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RELATIVE WHITNEY GERMS

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Let $f : (R^2,0) \to (R^2,0)$ a C^∞ germ. We denote by S the Y-axis and by T the X-axis. If $f = (f_1,f_2)$ we shall suppose that

(1) $f_1|_S \equiv 0$ and $f_2|_T \equiv 0$

(2) f is 2-generic

(3) $\overline{0}$ is a singular point

A germ with the above properties will be called a Relative Whitney germ.

If $Diff_{\overline{0}}(R^2)$ denotes the group of germs at the origin of diffeomorphisms of the plane, then G_S^* and G_T^* will denote the subgroups of $Diff_{\overline{0}}(R^2)$ which preserve S and T respectively.

<u>Theorem 1</u>

Let f be a Relative Whitney germ, then we get the following cases

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(1) If $\frac{\partial f_1}{\partial x}(\overline{0}) \neq 0$ and $\overline{0}$ is a fold point, there exist H_1 and H_2 in G_S^* such that $H_2 o f o H_1(x,y) =$ $= (x,y^2)$.

(2) If $\frac{\partial f_2}{\partial y}(\overline{0}) \neq 0$ and $\overline{0}$ is a fold point, there exist G_1 and G_2 in G_T^* such that $G_2 o f o G_1(x,y) =$ (x^2,y).

(3) If $\frac{\partial f_1}{\partial x}(\overline{0}) \neq 0$ and $\overline{0}$ is a simple cusp, there exist G_1 and G_2 in G_T^* such that $G_2 o f o G_1(x,y) =$ $= (x,y^3+xy)$.

(4) If $\frac{\partial f_2}{\partial y}(\overline{0}) \neq 0$ and $\overline{0}$ is a simple cusp, there exist H_1 and H_2 in G_S^* such that $H_2 o f o H_1(x,y) =$ $= (x^3+xy,y)$.

<u>Proof</u>

For (1) and (3), we let $\Phi(x,y) = (f_1(x,y),y)$ then $\Phi \in G_S^* \cap G_T^*$ and we start with $\tilde{f} = f o \Phi^{-1}$, $\tilde{f}(x,y) = (x,\tilde{f}_2(x,y))$. Since Φ preserves both axis we get

(a) $\frac{\partial^2 \tilde{f}_2}{\partial y^2}(\overline{0}) \neq 0$ if $\overline{0}$ is a fold

$$\frac{\partial^2\tilde f_2}{\partial x\partial y}(\overline{0}) \neq 0, \quad \frac{\partial^2\tilde f_2}{\partial y^2}(\overline{0}) = 0 \quad \text{and} \quad \frac{\partial^3\tilde f_2}{\partial y^3}(\overline{0}) \neq 0$$

if $\overline{0}$ is a simple cusp.

In case a) we define $\phi_1(x,y) = (x, \frac{\partial f_2}{\partial y}(x,y)), \phi_1 \in G_S^*$ and $\tilde f o \phi_1^{-1} = (x, \tilde{\tilde f}_2(x,y))$ has T as it singular set. Let $\phi_2(x,y) = (x, \tilde{\tilde f}_2(x,y) - \tilde{\tilde f}_2(x,0))$, $\phi_2 \in G_S^*$. Hence if we let $g(x,y) = \phi_2 o \tilde f o \phi_1^{-1}(x,y) = (x, g_2(x,y))$ then $g_2(x,0) \equiv 0$ and $\frac{\partial g_2}{\partial y}(x,0) \equiv 0$, and we can write

$$g(x,y) = (x, g_2(x,y)) = (x, y^2\,\tilde g_2(x,y)) \qquad \tilde g_2(\overline{0}) \neq 0$$

$$\text{Define} \quad \phi_3(x,y) = \begin{cases} (x, y\sqrt{\tilde g_2(x,y)} & \text{if } \tilde g_2(\overline{0}) > 0 \\ (x, y\sqrt{-\tilde g_2(x,y)} & \text{if } \tilde g_2(\overline{0}) < 0 \end{cases}$$

$$\phi_4(u,v) = \begin{cases} (u,v) & \text{if } \tilde g_2(\overline{0}) > 0 \\ (u,-v) & \text{if } \tilde g_2(\overline{0}) < 0 \end{cases}$$

then ϕ_3 and ϕ_4 belong to $G_S^* \cap G_T^*$ and

$$\phi_4 o (x,y^2) o \phi_3(x,y) = g(x,y) = \phi_2 o \tilde f o \phi_1^{-1}(x,y)$$

Hence $\quad \phi_4^{-1} o \phi_2 o \tilde f o \phi_1^{-1} o \phi_3^{-1}(x,y) = (x,y^2)$

In case 3) we start with $\tilde{f}(x,y) = (x,\tilde{f}_2(x,y))$

$$\frac{\partial^2\tilde{f}_2}{\partial x\partial y}(\bar{0}) \neq 0, \quad \frac{\partial^2\tilde{f}_2}{\partial y^2}(\bar{0}) = 0 \quad \text{and} \quad \frac{\partial^3\tilde{f}_2}{\partial y^3}(\bar{0}) \neq 0$$

Using Malgrange Preparation theorem we get

$$y^3 = 3a_2(x,\tilde{f}_2(x,y))y^2+a_1(x,\tilde{f}_2(x,y))y+a_o(x,\tilde{f}_2(x,y))$$

hence $a_o(x,0) \equiv 0$.

The above equation is equivalent to

$$(y-a(x,y))^3 + b(x,y)(y-a(x,y)) = c(x,y)$$

where

$$a(x,y) = a_2(x,\tilde{f}_2(x,y))-a_2(x,0)$$

$$b(x,y) = -[a_1(x,\tilde{f}_2(x,y))+3ya_2(x,0)+3(a_2(x,\tilde{f}_2(x,y))-a_2(x,0))^2]$$

$$c(x,y) = a_o(x,\tilde{f}_2(x,y)) - (a_2(x,\tilde{f}_2(x,y)) - a_2(x,0))^3 + \\ +[a_1(x,\tilde{f}_2(x,y)) + 3ya_2(x,\tilde{f}_2(x,y))+3(a_2(x,\tilde{f}_2(x,y) - \\ - a_2(x,0))^2].(a_2(x,\tilde{f}_2(x,y)) - a_2(x,0))$$

It is easy to see that

(1) $\frac{\partial c}{\partial y}(\bar{0}) \neq 0$

(2) $\frac{\partial c}{\partial x}(\bar{0}) = 0$

(3) $\frac{\partial b}{\partial x}(\bar{0}) \neq 0$

Let $\Psi_1(x,y) = (b(x,\tilde{f}_2(x,y)), y-(a_2(x,\tilde{f}_2(x,y)) - a_2(x,0))$

$\Psi_2(u,v) = (b(u,v), c(u,v))$

Hence Ψ_1 and Ψ_2 belong to G_T^* and

$$\Psi_2 \circ (x,\tilde{f}_2(x,y)) \circ \Psi_1^{-1}(u,v) = (u, v^3+uv)$$

For (2) and (4) we let $\Phi(x,y) = (x, f_2(x,y))$ $\Phi \in G_S^* \cap G_T^*$. We start with $\tilde{f} = f \circ \Phi^{-1}$, $\tilde{f}(x,y) = (\tilde{f}_1(x,y), y)$. Since Φ preserves both axis we get

(a) $\frac{\partial^2 \tilde{f}_1}{\partial x^2}(\overline{0}) \neq 0$ if $\overline{0}$ is a fold

(b) $\frac{\partial^2 \tilde{f}_1}{\partial x \partial y}(\overline{0}) \neq 0$, $\frac{\partial^2 \tilde{f}_1}{\partial x^2}(\overline{0}) \neq 0$ and $\frac{\partial^3 \tilde{f}_1}{\partial x^3}(\overline{0}) \neq 0$

if $\overline{0}$ is a simple cusp.

The proofs run similar to the ones above. □

<u>Theorem 2</u>

The subgroup of $\mathrm{Diff}_{\overline{0}}(R^2)$ generated by G_S^* and G_T^* coincides with $\mathrm{Diff}_{\overline{0}}(R^2)$.

Proof

Let $(g_1,g_2) \in \mathrm{Diff}_{\overline{0}}(R^2)$, $(k_1,k_2) \in \mathrm{Diff}_{\overline{0}}(R^2)$ with $g_1(\overline{0}) \neq 0$, $g_2(\overline{0}) = 0$, $k_2(\overline{0}) \neq 0$.

We define $\tilde{h}_1 \in m(2)$ such that

$$\begin{aligned} xk_1 + yk_2 &= \tilde{h}_1(xg_1 + yg_2, y) \\ &= \tilde{h}_1(xg_1+yg_2,y) - \tilde{h}_1(0,y) + \tilde{h}_1(0,y) \\ &= (xg_1+yg_2)h_1(xg_1+yg_2,y) + \tilde{h}_1(0,y) \end{aligned}$$

Since $k_2(\overline{0}) \neq 0$ and $g_2(\overline{0}) = 0$ we get $\frac{\partial \tilde{h}_1}{\partial y}(\overline{0}) \neq 0$.
Hence $\tilde{h}_1(0,y) = yh_2(0,y)$ with $h_2(0,0) \neq 0$ and

$$xk_1+yk_2 = (xg_1+yg_2)h_1(xg_1+yg_2,y) + yh_2(0,y)$$

Hence $(x, xh_1+yh_2)o(xg_1+yg_2,y) = (xg_1+yg_2, xk_1+yk_2)$ with $g_1(\overline{0}) \neq 0$, $g_2(\overline{0}) = 0$ and $k_2(\overline{0}) \neq 0$ (*)

That means that the subgroup of $\mathrm{Diff}_{\overline{0}}(R^2)$ generated by $G_S^* \cup G_T^*$ contains all the diffeomorphism with linear part of the form

$$\begin{vmatrix} e & 0 \\ f & g \end{vmatrix} \qquad (**)$$

Changing the order of composition in (*) we get that this subgroup contains also all the diffeomorphisms with linear part of the form

$$\begin{vmatrix} a & b \\ 0 & c \end{vmatrix} \qquad (***)$$

Finally, the subgroup of GL(2) generated by the matrices of the form (**) and (***) is the whole group GL(2).

□

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