

This paper will be concerned with high order product integration methods for the second kind Volterra equations of the types

$$y(t) + \int_0^t K(t,s)q(t,s)y(s)ds = f(t), \quad t \in [0,T], \tag{1.1}$$

with

$$q(t,s) := \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{[\ln(t/s)]}} \left(\frac{s}{t}\right)^{\mu} \frac{1}{s}, \tag{1.2}$$

and

$$y(t) - \int_0^t K(t,s)p(t,s)y(s)ds = g(t), \quad t \in [0,T], \tag{1.3}$$

with

$$p(t,s) := \left(\frac{s}{t}\right)^{\mu} \frac{1}{s}, \tag{1.4}$$

where $\mu > 0$, $K(t,s)$ is a smooth function and $f(t)$ and $g(t)$ are given functions. Particular cases of the above equations can arise from certain heat conduction problems. The special case of (1.1) when $K(t,s) = 1$ was originally due to Bartoshevich (1975) and has been studied theoretically by several authors. Sub-Sizonenko (1979) provided an analytic $\mathcal{L}_2$ solution and other solution expressions valid in weighted $\mathcal{L}_p$ spaces were derived by Rooney (1983) and Lamb (1980, 1985). Tang *et al* (1992) proved that if $\mu > 1$ then (1.1) has a unique solution $y \in C^m[0,T]$. Moreover the authors showed that, for an appropriate function g, (1.1) and (1.3) are equivalent (see Diogo *et al* (1991)). Han (1994) proved that in the case when $0 < \mu \leq 1$ equations (1.1) and (1.3) have a family of solutions in $C[0,T]$ of which only one has C^1 continuity. Further results in the space $C^m[0,T]$ were established by the author for the general equations (1.1) and (1.3), in the cases when $K(t,s) =$ constant and $K(t,s) = h(t)$.

Some attention has also been given to the numerical solution of (1.1) and (1.3). Unlike most weakly singular kernels, the functions q and p do not satisfy $\int_0^t q(t,s)ds \to 0$ as $t \to 0$ and $\int_0^t p(t,s)ds \to 0$ as $t \to 0$. In fact these integrals are constant and equal to $1/\sqrt{\mu}$ and $1/\mu$, respectively. This property creates some difficulties in the convergence analysis of discretisation methods for the above equations. Tang *et al* (1992) applied two low order product integration methods to equation (1.1). They obtained approximations to $y(t)$ of orders one and two with the Euler and Trapezoidal methods, respectively. In a further paper, Diogo *et al* (1991) considered a fourth order Hermite-type collocation method for the equivalent equation (1.3) which also involved the differentiated form of the equation. Recently, Lima & Diogo (1996) have developed an extrapolation method for (1.3), based on Euler's method. By introducing some appropriate function spaces they were also able to consider unbounded solutions. We note that although the above mentioned papers have dealt with the case $K(t,s) = 1$, the convergence results can be extended to the more general equations (1.1) and (1.3).

In the present work, we are concerned with constructing high order product integration methods for (1.3), under the assumption that the exact solution is sufficiently smooth. Related methods have been studied by several authors for equations with weakly singular kernels of the form $(t-s)^{-\alpha}, 0 < \alpha < 1$. Linz (1969) considered a method based on the product integration Simpson's rule and proved convergence of order three. By a sharper analysis de Hoog & Weiss (1973, 1974) were able to prove that, for $\alpha = 1/2$, the convergence order was in fact 7/2. Cameron & McKee (1984) extended the results of de Hoog & Weiss (1974) to higher order methods, for general values of α, $0 < \alpha < 1$.

For the sake of simplicity, we shall restrict ourselves here to the case when $K(t,s) = 1$ in (1.3). In Section 2 a class product integration methods based on interpolatory quadrature rules is introduced. A detailed convergence analysis is presented in Section 3 and a sufficient condition for convergence in terms of the weights is given. In Section 4 two particular methods are derived. Finally, the theoretical results are illustrated by some numerical examples.

We finish this Section with an existence and uniqueness result (see e.g. Diogo *et al* (1991)).

For a given non-negative integer m, let $V_m[0,T]$ denote the normed space of the real functions ϕ such that $\phi \in C^m[0,T]$ with

$$\|\phi\|_m := \max_{0 \le j \le m} \max_{t \in [0,T]} \left|\phi^{(j)}(t)\right|.$$

Theorem 1.1 Assume that $K(t,s) = 1$ in (1.1) and (1.3) and let $\mu > 1$. If the function f in (1.1) belongs to V_m then (1.1) possesses a unique solution $y \in V_m$. Similarly, if the function g in (1.3) belongs to V_m then (1.3) possesses a unique solution $y \in V_m$. Moreover, if g is given by

$$g(t) = f(t) - \int_0^t q(t,s) f(s) ds$$

then the solution of (1.1) is also the solution of (1.3).

2 High order product integration methods

In order to construct numerical methods for (1.3), let us define the grid

$$\{t_j = jh, \quad 0 \le j \le N; Nh = T\}$$

and consider the discretised form of (1.3)

$$y(t_i) - \int_0^{t_i} p(t_i, s) y(s) ds = g(t_i), \quad 0 \le i \le N. \tag{2.1}$$

To approximate the integral in (2.1), some $(n+1)$-point interpolatory formula is used repeatedly, followed by an end rule or a series of end rules when necessary. The coefficients of the resulting quadrature are calculated analytically.

Suppose first that i is a multiple of n, say $i = nr$. We can rewrite (2.1) as

$$y(t_i) - \sum_{j=0}^{r-1} \int_{t_{nj}}^{t_{n(j+1)}} \left(\frac{s}{t_i}\right)^{\mu} \frac{1}{s} y(s)\, ds = g(t_i). \tag{2.2}$$

Approximating $y(s), s \in (t_j, t_{j+n})$ by a polynomial of degree n, yields

$$\int_{t_j}^{t_{j+n}} \left(\frac{s}{t_i}\right)^{\mu} \frac{1}{s}\, y(s) ds \simeq \sum_{k=0}^{n} y(t_{j+k}) \int_{t_j}^{t_{j+n}} \left(\frac{s}{t_i}\right)^{\mu} \frac{1}{s} l_k(s) ds, \tag{2.3}$$

where the l_k are the Lagrange polynomials of degree n associated with $t_j, t_{j+1}, \cdots, t_{j+n}$, that is

$$l_k(s) = \prod_{\substack{i=0\\ i\ne k}}^{n} \frac{s - t_{j+i}}{t_{j+k} - t_{j+i}} \quad 0 \le k \le n.$$

By making the transformation $s = t_j + vh$, we have

$$\begin{aligned} \int_{t_j}^{t_{j+n}} \left(\frac{s}{t_i}\right)^{\mu} \frac{1}{s} l_k(s) ds &= h \int_0^n \frac{(t_j + vh)^{\mu-1}}{(t_i)^{\mu}} l_k(t_j + vh) dv \\ &= \frac{1}{i^{\mu}} \int_0^n (j+v)^{\mu-1} \rho_k(v) dv, \end{aligned} \tag{2.4}$$

where ρ_k is a polynomial of degree n, given by

$$\rho_k(v) := l_k(t_j + vh) = \prod_{\substack{i=0\\ i\ne k}}^{n} \frac{v - i}{k - i}. \tag{2.5}$$

Let us define

$$b_\gamma(j) := \int_0^n (j+v)^{\mu-1}\rho_\gamma(v)dv, \quad 0 \le \gamma \le n. \tag{2.6}$$

Substituting (2.4) into (2.3) and using (2.6), we obtain

$$\int_{t_j}^{t_{j+n}} \left(\frac{s}{t_i}\right)^\mu \frac{1}{s} y(s)ds \simeq \frac{1}{i^\mu}\sum_{k=0}^{n} y(t_{j+k})b_k(j). \tag{2.7}$$

Combining (2.7) and (2.2), we obtain for the case $i = nr$ the following approximate equation

$$y_{nr} - \frac{1}{(nr)^\mu}\sum_{j=0}^{r-1}\sum_{k=0}^{n} y_{nj+k}\, b_k(nj) = g(t_{nr}), \tag{2.8}$$

where the y_i are approximate values of $y(t_i)$.

When i is not a multiple of n, say $i = nr + \nu$, $\nu = 1, \cdots, n-1$, it is convenient to rewrite (2.1) in the form

$$y(t_i) - \int_0^{t_{i-n-\nu}} p(t_i, s)y(s)ds - \int_{t_{i-n-\nu}}^{t_i} p(t_i, s)y(s)ds = g(t_i). \tag{2.9}$$

The integral over $[0, t_{i-n-\nu}]$ is approximated by the main repeated formula. In order to approximate the integral over $[t_{i-n-\nu}, t_i]$, end rules based on polynomial interpolation of degrees $(n+1), (n+2), \cdots, (2n-1)$ are used. Thus

$$\int_{t_{i-n-\nu}}^{t_i} p(t_i, s)y(s)ds \simeq \sum_{j=0}^{n+\nu} y(t_{i-n-\nu+j}) \int_{t_{i-n-\nu}}^{t_i} p(t_i, s)\tilde{l}_j(s)ds, \tag{2.10}$$

where the $\tilde{l}_j$ are the Lagrange polynomials of degree $n+\nu$ associated with the points $t_{i-n-\nu+j}$, $0 \le j \le n+\nu$. Making $s = t_{i-n-\nu} + vh$, we obtain

$$\int_{t_{i-n-\nu}}^{t_i} \left(\frac{s}{t_i}\right)^\mu \frac{1}{s}\tilde{l}_j(s)ds = \frac{1}{i^\mu}\int_0^{n+\nu} (i-n-\nu+v)^{\mu-1}\tilde{\rho}_j(v)dv, \tag{2.11}$$

where

$$\tilde{\rho}_j(v) := \tilde{l}_j(t_{i-n-\nu} + vh) = \prod_{\substack{k=0 \\ k\ne j}}^{n+\nu} \frac{v-k}{j-k}, \quad 0 \le j \le n+\nu.$$

Let us define

$$d_k^\nu(i) := \int_0^{n+\nu} (i-n-\nu+v)^{\mu-1}\tilde{\rho}_k(v)dv, \quad 0 \le k \le n+\nu. \tag{2.12}$$

Combining (2.10) and (2.11) and using (2.12) results in

$$\int_{t_{i-n-\nu}}^{t_i} p(t_i, s)y(s)ds \simeq \frac{1}{i^\mu} \sum_{k=0}^{n+\nu} y(t_{i-n-\nu+k})\ d_k^\nu(i). \tag{2.13}$$

Substituting (2.13) into (2.9) and treating the first integral in (2.9) as the one in (2.2), gives the following approximate equation for the case when $i = nr + \nu$

$$y_{nr+\nu} - \frac{1}{(nr+\nu)^\mu} \cdot \left(\sum_{j=0}^{r-2} \sum_{k=0}^{n} y_{nj+k}\ b_k(nj) + \sum_{k=0}^{n+\nu} y_{nr-n+k}\ d_k^\nu(nr+\nu) \right) = g(t_{nr+\nu}). \tag{2.14}$$

Combining (2.8) and (2.14), we obtain the discretisation method

$$\mathbf{\Phi}_h \mathbf{y} = \mathbf{0}$$

with

$$[\mathbf{\Phi}_h \mathbf{y}]_i = \begin{cases} y_i - \tilde{y}_i, & 0 \leq i \leq n-1, \\ y_i - \sum_{j=0}^{i} \omega_{ij} y_j - g_i, & n \leq i \leq N, \end{cases} \tag{2.15}$$

where $\tilde{y}_i,\ 0 \leq i \leq n-1$, are given starting values. We can write (2.15) in matrix form as

$$\mathbf{\Phi}_h \mathbf{y} = (\mathbf{I} - \mathbf{A}_N)\mathbf{y} - \mathbf{g}, \tag{2.16}$$

with

$$\mathbf{y} = (y_0, y_1, \ldots, y_N)^T \text{and } \mathbf{g} = (\tilde{y}_0, \ldots, \tilde{y}_{n-1}, g_n, \cdots, g_N)^T.$$

Here $\mathbf{A}_N = \tilde{\mathbf{A}}_N\ \mathbf{D}_N$,with

$$\mathbf{D}_N = diag\{0, \ldots, 0, \frac{1}{n^\mu}, \frac{1}{(n+1)^\mu}, \ldots, \frac{1}{N^\mu}\},$$

the elements of $\tilde{\mathbf{A}}_N$ being integrals (or sums of integrals) of the types (2.6) and (2.12). We note that, since $\mu > 1$, all the integrals involved can be calculated analytically.

3 Convergence

In the sequel we shall assume that the solution y of (1.3) is sufficiently smooth. Let $\delta(h, t_i)$ denote the local consistency error for (1.3), defined by

$$\delta(h, t_i) := \int_0^{t_i} \left(\frac{s}{t_i}\right)^\mu \frac{1}{s} y(s)ds - \sum_{j=0}^{i} \omega_{ij} y(t_j). \tag{3.1}$$

We require the following definition of consistency.

Definition 3.1 The discretisation (2.15) is said to be consistent of order p with (1.3) at $t = t_i, i \geq n$, if there exists a constant C, independent of h, such that for $h \in (0, h_0), h_0 > 0$, we have

$$|\delta(h, t_i)| \leq Ch^p. \tag{3.2}$$

In order to investigate the consistency order of the discretisation (2.15), let us first suppose that i is a multiple of n. It follows, with $i = nr$, that

$$\begin{aligned} \delta(h, t_i) &= \int_0^{t_i} \left(\frac{s}{t_i}\right)^{\mu} \frac{1}{s} y(s) ds - \sum_{j=0}^{r-1} \int_{t_{nj}}^{t_{n(j+1)}} \left(\frac{s}{t_i}\right)^{\mu} \frac{1}{s} \left(\sum_{k=0}^{n} l_k(s) y(t_{nj+k})\right) ds \\ &= \sum_{j=0}^{r-1} \int_{t_{nj}}^{t_{n(j+1)}} \left(\frac{s}{t_i}\right)^{\mu} \frac{1}{s} \left(y(s) - \sum_{k=0}^{n} l_k(s) y(t_{nj+k})\right) ds. \end{aligned} \tag{3.3}$$

From the error formula for Lagrange interpolation

$$y(s) - \sum_{k=0}^{n} l_k(s) y(t_{nj+k}) = \frac{1}{(n+1)!} y^{(n+1)}(t_{nj} + n\theta_j h) \tau(s), \tag{3.4}$$

for some $\theta_j \in (0, 1)$ and where $\tau(s) := \prod_{k=0}^{n}(s - t_{nj+k})$. Using (3.4) and making the transformation $s = t_{nj} + nvh$, we obtain from (3.3) that

$$\begin{aligned} \delta(h, t_i) &= \frac{1}{(n+1)!} \sum_{j=0}^{r-1} nh \int_0^1 \frac{(t_{nj} + nvh)^{\mu-1}}{t_i{}^{\mu}} (y^{(n+1)}(t_{nj} + nvh) + O(h)) h^{n+1} \tau^*(nv) dv, \\ &= \frac{h^{n+1}}{(n+1)!} \int_0^1 \tau^*(nv)\; nh \sum_{j=0}^{r-1} \frac{(t_{nj} + nvh)^{\mu-1}}{t_i{}^{\mu}} y^{(n+1)}(t_{nj} + nvh) dv + O(h^{n+2}), \end{aligned} \tag{3.5}$$

where $\tau^*(s) := \prod_{l=0}^{n}(s - l)$.

If i is of the form $i = nr + \nu,\ 1 \leq \nu \leq n - 1$, then an end rule of degree $n + \nu$ will be used over $[t_{i-n-\nu}, t_i]$. The error associated with this rule is

$$E_{n+\nu}(h, t_i) = \frac{1}{(n + \nu + 1)!} \int_{t_{i-n-\nu}}^{t_i} \left(\frac{s}{t_i}\right)^{\mu} \frac{1}{s} y^{(n+\nu+1)}(\theta_i) \lambda(s) ds, \tag{3.6}$$

with $\lambda(s) := \prod_{k=0}^{n+\nu}(s - t_{i-n-\nu+k})$ and where $\theta_i \in (t_{i-n-\nu}, t_i)$. Making the transformation $s = t_{i-n-\nu} + vh$, gives

$$E_{n+\nu}(h, t_i) = \frac{h^{n+\nu+1}}{(n + \nu + 1)!} \int_0^{n+\nu} \frac{(i - n - \nu + v)^{\mu-1}}{i^{\mu}} y^{(n+\nu+1)}(\theta_i) \lambda^*(v) dv, \tag{3.7}$$

with $\lambda^*(v) = \prod_{k=0}^{n+\nu}(v-k)$. Then $E_{n+\nu}(h, t_i) = O(h^{n+\nu+1})$, which will clearly not affect the global consistency error. We see that when $i = nr+\nu$, the order will be determined by (3.5), with i replaced by $nr+\nu$.

In order to obtain the correct order of consistency, we need to analyse the sum

$$nh \sum_{j=0}^{r-1} \frac{(t_{nj}+nvh)^{\mu-1}}{t_i^{\mu}} y^{(n+1)}(t_{nj}+nvh). \tag{3.8}$$

Two cases will be considered, depending on the smoothness of the functions involved. The following lemmas will be the basis of the results of this Section.

Lemma 3.1

If $\psi(t) := t^{\mu-1}y^{(n+1)}(t) \in C^1[0,T]$ then, for each $s \in [0,1]$,

$$h \sum_{l=0}^{r-1} (t_l+sh)^{\mu-1} y^{(n+1)}(t_l+sh) = \int_0^{t_r} t^{\mu-1} y^{(n+1)}(t)dt + (t_r)^{\mu}\, O\left(\frac{h}{t_r}\right). \tag{3.9}$$

Proof

Defining $\phi(u) := y(u\, t_r)/t_r^{n+1}$, we may write

$$\int_0^{t_r} t^{\mu-1} y^{(n+1)}(t)dt = t_r{}^{\mu} \int_0^1 u^{\mu-1}\phi^{(n+1)}(u)du. \tag{3.10}$$

We now apply the Euler-MacLaurin summation formula to the integral on the right hand side of (3.10) (see e.g. de Hoog & Weiss (1973)). Letting $\bar{h}_r := h/t_r = 1/r$, then we have, for each $s \in [0,1]$,

$$\int_0^{t_r} t^{\mu-1} y^{(n+1)}(t)dt = t_r{}^{\mu}\left[\bar{h}_r \sum_{l=0}^{r-1}((l+s)\bar{h}_r)^{\mu-1}\phi^{(n+1)}((l+s)\bar{h}_r) + O(\bar{h}_r)\right]. \tag{3.11}$$

Expressing (3.11) in terms of h gives (3.9). □

When $i = nr$ we use (3.9) in (3.5), with h replaced by nh, to give

$$\delta(h, t_i) = \frac{h^{n+1}}{(n+1)!}\int_0^1 \tau^*(nv)\left[\int_0^{t_i} \frac{t^{\mu-1}}{t_i^{\mu}} y^{(n+1)}(t)dt + O\left(\frac{h}{t_i}\right)\right] dv + O(h^{n+2}). \tag{3.12}$$

We see that when n is even, that is, with rules verifying $\int_0^1 \tau^*(nv)dv = 0$, then $\delta(h, t_i) = O(h^{n+2})$. If $i = nr+\nu$, $1 \le \nu \le n-1$, we can still use lemma 3.1, giving again an error of order $O(h^{n+2})$ when n is even. Thus, methods based on $(n+1)$-point interpolatory formulas are consistent of order $n+2$ if n is even, while if n is odd, consistency of order $n+1$ follows.

Lemma 3.2

If $y \in C^{n+1}[0,T]$ and $\mu > 1$ is not an integer then, for each $s \in [0,1]$,

$$h\sum_{l=0}^{r-1}(t_l+sh)^{\mu-1}y^{(n+1)}(t_l+sh) = \int_0^{t_r} t^{\mu-1}y^{(n+1)}(t)dt + h^\mu\gamma(1-\mu,s)y^{(n+1)}(0) + t_r{}^\mu\left(\frac{h}{t_r}\right)\gamma(0,1-s)y^{(n+1)}(t_r) + t_r{}^\mu O\left(\frac{h}{t_r}\right). \quad (3.13)$$

Proof

Defining $g(u) := y^{(n+1)}(u\, t_r)$, we may write

$$\int_0^{t_r} t^{\mu-1}y^{(n+1)}(t)dt = t_r{}^\mu \int_0^1 u^{\mu-1}g(u)du. \quad (3.14)$$

We now make use of the generalized Euler-Maclaurin summation formula for functions with algebraic singularities given by Lyness & Ninham (1967). Letting $\bar{h}_r := h/t_r = 1/r$, we have, for each $s \in [0,1]$,

$$\int_0^{t_r} t^{\mu-1}y^{(n+1)}(t)dt = t_r{}^\mu\Big[\bar{h}_r\sum_{l=0}^{r-1}((l+s)\bar{h}_r)^{\mu-1}\, g((l+s)\bar{h}_r) - \bar{h}_r^\mu\gamma(1-\mu,s)g(0) + \bar{h}_r\gamma(0,1-s)g(1) + O(\bar{h}_r)\Big]. \quad (3.15)$$

Here $\gamma(a,s)$ is the periodic generalized zeta function defined by

$$\gamma(a,s) := \zeta(a,\bar{s}), \quad s-\bar{s} = \text{integer}, \quad 0 < \bar{s} \le 1$$

where $\zeta(a,s)$ is the generalized Riemann zeta function

$$\zeta(a,s) = \sum_{k=0}^{\infty}\frac{1}{(s+k)^a},$$

$$\zeta(-a,s) = \frac{2a!}{(2\pi)^{a+1}}\sum_{k=1}^{\infty}\frac{\sin[2\pi sk - \pi a/2]}{k^{a+1}}, \quad \text{Re } a > 0.$$

From (3.15), the desired result follows. □

When $i = nr$, we use (3.13) in (3.5) which gives

$$\delta(h,t_i) = \frac{h^{n+1}}{(n+1)!}\int_0^1 \tau^*(nv)\Big[\int_0^{t_i}\frac{t^{\mu-1}}{t_i^\mu}y^{(n+1)}(t)dt + \left(\frac{h}{t_i}\right)^\mu\gamma(1-\mu,v)y^{(n+1)}(0) + \left(\frac{h}{t_i}\right)\gamma(0,1-v)y^{(n+1)}(t_i) + O\left(\frac{h}{t_i}\right)\Big]dv + O(h^{n+2}). \quad (3.16)$$

From the above equation we conclude that methods based on $(n+1)$-point interpolatory rules are consistent of order $n+1$. However, the numerical results seem to indicate that, when n is even and $1<\mu<2$, one may get order $n+\mu$ for some particular equations (cf. Section 5).

In order to prove convergence we require the following lemma from Linz (1969).

Lemma 3.3 Let $\{\epsilon_i\}_{i=0}^{\infty}$ be real numbers verifying

$$|\epsilon_i| \leq \sum_{j=0}^{i} |\alpha_{ij}||\epsilon_j| + E, \quad E > 0, \tag{3.17}$$

with $|\epsilon_i| \leq \eta, \quad 0 \leq i \leq s-1$, and

$$\sum_{j=0}^{i-1} |\alpha_{ij}| \leq \alpha < 1, \quad i = 1, 2, \ldots \tag{3.18}$$

then

$$|\epsilon_i| \leq \frac{E + \alpha\eta}{1-\alpha}, \quad i = s, s+1, \ldots \tag{3.19}$$

A further definition will be needed.

Definition 3.2 The starting values are said to be convergent of order p if there exists a constant C_3, independent of h, such that

$$|y(t_i) - y_i| \leq C_3 h^p, \quad 0 \leq i \leq n-1. \tag{3.20}$$

We now state the following convergence result.

Theorem 3.1 Let $y \in C^{2n}[0,T]$ and let the weights ω_{ij} of the discretisation (2.15) satisfy the condition

$$\sum_{j=0}^{i-1} \frac{|\omega_{ij}|}{1-|\omega_{ii}|} \leq \alpha < 1, \quad i = 1, 2, \ldots. \tag{3.21}$$

If the discretisation (2.15) is consistent of order p ($p \leq n+2$) and the starting values are convergent of order p, then it is convergent of order p.

Proof:

Let us define $e_i := y(t_i) - y_i$. We have

$$\begin{aligned} e_i &= \int_0^{t_i} p(t_i,s)y(s)ds - \sum_{j=0}^{i} \omega_{ij} y_j \\ &= \sum_{j=0}^{i} \omega_{ij}(y(t_j) - y_j) + \int_0^{t_i} p(t_i,s)y(s)ds - \sum_{j=0}^{i} \omega_{ij} y(t_j), \end{aligned} \tag{3.22}$$

which gives, after using (3.1) and taking modulus,

$$|e_i|\,(1-|\omega_{ii}|) \le \sum_{j=0}^{i-1} ||\omega_{ij}||e_j| + |\delta(h,t_i)|. \tag{3.23}$$

Let us now prove that

$$|\omega_{ii}| \le 1/\mu. \tag{3.24}$$

If i is a multiple of n, say, $i = nr$, then from (2.8) and (2.6)

$$\omega_{ii} = \frac{b_n(nr)}{(nr)^\mu} = \int_0^n \frac{(nr-n+v)^{\mu-1}}{(nr)^\mu}\rho_n(v)dv, \tag{3.25}$$

which gives

$$|\omega_{ii}| \le \int_0^n \frac{(nr-n+v)^{\mu-1}}{(nr)^\mu}dv = \frac{r^\mu-(r-1)^\mu}{\mu r^\mu} \le \frac{1}{\mu}. \tag{3.26}$$

If $i = nr+\nu$, $\nu = 1,2,\dots,n-1$, then from (2.14) and (2.12)

$$\omega_{ii} = \frac{d^\nu_{n+\nu}}{(nr+\nu)^\mu} = \int_0^{n+\nu}(nr-n+v)^{\mu-1}\tilde{\rho}_{n+\nu}(v)dv \tag{3.27}$$

and so

$$|\omega_{ii}| \le \int_0^{n+\nu}\frac{(nr-n+v)^{\mu-1}}{(nr+\nu)^\mu}dv = \frac{(nr+\nu)^\mu-(nr-n)^\mu}{\mu(nr+\nu)^\mu} \le \frac{1}{\mu}. \tag{3.28}$$

Then from (3.2) and (3.23) we obtain, for some constant C independent of h,

$$|e_i| \le \sum_{j=0}^{i-1}\frac{|\omega_{ij}|}{1-|\omega_{ii}|}\,|e_j| + Ch^p. \tag{3.29}$$

Finally, an application of lemma 3.3 yields the desired result. □

Remark 3.1

As we see from Section 2, the weights ω_{ij} are a linear combination of a finite number of moment integrals of the kernel. For weakly singular kernels of the form $(t-s)^{-\alpha}$, $0<\alpha<1$, each of these integrals can be made arbitrarily small by taking h sufficiently small. Thus a condition on the weights of the type of (3.21) follows naturally from the nature of the kernel itself and is easily proved.

In the case of equation (1.3), it is not possible to use the same kind of arguments. We also note that while for $n=1$ it is quite straightforward to prove condition (3.21), for $n>1$ is a much harder task. In fact, a theoretical proof of (3.21) for a general $n>1$ has not been available yet. However, the numerical results for the cases $n=2,3$ and 4 suggest that the above condition is indeed true. Moreover, a relation like (3.21), with $\alpha = 1/\mu$, seems to hold for $i \ge i_0$, with $i_0 \ge 1$.

4 Two product integration methods

The product trapezoidal method

In the case when $n = 1$ the integral in (2.1) is approximated by the product trapezoidal rule applied repeatedly over $[0, t_i]$. We then have the following approximate equation (cf. (2.8))

$$y_i - \frac{1}{i^\mu}\sum_{j=0}^{i-1}(y_j\ b_0(j) + y_{j+1}\ b_1(j)) = g(t_i), \quad 0 \le i \le N, \tag{4.1}$$

with

$$b_0(j) = \int_0^1 (j+v)^{\mu-1}(1-v)dv, \tag{4.2}$$

$$b_1(j) = \int_0^1 (j+v)^{\mu-1}v dv. \tag{4.3}$$

The matrix $\mathbf{A}_N$ in (2.16) will be such that $\mathbf{A}_N = \tilde{\mathbf{A}}_N\ \mathbf{D}_N$, with

$$\mathbf{D}_N = diag\{0, 1, \frac{1}{2^\mu}, \frac{1}{3^\mu}, \frac{1}{4^\mu}, \ldots, \frac{1}{N^\mu}\}$$

and

$$\tilde{\mathbf{A}}_N = \begin{pmatrix} 0 & & & & & \\ b_0(0) & b_1(0) & & & & \\ b_0(0) & b_1(0)+b_0(1) & b_1(1) & & & \\ b_0(0) & b_1(0)+b_0(1) & b_1(1)+b_0(2) & b_1(2) & & \\ b_0(0) & b_1(0)+b_0(1) & b_1(1)+b_0(2) & b_1(2)+b_0(3) & b_1(3) & \\ & & & & & \\ \vdots & & & & & \ddots \end{pmatrix}.$$

In order to prove (3.21), we first note that the weights are positive. Then from (4.1) we obtain

$$\begin{aligned} \sum_{j=0}^{i-1}\omega_{ij} &= \frac{1}{i^\mu}\left[\sum_{j=0}^{i-2}(b_0(j)+b_1(j)) + b_0(i-1)\right] \\ &= \frac{i^{\mu+1}-(i-1)^{\mu+1}}{\mu(\mu+1)i^\mu}. \end{aligned} \tag{4.4}$$

Further from (4.1) and (4.3) we see that

$$\begin{aligned} w_{ii} &= \frac{1}{i^\mu}\int_0^1 (i-1+v)^{\mu-1}v dv \\ &= \frac{1}{\mu} - \frac{1}{\mu(\mu+1)}\frac{i^{\mu+1}-(i-1)^{\mu+1}}{i^\mu}. \end{aligned} \tag{4.5}$$

Finally, using (4.4) and (4.5) we arrive at

$$\sum_{j=0}^{i-1} \frac{\omega_{ij}}{1-\omega_{ii}} = \frac{i^{\mu+1}-(i-1)^{\mu+1}}{i^{\mu+1}-(i-1)^{\mu+1}+(\mu^2-1)\, i^{\mu}}. \tag{4.6}$$

This can be easily seen to be bounded by $1/\mu$, for $i \geq 1$. Therefore (3.21) is satisfied with $\alpha = 1/\mu$.

The product Simpson's method

We consider the case when $n = 2$. To approximate the integral in (2.1), the product Simpson's rule is used repeatedly over $[0, t_i]$ if i is even. When i is odd, we use the product Simpson's rule over $[0, t_{i-3}]$ and the product three-eights rule will be used over $[t_{i-3}, t_i]$. In this case, (2.8) and (2.14) take the form, respectively,

$$y_{2r} - \frac{1}{(2r)^{\mu}} \sum_{j=0}^{r-1} \sum_{k=0}^{2} y_{2j+k}\, b_k(2j) = g(t_{2r}) \tag{4.7}$$

and

$$y_{2r+1} - \frac{1}{(2r+1)^{\mu}} \left(\sum_{j=0}^{r-2} \sum_{k=0}^{2} y_{2j+k}\, b_k(2j) + \sum_{k=0}^{3} y_{2r-2+k}\, d_k(2r+1) \right) = g(t_{2r+1}), \tag{4.8}$$

with

$$\begin{aligned} b_0(j) &= \frac{1}{2} \int_0^2 (v-1)(v-2)(j+v)^{\mu-1} dv, \\ b_1(j) &= -\int_0^2 v(v-2)(j+v)^{\mu-1} dv, \\ b_2(j) &= \frac{1}{2} \int_0^2 v(v-1)(j+v)^{\mu-1} dv, \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} d_0(i) &= -\frac{1}{6} \int_0^3 (v+i-3)^{\mu-1}(v-1)(v-2)(v-3) dv, \\ d_1(i) &= \frac{1}{2} \int_0^3 (v+i-3)^{\mu-1} v(v-2)(v-3) dv, \\ d_2(i) &= -\frac{1}{2} \int_0^3 (v+i-3)^{\mu-1} v(v-1)(v-3) dv, \\ d_3(i) &= \frac{1}{6} \int_0^3 (v+i-3)^{\mu-1} v(v-1)(v-2) dv. \end{aligned} \tag{4.10}$$

Here the matrix $\tilde{\mathbf{A}}_N$ has the form

$$\tilde{\mathbf{A}}_N = \begin{pmatrix} 0 \\ 0 & 0 \\ b_0(0) & b_1(0) & b_2(0) \\ d_0(3) & d_1(3) & d_2(3) & d_3(3) \\ b_0(0) & b_1(0) & b_2(0)+b_0(2) & b_1(2) & b_2(2) \\ b_0(0) & b_1(0) & b_2(0)+d_0(5) & d_1(5) & d_2(5) & d_3(5) \\ b_0(0) & b_1(0) & b_2(0)+b_0(2) & b_1(2) & b_2(2)+b_0(4) & b_1(4) & b_2(4) \\ b_0(0) & b_1(0) & b_2(0)+b_0(2) & b_1(2) & b_2(2)+d_0(7) & d_1(7) & d_2(7) & d_3(7) \\ \vdots & & & & & & & & \ddots \end{pmatrix}.$$

We note here that, unlike in the Trapezoidal method, the weights ω_{ij} may not be positive for all i, j and this makes the proof of (3.21) harder. However, as pointed out in remark 3.1, the numerical results support the conjecture that the above condition is valid with $\alpha = 1/\mu$ and $i \geq i_0$, for some integer i_0.

5 Numerical results

In order to illustrate the theoretical results of Section 5 we have considered the numerical solution of equation (1.3), with $g(t) = t^\alpha\ (\mu - 1)/\mu$ and exact solution $y(t) = t^\alpha$, $t \in [0, 2]$. The following choices of α and μ have been considered.

Example 5.1: $\mu = 1.5$ and $\alpha = 6.5$;

Example 5.2: $\mu = 1.3$ and $\alpha = 3$;

Example 5.3: $\mu = 1.5$ and $\alpha = 8.5$;

Example 5.4: $\mu = 1.3$ and $\alpha = 5$.

In Tables 5.1–5.6 the errors $|e_i| = |y(t_i) - y_i|$ produced with several numerical methods of the type (2.15) are displayed for the cases $n = 1, 2, 3, 4$. Table 5.1 shows the results obtained with the product trapezoidal method applied to example 5.1. Here lemma 3.1 can be used with $n = 1$ and $\psi(t) = \text{constant} \times t^4$, giving consistency of order two. Then theorem 3.1 gives second order of convergence and this is confirmed by the numerical results.

Table 5.2 contains the errors obtained with the product Simpson's method for example 5.1. In this case lemma 3.1 can be applied with $n = 2$ and $\psi(t) = \text{constant} \times t^3$ and from (3.12) we get consistency of order four. The results indicate that the convergence order is four and this is in agreement with the theoretical prediction (cf. remark 3.1). In order to illustrate an application of lemma 3.2, we have applied the product Simpson's method to example 5.2. Here lemma 3.1 is no longer applicable as $\psi(t) = \text{constant} \times t^{0.3}$ is not in $C^1[0,2]$. By using lemma 3.2 and taking into account remark 3.1, one would expect to get convergence of order three. Here the numerical results seem to indicate that the convergence order might be slightly higher, maybe 3.3. We note that this is just $2 + \mu$. Further numerical results for other values of μ, $1 < \mu < 2$, seem to support this conjecture.

To obtain the values in Table 5.4, we implemented a method based on the repeated use of the product three-eights rule (as main rule) which was applied to example 5.1. Here lemma 3.1 can be applied with $n = 3$ and $\psi(t) = \text{constant} \times t^4$. The results are in agreement with the theoretical prediction, showing convergence of order four.

A further method corresponding to $n = 4$ was also implemented and applied to examples 5.3 (Table 5.5) and 5.4 (Table 5.6). With example 5.3 we can use lemma 3.1 (here $\psi(t) = \text{constant} \times t^4$), getting consistency of order six. However in the case of example 5.4 lemma 3.1 is no longer applicable, as $\psi(t) = t^{0.3}$ is not in $C^1[0,2]$. Here the use of lemma 3.2 gives consistency of order five. As in the case of Simpson's method applied to example 5.2, the numerical results suggest that convergence order could be a bit higher, say 5.3. In general we conjecture that, for n even, some particular cases (like when $y^{(n+1)}(t) = \text{constant}$) may exhibit convergence order $n + \mu$ if $1 < \mu < 2$, instead of the predicted order $n + 1$ of lemma 3.2. A more detailed analysis would be required to determine the exact order of consistency.

For the sake of comparison we have considered the solution of (1.3) by spline collocation and iterated collocation methods. A comprehensive study of collocation methods for Volterra equations can be found in Brunner & van der Houwen (1984). For the use of iterated collocation methods we refer to Brunner (1984). Tables 5.7 and 5.8 illustrate the performance of two spline collocation methods corresponding to $m = 2$ and $m = 3$, respectively, applied to example 5.1. A further column in these Tables shows the iterated collocation error at $T = 2$. It seems that the collocation results are of similar quality as the ones produced with product integration methods of the same order. In particular, a considerable increase in accuracy can be achieved with iterated collocation. The theoretical study of these methods will be pursued elsewhere.

Table 5.1: *The trapezoidal method*
example 5.1

t_i	0.5	1.0	2.0	rate
$h = 0.05$	$6.6D-5$	$1.5D-3$	$3.4D-2$	
$h = 0.025$	$1.6D-5$	$3.7D-4$	$8.4D-3$	2.0
$h = 0.0125$	$4.1D-6$	$9.3D-5$	$2.1D-3$	2.0
$h = 0.00625$	$1.0D-6$	$2.3D-5$	$5.3D-4$	2.0

Table 5.2: *Simpson's method*
example 5.1

t_i	0.5	1.0	2.0	rate
$h = 0.05$	$2.6D-6$	$1.7D-5$	$1.0D-4$	
$h = 0.025$	$1.9D-7$	$1.1D-6$	$6.7D-6$	3.94
$h = 0.0125$	$1.3D-8$	$7.4D-8$	$4.2D-7$	3.97
$h = 0.00625$	$8.1D-10$	$4.7D-9$	$2.7D-8$	3.99

Table 5.3: *Simpson's method*
example 5.2

t_i	0.5	1.0	2.0	rate
$h = 0.05$	$1.0D-5$	$9.2D-6$	$8.1D-6$	
$h = 0.025$	$1.2D-6$	$1.0D-6$	$8.5D-7$	3.24
$h = 0.0125$	$1.3D-7$	$1.1D-7$	$8.9D-8$	3.27
$h = 0.00625$	$1.3D-8$	$1.1D-8$	$9.2D-9$	3.28

Table 5.4: *A method based on the three-eights rule*
example 5.1

t_i	0.5	1.0	2.0	rate
$h = 0.05$	$5.8D-7$	$5.1D-6$	$5.4D-5$	
$h = 0.025$	$5.6D-8$	$6.0D-7$	$4.0D-6$	3.75
$h = 0.0125$	$6.6D-9$	$4.4D-8$	$2.9D-7$	3.78
$h = 0.00625$	$4.9D-10$	$3.2D-9$	$1.9D-8$	3.94

Table 5.5: *A method based on a four- point rule*
example 5.3

t_i	0.5	1.0	2.0	rate
$h = 0.05$	$2.6D-8$	$6.4D-7$	$3.8D-6$	
$h = 0.025$	$1.8D-9$	$1.1D-8$	$6.2D-8$	5.90
$h = 0.0125$	$2.9D-11$	$1.7D-10$	$9.9D-10$	5.96
$h = 0.00625$	$4.7D-13$	$2.7D-12$	$1.5D-11$	6.00

Table 5.6: *A method based on a four-point rule*
example 5.4

t_i	0.5	1.0	2.0	rate
$h = 0.05$	$4.1D-8$	$4.7D-8$	$4.0D-8$	
$h = 0.025$	$1.5D-9$	$1.3D-9$	$1.1D-9$	5.20
$h = 0.0125$	$4.0D-11$	$3.3D-11$	$2.8D-11$	5.20
$h = 0.00625$	$1.0D-12$	$8.7D-13$	$7.5D-13$	5.23

Table 5.7: *A third order collocation method* (CO2) *and iterated collocation* (IT2) *example 5.1*

	CO2			IT2
t_i	0.5	1.0	2.0	2.0
h= 0.05	3.3D-5	5.1D-4	6.6D-3	2.1D-4
h= 0.025	5.6D-6	7.3D-5	8.8D-4	2.6D-5
h= 0.0125	8.0D-7	9.7D-6	1.1D-4	3.3D-6

Table 5.8: *A fourth order collocation method* (CO3) *and iterated collocation* (IT3) *example 5.1*

	CO3			IT3
t_i	0.5	1.0	2.0	2.0
h=0.05	9.3D-7	5.6D-6	3.2D-5	1.4D-6
h=0.025	6.2D-8	3.6D-7	2.1D-6	8.8D-8
h= 0.0125	4.0D-9	2.3D-8	1.3D-7	5.5D-9

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