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Global Attractors for Parabolic Problems in Fractional Power Spaces

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1. Introduction

Let Ω be a bounded smooth domain of $\mathbb{R}^n$, $n \leq 3$. In this paper we consider parabolic problems of the form

$$\left.\begin{aligned} &u_t = d\Delta u - \gamma u + f(u), \qquad \text{in} \quad \Omega, \\ &\frac{\partial u}{\partial n} = 0, \qquad \text{in} \quad \partial\Omega. \end{aligned}\right\} \tag{1.1}$$

We are interested on global well posedness and on existence of global attractors for such problems. To simplify the presentation we assume that u is a scalar and that the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is a C^1 function satisfying

$$\limsup_{|u|\to\infty} \frac{f(u)}{u} \leq -\delta < 0. \tag{1.2}$$

This dissipativeness condition will play a fundamental role in proving that there is an absorbing set for (1.1).

To proceed describing the results we introduce some terminology. Let $X = L^2(\Omega)$ and $A : D(A) \subset X \to X$ be the self adjoint operator defined by

$$D(A) = \{\phi \in H^2(\Omega) : \tfrac{\partial \phi}{\partial n} = 0 \text{ in } \partial\Omega\},$$

$$A\phi = -d\Delta\phi + \gamma\phi, \quad \forall \phi \in D(A).$$

This operator is sectorial, and we can define its fractional powers A^α, $0 \leq \alpha$, and the associated fractional power spaces $X^\alpha = D(A^\alpha)$ endowed with the graph norm. We will be concerned only with $\alpha < 1$.

It is well known that under some growth assumptions on the nonlinearity f, the problem (1.1) has a global attractor in $H^1(\Omega)$. More specifically, if f satisfies

$$|fu) - f(v)| \leq c(e^{\theta|u|^\eta} + e^{\theta|v^\eta})|u-v|, \quad \eta < 2 \text{ and } \theta > 0, \text{ if } n = 2,$$

$$|fu) - f(v)| \leq c(1 + |u|^2 + |v|^2)|u - v|, \quad \text{if } n = 3.$$

Then, the problem (1.1) has a global attractor in $H^1(\Omega)$ (see, for example, Hale [1989], Carvalho [1992] and Carvalho and Oliveira [1992]).

These growth assumptions are necessary to obtain local existence of solutions for (1.1) and also play a role in obtaining some energy estimates necessary to guarantee that the solution operator for (1.1) defines a global dynamical system which is bounded dissipative.

It would be interesting to pose the problem (1.1) in a space for which no growth assumption on the nonlinearity f were required for local existence and in which we were able to prove the existence of a global attractor.

We will consider spaces X^α which are embedded in $L^\infty(\Omega)$. The following result identifies the values of α that we will be considering and its proof can be found in Henry [1981], Theorem 1.6.1.

Theorem 1.1. *Suppose that $\Omega \subset \mathbb{R}^n$ is an open bounded set having the C^2 extension property. Then, for $0 \leq \alpha \leq 1$,*

$$X^\alpha \subset W^{1,2}(\Omega) \quad \text{when} \quad \frac{1}{2} \leq \alpha,$$

$$X^\alpha \subset C^\nu(\Omega) \quad \text{when} \quad 0 \leq \nu + \frac{n}{2} < 2\alpha.$$

Furthermore, the embedding is compact whenever the inequality is strict.

Therefore, if we assume that $\alpha > \max\{\frac{n}{4}, \frac{1}{2}\}$ for $n = 2, 3$, or $\alpha \geq \frac{1}{2}$ for $n = 1$, we have that

$$X^\alpha \subset H^1(\Omega) \cap L^\infty(\Omega). \tag{1.3}$$

Hereafter we assume that (1.3) holds. The next lemma is the main reason why we are interested in working with such spaces.

Lemma 1.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1 function and $f^e : X^\alpha \to X$ be the map defined by*

$$f^e(\phi)(x) = f(\phi(x)).$$

Then, f^e is a well defined compact map which is Lipschitz continuous in bounded sets of X^α. Furthermore, for any $r > 0$ there exists a constant N_1, depending only on r, such that

$$\|f^e(\phi)\|_{L^\infty(\Omega)} \leq N_1,$$

whenever $\|\phi\|_{X^\alpha} \leq r$.

The proof of this result is rather trivial and we omit it. This lemma is telling us that the problem (1.1) is locally well posed in X^α even if no growth assumption is made on the nonlinearity f.

Our aim is to show that in such spaces the problem (1.1) has a global attractor and to obtain some good estimates for the size of the attractor in the uniform norm.

Hale [1986] proved the existence of a local attractor for (1.1), which coincides with the embedding of the attractor for $\dot{u} = f(u)$ into the subspace of constant functions of X^α, $\alpha > \frac{3}{4}$, if the diffusion coefficient d is large (see also, Hale and Rocha [1987a,b] and Hale and Sakamoto [1989]). However, the techniques employed by Hale [1986] would only apply to global attractors if some a priori bound on the size of the absorbing set could be obtained and only if the diffusion coefficient is large (see Carvalho [1992] and Carvalho and Oliveira [1992]).

We prove the existence of a global attractor for the problem (1.1) regardless of the size of d. We also give a priori bounds on the size of the basin of attraction which will make the results of Carvalho [1992] and Carvalho and Oliveira [1992] applicable to the case $\alpha \neq \frac{1}{2}$ as in Hale [1986], Hale and Rocha[1987a,b] and Hale and Sakamoto [1989] (see also Fusco [1987]).

To carry on this project we need to obtain that the solution operator associated to (1.1) is globally defined, that orbits of bounded subsets of X^α, under the flow definded by (1.1), are bounded subsets of X^α and that there is a bounded set that attracts points of X^α. Since the solution operator associated to (1.1) is compact, Theorem 3.4.6 in Hale [1989] would guarantee the existence of a global attractor.

The tools employed are the invariance theory as in Henry [1981] and the theory of invariant regions of Chueh, Conley and Smoller [1977].

We believe that the hypothesis (1.2) can be relaxed a little to include the case when the nonlinearity f satisfies $s\,f(s) - \gamma s^2 < 0$, $|s| \geq \xi$, for some $\xi > 0$. However, we have not been able to prove that, in this case, orbits of bounded subsets of X^α under the flow definded by (1.1) are bounded subsets of X^α.

We remark that other boundary conditions can be considered with little change; hence, we will restrict the presentation to the homogeneous Neumann boundary conditions case. We also remark that, if the nonlinearity depends on the spatial variable, the results hold with almost no change in the proofs.

In Section 2 we obtain global existence of solutions and that orbits of bounded sets are bounded for systems of reaction diffusion equations with dispersion of the form

$$\left.\begin{aligned} &u_t = D\Delta u - \gamma u + \textstyle\sum_{j=1}^{n} B_j(x)\frac{\partial u}{\partial x_j} + f(u), \qquad \text{in} \quad \Omega, \\ &\frac{\partial u}{\partial n} = 0, \qquad \text{in} \quad \partial\Omega. \end{aligned}\right\} \tag{1.1$'$}$$

where $u = (u_1, u_2, \cdots, u_N)^\top$, $N \geq 1$, $D = \text{diag}(d_1, \cdots, d_N)$, $d_i > 0$, $1 \leq i \leq N$, and $B_j = \text{diag}(b_j^1, \cdots, b_j^N)$ is continuous in $\bar{\Omega}$, $1 \leq j \leq n$. The nonlinearity $f = (f_1, \cdots, f_N)^\top : \mathbb{R}^N \to \mathbb{R}^N$ is assumed to be a C^1 function satisfying

$$\limsup_{|u_i| \to \infty} \frac{f_i(u)}{u_i} \leq -\delta < 0, \tag{1.4}$$

uniformly with respect to $\hat{u}_i = (u_1, \cdots, u_{i-1}, u_{i+1}, \cdots, u_N) \in \mathbb{R}^{N-1}$, $1 \leq i \leq N$.

In Section 3 we show the existence of a global attractor neglecting the dispersion terms in (1.1)′, in Section 4 we consider the structure of gradient systems that such problems have and in Section 5 we consider a class of cooperative systems arising in thin domains around a point.

Recall (see Hale [1989] , for example) that, if $T(t) : X \to X$, $t \geq 0$, is a semigroup of transformations on a Banach space X, then a set $\mathcal{A}$ is an attractor if it is a compact invariant set that attracts a neighborhood $\mathcal{O}$ of itself; that is, $\mathcal{A}$ is compact, $T(t)\mathcal{A} = \mathcal{A}$ for $t \geq 0$ and there is a neighborhood $\mathcal{O}$ of $\mathcal{A}$ such that dist $(T(t)U, \mathcal{A}) \to 0$ as $t \to \infty$. The set $\mathcal{A}$ is a global attractor if it is an attractor which attracts each bounded set of X.

2. Global Existence and Boundedness

In this section we prove that solutions of (1.1)′ with initial data in X^α are globally defined and that orbits of bounded subsets of X^α under the flow defined by (1.1)′ are also bounded subsets of X^α. To prove this result we will use the following lemma

Lemma 2.1. *Let A be a sectorial operator and $f^e : X^\alpha \to X$ be a bounded map which is Lipschitz continuous in bounded subsets of X^α. Then, the problem*

$$\begin{aligned} \dot{u} + Au &= f^e(u) \\ u(0) &= u_0 \in X^\alpha, \end{aligned} \tag{2.1}$$

has a local solution $T(t)u_0$ defined in a maximal interval of existence $[0, t_{\max})$. Furthermore, either $\|T(t)u_0\|_\alpha \to \infty$ when $t \to \infty$ or $t_{\max} = +\infty$.

For a proof of this lemma see Henry [1981]. We observe that it is not enough that the nonlinearity f^e be locally Lipschitz continuous. We assume that f is Lipschitz continuous in bounded sets, which is suitable for our applications. Some extra hypothesis is necessary because the phase space is not locally compact.

We know that $A = \text{diag}(A_1, \cdots A_N)$ defined by

$$D(A_i) = \{\phi \in H^2(\Omega) : \tfrac{\partial \phi}{\partial n} = 0\},$$

$$A_i\phi = d_i\Delta\phi - \gamma\phi + \textstyle\sum_{j=1}^n b_j^i(x)\frac{\partial \phi}{\partial x_j},$$

generates an analytic semigroup on X^α and that it satisfies the following estimates

$$\|e^{-At}u_0\|_{X^\alpha} \le Me^{\epsilon t}\|u_0\|_{X^\alpha}, \quad t \ge 0,$$
$$\|e^{-At}u_0\|_{X^\alpha} \le Me^{\epsilon t}t^{-\alpha}\|u_0\|_X, \quad t > 0, \tag{2.2}$$

for some $\epsilon \in \mathbb{R}$, $M \ge 1$.

Writing the problem $(1.1)'$ in the form (2.1) and using the variation of constants formula, we can view its solution through $u_0 \in X^\alpha$ as

$$T(t)u_0 = e^{-A\,t}u_0 + \int_0^t e^{-A(t-s)} f^e(T(s)u_0)ds,$$

where $(f^e(\phi))(x) = f(\phi(x))$, for all $\phi \in X^\alpha$. If $\|T(t)u_0\|_{L^\infty(\Omega)} \le N_1$, $t \in [0, t_{\max})$, for some $N_1 > 0$; then, we have that

$$\|T(t)u_0\|_{X^\alpha} \le M\|u_0\|_\alpha + MK\int_0^t e^{\epsilon(t-s)}(t-s)^{-\alpha}ds, \tag{2.3}$$

where $K = \max_{t\in[0,t_{\max})}\|f^e(T(t)u_0)\|_{L^2(\Omega)}$. Therefore, if we are able to obtain estimates in $L^\infty(\Omega)$ for $T(t)u_0$, $0 \le t < t_{\max}$, a similar estimate can be obtained in X^α and the solutions are globally defined.

To obtain such estimates in $L^\infty(\Omega)$ we introduce the notion of invariant regions as in Smoller [1983].

Definition 2.2. *A set $\Sigma \subset \mathbb{R}^N$ is called a positively invariant region for the local solution of $(1.1)'$ if any solution $T(t)u_0$ satisfying $u_0(x) \in \Sigma$, $\forall x \in \Omega$ is such that $(T(t)u_0)(x) \in \Sigma$, $\forall x \in \Omega$ and for all t in the maximal interval of existence of the solution.*

Our next result characterizes some of the invariant regions of the problem $(1.1)'$. Its proof follows Smoller [1983] and is presented here for the sake of completeness.

Theorem 2.3. *Let $\bar{\xi}_j, \xi_j > 0$, $1 \le j \le N$ be such that $u_j f_j(u) < 0$ for all $u \in \mathbb{R}^N$ with $u_j \notin [-\bar{\xi}_j, \xi_j]$. Then, the rectangle $\Sigma = [-\bar{\xi}_1, \xi_1] \times [-\bar{\xi}_2, \xi_2] \times \cdots \times [-\bar{\xi}_N, \xi_N]$ is an invariant region for the local solution of $(1.1)'$.*

Proof. If there is a solution $v(x,t) = (v^1(x,t), v^2(x,t), \cdots, v^N(x,t))$ of $(1.1)'$ with initial data $v(x,0) = (v^1(x,0), v^2(x,0), \cdots, v^N(x,0)) \in \Sigma$ for all $x \in \bar{\Omega}$, that does not stay in $\Sigma_1 = (-\infty, \xi_1] \times \mathbb{R}^{N-1}$ for all $t \in [0, t_{\max})$, then there is a t_0 and $x_0 \in \Omega$ such that

$$v^1(x,t) < \xi_1, \quad 0 \le t < t_0, \quad x \in \Omega, \quad \text{and} \quad v^1(x_0,t_0) = \xi_1.$$

Therefore, if $v^1(x_0,t) < \xi_1$, $\forall t \in [0,t_0)$ and $v^1(x_0,t_0) = \xi_1$ implies that $v^1_t(x_0,t_0) < 0$; then, Σ_1 is invariant.

Consider the following

$$v^1_t(x_0,t_0) = d_1\Delta v^1(x_0,t_0) - \gamma v^1(x_0,t_0) + \sum_{j=1}^n B^1_j(x_0)\frac{\partial v^1}{\partial x_j}(x_0,t_0) + f_1(v(x_0,t_0)). \tag{2.4}$$

We claim that $\nabla v^1 = 0$ at (x_0, t_0). In fact, if $\frac{\partial v_1}{\partial x_i} > 0$ at (x_0, t_0), for some $1 \leq i \leq n$; then, $v(x_0, t_0) = \xi_1$ and $v^1(x, t_0) > \xi_1$ for some x with $|x - x_0|$ small. This implies that $v^1(x,t) > \xi_1$ for x near x_0 and $t < t_0$ near t. This contradicts the definition of t_0 and $\frac{\partial v^1}{\partial x_i} \leq 0$. Using the same reasoning we obtain that $\frac{\partial v^1}{\partial x_i} > 0$ at (x_0, t_0), for some $1 \leq i \leq n$ leads to a contradiction and the claim is proved.

Similarly, $v^1_{x_i x_i} \leq 0$, for all $1 \leq i \leq n$. Therefore $\Delta v^1(x_0, t_0) \leq 0$ and from expression (2.4), we have that

$$v^1_t(x_0, t_0) \leq f_1(v(x_0, t_0)) - \gamma v^1(x_0, t_0).$$

Since $f_1(v(x_0, t_0)) - \gamma v^1(x_0, t_0) < 0$, we conclude that $v^1_t(x_0, t_0) < 0$.

From the reasoning at the beginning of the proof we have that $(-\infty, \xi_1] \times \mathbb{R}^{N-1}$ is an invariant region for the local solution of (1.1)$'$. In the same way we obtain that $[-\xi_1, \infty) \times \mathbb{R}^{N-1}$ is also an invariant region. From the fact that the intersection of invariant regions is still invariant, the proof is completed.

This theorem shows that for any $u_0 \in X^\alpha$ the local solution $T(t)u_0$ of (1.1)$'$ through u_0 satisfies

$$\|T(t)u_0\|_{L^\infty(\Omega)} \leq N_1,$$

for some $N_1 > 0$, whenever defined.

It follows from Lemma 2.1 and from expression (2.4), that the problem (1.1)$'$ defines a global dynamical system in X^α.

If in addition the semigroup generated by A satisfies (2.2) for some $\epsilon = -\zeta < 0$ then our next computations show that orbits of bounded subsets of X^α are bounded subsets of X^α. To do this we once more resort to the variation of constants formula and to Lemma 2.1.

Let B be a bounded subset of X^α. Then, from the fact that X^α is embedded in $L^\infty(\Omega)$ and from the variation of constant formula we have that there are constants $K_i > 0$, $1 \leq i \leq 4$ depending only on B such that

$$\begin{aligned}\|T(t)u_0\|_{X^\alpha} &\leq K_1 + K_2 \textstyle\int_0^t (t-s)^{-\alpha} e^{-\zeta(t-s)} \|f(T(s)u_0)\|_{L^\infty(\Omega)} ds \\ &\leq K_1 + K_3 \textstyle\int_0^t (t-s)^{-\alpha} e^{-\zeta(t-s)} ds,\end{aligned}$$

$\forall t \geq 0$ and $\forall u_0 \in B$. We used that $\|T(t)u_0\|_{L^\infty(\Omega)} \leq K_4$ which follows from the embedding of X^α into $L^\infty(\Omega)$ and from Theorem 2.3.

In what follows we consider an example of a reaction diffusion equation with dispersion for which we know that orbits of bounded sets are bounded.

Consider a system of reaction diffusion equations of chemical kinetics (see, for example, Chueh, Conley and Smoller[1977])

$$
\begin{aligned}
&w_t = d_1 \Delta w + \textstyle\sum_{i=1}^{n} \beta_i^1 \frac{\partial w}{\partial x_i} + g(w,v) \quad \text{in} \quad \Omega \\
&v_t = d_1 \Delta v + \textstyle\sum_{i=1}^{n} \beta_i^2 \frac{\partial v}{\partial x_i} + h(w,v) \quad \text{in} \quad \Omega \\
&\frac{\partial w}{\partial n} = \frac{\partial v}{\partial n} = 0 \quad \text{in} \quad \partial\Omega
\end{aligned}
\tag{2.5}
$$

where $\Omega \subset \mathbb{R}^n$, $n \leq 3$, is as in Section 1, $f, g : \mathbb{R}^2 \to \mathbb{R}$ are C^1–functions satisfying

$$
\limsup_{|u|\to\infty} \frac{g(u,v)}{u} = -\infty, \qquad \limsup_{|v|\to\infty} \frac{h(u,v)}{v} = -\infty,
$$

where the first limit is uniform with respect to v and the second is uniform with respect to w.

Then, the system (2.5) can be rewritten as

$$
\begin{aligned}
&u_t = D\Delta u - \gamma u + \textstyle\sum_{j=1}^{n} B_j \frac{\partial u}{\partial x_j} + f(u) \quad \text{in} \quad \Omega \\
&\frac{\partial u}{\partial n} = 0 \quad \text{in} \quad \partial\Omega
\end{aligned}
$$

where $u = (w,v)^{\mathsf{T}} \in \mathbb{R}^2$, $B_j = \operatorname{diag}(\beta_1^j, \beta_2^j)$, $j = 1,2$, $f(u) = \begin{pmatrix} g(w) + \gamma w \\ h(v) + \gamma v \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$ satisfy (1.4) and γ is chosen as follows.

Consider the operator A defined by $A = \operatorname{diag}(A_1, A_2)$,

$$
D(A_j) = \{\phi \in H^2(\Omega) : \tfrac{\partial \phi}{\partial n} = 0\},
$$

$$
A_j\phi = d_j \Delta\phi - \gamma\phi + \textstyle\sum_{i=1}^{n} \beta_i^j \frac{\partial \phi}{\partial x_i}, \quad j = 1,2.
$$

Let γ be such that the analytic semigroup generated by A decays exponentially to zero as $t \to \infty$. The results in this section imply that the solution operator $S(t)$ for (2.5) is defined globally and orbits of bounded sets under $\{S(t) : t \geq 0\}$ are bounded subsets of X^α.

3. Point Dissipativeness and Global Attractors

In this section we use the invariance theory as in Henry [1981] to prove that there is a bounded set in X^α which attracts points of X^α. Unfortunately, the techniques employed in this section will not work for systems of reaction diffusion equations with dispersion due to the fact that we will not be able to find a Liapunov function for such systems. We state the results of the invariance theory that we will use, starting with the definition of Liapunov function.

Definition 3.1. *Let $\{S(t),\ t \geq 0\}$ be a dynamical system on X^α. A Liapunov function is a continuous real valued function $V : X^\alpha \to \mathbb{R}$ such that*

$$
\dot{V}(\phi) = \limsup_{t\to 0^+} \frac{V(S(t)\phi) - V(\phi)}{t} \leq 0
$$

for all $\phi \in X^\alpha$.

The next theorem is a classical result from invariance theory and will be the main tool in the proof of point dissipativeness.

Theorem 3.2. *Suppose that $u_0 \in X^\alpha$ and $\{S(t)u_0,\ t \geq 0\}$ lies in a compact set in X^α; then, $\omega(u_0)$ is nonempty, compact, invariant, connected, and dist$(S(t)u_0, \omega(u_0)) \to 0$ as $t \to +\infty$.*

The following result has a classical and simple proof but we present it to make sure that our Liapunov function (see latter in this section) is suitable.

Theorem 3.3. *Let V be a Liapunov function on X^α and define $E = \{\phi \in X^\alpha : \dot{V}(\phi) = 0\}$, $\mathcal{M}$ the maximal invariant subset of E. If $\{S(t)u_0,\ t \geq 0\}$ lies in a compact set in X^α, then $S(t)u_0 \to \mathcal{M}$ as $t \to +\infty$.*

Proof. By hypothesis, $V(S(t)u_0)$ is nonincreasing for $t \geq 0$ and is bounded bellow (since orbits of points are precompact) so $\ell = \lim_{t\to\infty} V(S(t)u_0)$ exists. If $y \in \omega(u_0)$, then $V(y) = \ell$, so also $V(S(t)y) = \ell$, $t \geq 0$, and so $\dot{V}(y) = 0$. Thus $\omega(u_0) \subset E$, so $\omega(u_0) \subset \mathcal{M}$ and the result is proved.

We will apply these results and the results of Section 2 to obtain the existence of a global attractor for systems of reaction diffusion equations without dispersion; that is, we consider the problem

$$\left.\begin{aligned} &u_t = D\Delta u - \gamma u + f(u), \quad \text{in } \Omega, \\ &\frac{\partial u}{\partial n} = 0, \quad \text{in } \partial\Omega. \end{aligned}\right\} \tag{1.1}''$$

where $u = (u_1, u_2, \cdots, u_N)^\top$, $N \geq 1$, $D = \text{diag}(d_1, \cdots, d_N)$, $d_i > 0$, $1 \leq i \leq N$ and $\gamma > 0$. The nonlinearity $f = (f_1, \cdots, f_N)^\top : \mathbb{R}^N \to \mathbb{R}^N$ is assumed to be a C^1 function satisfying (1.4) and

$$\frac{\partial f_i(u)}{\partial u_j} = \frac{\partial f_j(u)}{\partial u_i} \quad \forall u \in \mathbb{R}^N \tag{3.1}$$

To prove that the solution operator for $(1.1)''$ is point dissipative we must first prove that the orbit of a point $\phi \in X^\alpha$ is a compact subset of X^α. This is a consequence of the following result.

Theorem 3.4. *In the problem (2.1) assume that the nonlinearity f^e is Lipschitz continuous in bounded subsets of X^α and that A is a sectorial operator with compact resolvent. If $T(t)\phi$ is a solution of $(1.1)''$ on $[0, \infty)$ with $\|T(t)\phi\|_\alpha$ bounded as $t \to \infty$; then, $\{T(t)\phi,\ t \geq 0\}$ is a compact subset of X^α. Furthermore, if B is a bounded subset of X^α and $T(t)B$ remains in a bounded subset of X^α as $t \to \infty$; then, $\{T(t)B,\ t \geq 1\}$ is a compact subset of X^α.*

The proof of the above result is very simple and can be easily adapted from Henry [1981], Theorem 3.3.6.

Now since we have that all the hypotheses of Theorem 3.4 are satisfied for problem $(1.1)''$ we can conclude that orbits of points under the flow defined by $(1.1)''$ are compact subsets of X^α and from Theorem 3.2 the ω−limit set of any point u_0 is a nonempty, compact, invariant set that attracts u_0 under the flow defined by $(1.1)''$.

From Theorem 3.3 we know that the set $\mathcal{M}$ attracts points of X^α. We need to find a Liapunov function V for which E is a bounded subset of X^α and point dissipativeness will follow.

Let $V : X^\alpha \to \mathbb{R}$ be the function defined by

$$V(\phi) = \frac{1}{2}\int_\Omega \langle D\nabla\phi, \nabla\phi\rangle dx + \frac{\gamma}{2}\int_\Omega |\phi|^2 dx - \int_\Omega F(\phi)dx$$

where $F(u) = \frac{1}{N}\sum_{i=1}^N \int_0^{u_i} f(u_1,\cdots,u_{i-1},s,u_{i+1},\cdots,u_N)ds$. Then V is continuous and

$$\dot V(\phi) \le 0, \quad \forall\phi \in X^\alpha.$$

Therefore, we must prove that $E = \{\phi \in X^\alpha : \dot V(\phi) = 0\}$ is a bounded set in X^α.

The set E is the set of equilibrium points of (1.1)$''$ and therefore, any function $\phi \in E$ it must satisfy that, $\phi \in X^\alpha$ and

$$\left.\begin{aligned} &D\Delta\phi - \gamma\phi + f(\phi) = 0, \quad \text{in } \Omega,\\ &\frac{\partial\phi}{\partial n} = 0, \quad \text{in } \partial\Omega. \end{aligned}\right\} \tag{3.2}$$

To prove that E is a bounded subset of X^α we proceed in the following way. First we prove that there exists a constant $c > 0$ such that $\|\phi\|_{L^\infty(\Omega,\mathbb{R}^N)} \le c \ \forall\phi \in E$ and then we use (2.1) to prove that $\|D\Delta\phi\|_{L^\infty(\Omega,\mathbb{R}^N)} \le \max_{|s|\le c}|-\gamma s + f(s)|$ and the result will follow.

Lemma 3.5. *There exists a constant $\xi > 0$ such that $\|\phi\|_{L^\infty(\Omega,\mathbb{R}^N)} \le \xi$ for every $\phi \in E$.*

Proof. Let ξ_i be such that $sf(u_1,\cdots,u_{i-1},s,u_{i+1},\cdots,u_N) < 0 \ \forall s$ such that $|s| \ge \xi_i$. Then, suppose that $\phi := (\phi_1,\cdots,\phi_N) \in E$ and that $\max_{x\in\bar\Omega}\phi_k(x) = \phi_k(y) \ge \xi_k$, for some k. Thus, at y

$$\phi_k d_k \Delta\phi_k - \gamma\phi_k^2 + \phi_k f_k(\phi) = 0$$

and $\Delta\phi_k > 0$ since $\xi_k > 0$, but $\Delta\phi_k(y) \le 0$. This is a contradiction and $\max_{x\in\bar\Omega}\phi_k(x) \le \xi_k$, $1 \le k \le N$. In the same way we obtain that $\min_{x\in\bar\Omega}\phi_k(x) \ge -\xi_k$, $1 \le k \le N$, and the result is proved.

Corollary 3.6. *The set E is a bounded subset of X^α.*

Remark. *It is very important, for some applications, to be able to obtain some a priori estimates that do not depend on the size of the diffusion coefficient. See for example Carvalho* [1992] *and Carvalho and Oliveira* [1992]. *This is an advantage that this technique (the use of invariant regions) has over the techniques employed in Hale* [1989], *page 77, even in the case $\alpha = \frac{1}{2}$. The estimates obtained there, for the size of E, strongly depend on the size of the diffusion coefficient. It is also important to obtain good $L^\infty(\Omega)$ bounds on the attractor. Since models are approximation of a real phenomenon, it is important to know that at least in the basin of attraction the approximation must be as accurate as we can get and outside this basin it does not matter much.*

Corollary 3.7. *The solution operator* $\{T(t),\ t \geq 0\}$ *for* $(1.1)''$ *is point dissipative and therefore it has a global attractor.*

The proof of point dissipativeness follows from Theorem 3.3 and Corollary 3.6 and the proof of existence of a global attractor follows from the results in this section, in Section 2 and from Theorem 3.4.6 in Hale [1989].

4. Gradient Systems

In this section we consider the possibility that the problem (1.1) has the structure that the so called gradient systems have. Such special class of dynamical systems, for which the flow on the attractor can be better understood, are considered in Hale [1989], for example.

Definition 4.1. *Let* Y *be a Banach space. A strongly continuous* C^r*−semigroup* $T(t) : Y \to Y$, $t \geq 0$, $r \geq 0$, *is said to be a gradient system if*

1. *Each bounded positive orbit is precompact.*
2. *There exists a Liapunov function for* $T(t)$*; that is, there is a continuous function* $V : Y \to \mathbb{R}$ *with the property that*
   (a) $V(y)$ *is bounded below.*
   (b) $V(u) \to \infty$ *as* $\|y\|_Y \to \infty$.
   (c) $V(T(t)y)$ *is nonincreasing in* t *for each* $y \in Y$.
   (d) *If* y *is such that* $T(t)y$ *is defined for* $t \in \mathbb{R}$ *and* $V(T(t)y) = V(y)$ *for* $t \in \mathbb{R}$, *then* y *is an equilibrium point.*

Observe that the Liapunov function defined in Section 3 does not satisfy the property *2b*. However we still obtain that the dynamics in the attractor for (1.1) can be described as well as the dynamics of a gradient system.

To obtain these results we consider an auxiliary system, namely

$$\left.\begin{aligned} &u_t = D\Delta u - \gamma u + \tilde{f}(u), \qquad \text{in} \quad \Omega, \\ &\frac{\partial u}{\partial n} = 0, \qquad \text{in} \quad \partial\Omega, \end{aligned}\right\} \tag{4.1}$$

where $\tilde{f}$ is obtained from f in the following way. The attractor $\mathcal{A}$ for (1.1) in X^α is a bounded subset of $L^\infty(\Omega, \mathbb{R}^N)$ and therefore, we can cut the nonlinearity f is such a way that all the properties of f are preserved plus $\tilde{f}$ is globally Lipschitz continuous and the dynamics in the attractor remains unchanged. That is, the problem (4.1) has a global attractor in X^α which coincides with $\mathcal{A}$.

The problem (4.1) is well posed in $X^{\frac{1}{2}}$ and has an attractor $\tilde{\mathcal{A}}$ in $X^{\frac{1}{2}}$. Therefore we must have $\mathcal{A} \subset \tilde{\mathcal{A}}$. Since $\tilde{\mathcal{A}} \subset X^\alpha$ and we have that it is invariant it must be contained in $\mathcal{A}$ and they are the same. This reasoning proves the following result.

Theorem 4.2. *If $\{T(t), t \geq 0\}$ is the semigroup defined by (1.1) in X^α, $\mathcal{A}$ denotes its attractor, and E the set of equilibrium points, then $\mathcal{A} = W^u(E) = \{y \in X^\alpha : T(-t)y$ is defined for $t \geq 0$ and $T(-t)y \to E$ as $t \to \infty\}$. If, in addition, every element of E is hyperbolic, then E is a finite set and $\mathcal{A} = \cup_{x \in E} W^u(x)$.*

Corollary 4.3. *Let $\Sigma_i = [-\bar{\xi}_i, \xi_i]$, where ξ_i, $\bar{\xi}_i$ are positive constants, $1 \leq i \leq N$. If $s_i\, f_i(u_1, \cdots, u_{i-1}, s, u_{i+1}, \cdots, u_N) < 0$ whenever $s_i \notin \Sigma_i$; then, $\phi(x) \in \Sigma := \Sigma_1 \times \cdots \times \Sigma_N$, $\forall x \in \Omega$ and $\forall \phi \in \mathcal{A}$.*

Proof. The proof of this result follows in two steps. First we observe that the equilibrium points satisfy that $\phi(x) \in \Sigma$, $\forall x \in \Omega$ (as in Lemma 3.5). The second part follows from the fact that the α−limit set of points in $\mathcal{A}$ are subsets of the set of equilibrium points and that any interval containing Σ is invariant. More specifically, let $\epsilon > 0$ and Σ_ϵ be an ϵ−neighborhood of Σ. If $\phi \in \Sigma_\epsilon$ then there is a $\tau < 0$ and $e \in E$ such that $\|T(\tau)\phi - e\|_{L^\infty(\Omega, \mathbb{R}^N)} \leq K\|T(\tau)\phi - e\|_{X^\alpha} < \frac{\epsilon}{2}$, where K is the embedding constant of $X^\alpha \subset L^\infty(\Omega, \mathbb{R}^N)$. This implies that $(T(\tau)\phi)(x) \in \Sigma_\epsilon$, $\forall x \in \bar{\Omega}$. Since Σ_ϵ is invariant we have that $\phi \in \Sigma_\epsilon$.

This proves that, for any $\epsilon > 0$ and $\phi \in \mathcal{A}$, $\phi(x) \in \Sigma_\epsilon$ $\forall x \in \bar{\Omega}$ and the result follows.

5. Systems Arising from Thin Domains Problems

In this section we consider a class of systems of weakly coupled parabolic partial differential equations of the form

$$\left.\begin{aligned} &u_t = D\Delta u - \tilde{A}u + f(u), \qquad \text{in } \Omega, \\ &\frac{\partial u}{\partial n} = 0, \qquad \text{in } \partial\Omega. \end{aligned}\right\} \tag{5.1}$$

where $u = (u_1, u_2, \cdots, u_N)^\top$, $N \geq 1$, $D = \text{diag}(d_1, \cdots, d_N)$, $d_i > 0$, $1 \leq i \leq N$ and the nonlinearity $f := (f(u_1), \cdots, f(u_N))^\top : \mathbb{R}^N \to \mathbb{R}^N$ is assumed to be a C^1 function satisfying

$$u_i f_i(u) < 0 \qquad \forall u \in \mathbb{R}^N, \quad u_i \notin [\bar{\xi}, \xi], \quad 1 \leq i \leq N. \tag{5.2}$$

The matrix $\tilde{A}$ is taken as $\tilde{A} = M^{-1}B$ where $M = \text{diag}(L_1, \cdots, L_N)$, with $0 \leq L_i \leq 1$ for $1 \leq i \leq N$ and B be is the tridiagonal symmetric matrix

$$B = \begin{bmatrix} m_1 & r_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ r_1 & m_2 & r_2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & r_2 & m_3 & r_3 & 0 & \cdots & 0 & 0 \\ 0 & 0 & r_3 & m_4 & r_4 & \cdots & 0 & 0 \\ \vdots & \vdots & & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & & & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & r_{N-3} & m_{N-2} & r_{N-2} & 0 \\ 0 & 0 & \cdots & 0 & 0 & r_{N-2} & m_{N-1} & r_{N-1} \\ 0 & 0 & \cdots & 0 & 0 & 0 & r_{N-1} & m_N \end{bmatrix} \tag{5.3}$$

where

$$\begin{aligned} m_1 &= \tfrac{a_2}{2l_2} + \tfrac{a_1\rho}{\rho l_1 + a_1(1-\rho)}, \\ m_N &= \tfrac{a_N}{2l_N} + \tfrac{a_{N+1}\sigma}{\sigma l_{N+1} + a_{N+1}(1-\sigma)}, \\ m_k &= \tfrac{a_k}{2l_k} + \tfrac{a_{k+1}}{2l_{k+1}}, \quad k = 2, \cdots, N-1, \\ r_k &= \tfrac{-a_{k+1}}{2l_{k+1}}, \quad k = 1, \cdots, N-1. \end{aligned}$$

and $a_k > 0$, $l_k > 0$ for $1 \le k \le N+1$.This problem arises as a limiting problem for reaction diffusion equations in thin domains around a point (see Hale and Raugel[1992], Carvalho[1992] and Carvalho and Oliveira[1992] for details).

To prove that problem (5.1) has a global attractor we have to obtain an invariant region for equation (5.1) as in Theorem 2.3, find a Liapunov function for it and show that the set of equilibrium solutions are bounded in the $L^\infty(\Omega)$ norm as in Section 3.

Assume that f satisfies (3.1) an let

$$F(u) = \sum_{i=1}^{N} \frac{1}{N} \int_0^{u_1} f(u_1, \cdots, u_{i-1}, s, u_{i+1}, \cdots, u_N) ds$$

and $V : X^\alpha \to \mathbb{R}$ be the function defined by

$$V(\phi) = \int_\Omega (\sum_{i=1}^{N} \frac{d_i}{2} \|\nabla \phi_i\|^2 + \langle \tilde{A}\phi, \phi \rangle + F(\phi)) dx$$

where $\langle \cdot, \cdot \rangle$ stands for the usual inner product in $\mathbb{R}^n$.

The function V is continuous and

$$\dot{V}(\phi) \le 0, \quad \forall \phi \in X^\alpha.$$

Thus if $\phi \in E = \{\phi \in X^\alpha : \dot{V}(\phi) = 0\}$ we have that $\phi \in X^\alpha$ and

$$\left. \begin{aligned} & D\Delta\phi - \tilde{A}\phi + f(\phi) = 0, \quad \text{in } \Omega, \\ & \frac{\partial \phi}{\partial n} = 0, \quad \text{in } \partial\Omega. \end{aligned} \right\} \tag{5.4}$$

We now prove the $L^\infty(\Omega)$ boundedness of the equilibrium solutions and the existence of the invariant regions for (5.1).

The following result is a consequence of the theory of invariant regions (see Smoller[1982], page 202) and appears in Carvalho and Oliveira[1992] when Ω is an interval.

Lemma 5.1 *The rectangle $[\bar{\rho}, \rho]^N$ is an invariant region for* (5.1) *whenever $\bar{\rho} \ge \bar{\xi}$ and $\rho \ge \xi$.*

To prove that E is a bounded subset of X^α we proceed as in Section 3. First we prove that there exists a constant $c > 0$ such that $\|\phi\|_{L^\infty(\Omega, \mathbb{R}^N)} \le c \quad \forall \phi \in E$ and then we obtain that

$$\|D\Delta\phi - \tilde{A}\phi\|_{L^\infty(\Omega, \mathbb{R}^N)} \le \tilde{c}, \quad \forall \phi \in E.$$

Lemma 5.2 *If E is the set of equilibrium solutions of* (5.1) *there exists a constant $\xi > 0$ such that $\|\phi\|_{L^\infty(\Omega)} \le \xi$ for every $\phi \in E$.*

Proof. Let ξ be such that $s_j f_j(s) < 0\ \forall s$ such that $|s_j| \ge \xi$. Suppose that $\phi := (\phi_1, \phi_2, \cdots, \phi_N) \in E$ and let $\max_{x\in\bar\Omega} \phi_j(x) = \phi_j(y_j)$. We claim that $\phi_j(y_j) \le \xi$ for all j.

Suppose $\phi_j(y_j) \le \xi$ for $j = 1, 2, \cdots k-1$ and $\phi_k(y_k) > \xi$ for some $1 \le k \le N$. From the k^{th} equation we obtain

$$0 = \phi_k(y_k) d_k \Delta\phi_k(y_k) - \tfrac{r_k}{L_k}\phi_k(y_k)[\phi_{k+1}(y_k) - \phi_k(y_k)]$$

$$+\tfrac{r_{k-1}}{L_k}\phi_k(y_k)[\phi_k(y_k) - \phi_{k-1}(y_k)] + \phi_k(y_k) f(\phi)$$

At y_k the maximum principle implies (see Protter and Weimberger [1967], page 65)

$$\phi_k(y_k) d_k \Delta\phi_k(y_k) \le 0$$

and

$$\phi_k(y_k) f(\phi(y_k)) < 0.$$

Since $\phi_{k-1}(y_k) \le \phi_{k-1}(y_{k-1}) \le \xi$, we also have

$$\frac{r_{k-1}}{L_k}\phi_k(y_k)[\phi_k(y_k) - \phi_{k-1}(y_k)] < \frac{r_{k-1}}{L_k}\phi_k(y_k)[\xi - \phi_{k-1}(y_{k-1})] \le 0$$

These inequalities together with equation (5.5) imply that

$$-\frac{r_k}{L_k}\phi_k(y_k)[\phi_{k+1}(y_k) - \phi_k(y_k)] > 0$$

It follows that $\phi_{k+1}(y_k) > \phi_k(y_k)$, and

$$\begin{cases} \phi_{k+1}(y_{k+1}) \ge \phi_{k+1}(y_k) > \phi_k(y_k) > \xi \\ \phi_{k+1}(y_{k+1}) \ge \phi_{k+1}(y_k) > \phi_k(y_k) \ge \phi_k(y_{k+1}). \end{cases}$$

Using the same argument through the $(N-1)^{th}$ equation we obtain

$$\begin{array}{l} \phi_N(y_N) > \xi \\ \phi_N(y_N) > \phi_{N-1}(y_N) \end{array}$$

and the N^{th} equation gives

$$0 = \phi_N(y_N) d_N \Delta\phi_N(y_N) - \tfrac{m_N - r_{N-1}}{L_N}\phi_N(y_N)[0 - \phi_N(y_N)]$$

$$+\tfrac{r_{N-1}}{L_N}\phi_N(y_N)[\phi_N(y_N) - \phi_{N-1}(y_N)] + \phi_N(y_N) f(\phi(y_N)).$$

The same reasoning as before implies that

$$-\frac{m_N - r_{N-1}}{L_N}(\phi_N(y_N))^2 > 0$$

This contradiction implies $\max_{x\in\bar\Omega} \phi_j(x) \le \xi$ for $1 \le j \le N$. In the same way we obtain that $\min_{x\in\bar\Omega} \phi_j(x) \ge -\xi$ for $1 \le j \le N$ and the result is proved.

Corollary 5.3 *Suppose that* (3.1) *and* (5.2) *hold. Then, the set E is a bounded subset of X^α, the solution operator $\{T(t), t \geq 0\}$ for* (5.1) *is point dissipative and it has a global attractor $\mathcal{A}$. In addition*

$$\phi(x) \in [\bar{\xi}, \xi]^N, \quad \forall x \in \bar{\Omega}$$

for alll $\phi \in \mathcal{A}$ and

$$\mathcal{A} = \omega^u(E)$$

Furthermore if each element of E is hyperbolic, E is finite and

$$\mathcal{A} = \cup_{x \in E} W^u(x).$$

If in addition

$$\frac{\partial f_i}{\partial u_j} > 0, \quad \text{for} \quad i \neq j, \tag{5.5}$$

then (5.1) is a cooperative system if Ω is a convex domain (see Kishimoto and Weinberger [1985]). For such systems the following result holds

Proposition 5.4 *Let $\bar{u}$ be a nonconstant equilibrium solution of* (5.1). *Suppose that* (5.6) *holds on the range of $\bar{u}$. Then $\bar{u}$ is unstable.*

Corollary 5.5 *If* (5.6) *holds in Σ then every nonconstant equilibrium solution for* (5.1) *is unstable; that is, if $\bar{u} \in E$ is stable then $\bar{u}$ is constant.*

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