ISSN - 0103-2577

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JOSEMAR RODRIGUES

Nº 120

N O T A S D O I C M S C

São Carlos (SP)
Sept. 1992

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Josemar Rodrigues
ICMSC - USP - 13560 - 970 - São Carlos - SP - Brasil

Abstract

The purpose of this paper is to consider the estimation problem for the exponential mean time to failure which reflects both goodness of fit and precision. This can be obtained by using the Weighted Balanced Loss Function (WBLF) which generalizes the Balanced Loss Function (Zellner, 1991) used for estimation of a scalar mean. The optimal estimator relative to WBLF creates a kind of balance between Bayesian and non-Bayesian estimators.

Key Words: Admissibility; Bayesian and non-Bayesian estimators.

1 Introduction

In life testing research the most widely used life distribution is the exponential model with probabilty density function

$$f(x \mid \theta) = \frac{1}{\theta}\exp\{-\frac{x}{\theta}\}, \quad x \geq 0, \quad \theta > 0. \tag{1}$$

The reliability function $R(t) = \exp\{-\theta t\}$ and the failure rate $\gamma = \frac{1}{\theta}$. Thus, for a situation where failure rate is constant the exponential model would be a ideal choice.

In this paper estimation procedures for the mean time to failure θ, using a Weighted Balanced Loss Function (WBLF), are considered in detail. The

WBLF is an extension of the Balanced Loss Function (BLF) for a scalar mean introduced by Zellner (1991) in his recent paper which reflects both goodness of fit and precision of the estimator of θ. Let $x' = (x_1, \ldots, x_n)$, the observed life test vector from exponential distribution defined by (1) where $E_\theta(X) = \theta$ and $V_\theta(X) = \theta^2$. Our problem is to estimate θ, assuming a noninformative and informative priors for θ.

In Section 2, the WBLF is introduced and the optimal estimate relative to WBLF is derived. It will be seen that Bayesian and non-Bayesian standard estimators for the mean time to failure, θ, are dominated in terms of WBLF by an optimal estimator relative to WBLF. The optimal estimator relative to WBLF is also a kind of balance or weighted average between Bayesian and non-Bayesian estimators of θ.

2 Estimation of the exponential mean time to failure relative to Weighted Balanced Loss Function (WBLF)

A WBLF for θ, denoted by $L_B(\hat{\theta}, \theta)$, where $\hat{\theta}$ is some estimate, is given by

$$L_B(\hat{\theta}, \theta) = w \frac{\sum_i^n (X_i - \hat{\theta})^2}{n Var_{\hat{\theta}}[X]} + (1 - w) \frac{(\hat{\theta} - \theta)^2}{Var_{\hat{\theta}}[X]} \tag{2}$$

and w having a given value such that $0 \leq w \leq 1$. Taking $Var_{\hat{\theta}}[X] = 1$ we obtain the BLF introduced by Zellner (1991). As emphasized by Zellner (1991), the first term on the r.h.s of (2) represents goodness of fit while the second term represents precision of estimation. In this paper, from (1), we have that $Var_{\hat{\theta}}[X] = \hat{\theta}^2$. We shall show in this section how optimal estimators are obtained from (2) and how sensitive they are to the choice of the value of w. Usually, many employ $w = 0$ or $w = 1$. More discussions of the loss function in (2) will be given by Lemma 2.

Definition 1:

Conditional on the data and prior information, θ_{**} is optimal for θ if it minimizes the posterior expected balanced loss, that is,

$$EL_B(\theta_{**}, \theta) = \min_{\hat{\theta}} EL_B(\hat{\theta}, \theta). \tag{3}$$

The following Lemma will be very useful to find the optimal estimator θ_{**}.

Lemma 1:

Given $\hat{\theta}$ and constants A,B and C, we have that:

$$\frac{A\hat{\theta}^2 - 2B\hat{\theta} + C}{n\hat{\theta}^2} = \lambda + \frac{A - n\lambda}{n\hat{\theta}^2}(\hat{\theta} - \theta_X)^2 \tag{4}$$

where

$$\theta_X = \frac{B}{A - n\lambda} \quad \text{and} \tag{5}$$

$$\lambda = \frac{1}{n}(A - \frac{B^2}{C}). \tag{6}$$

Proof:

It is easy to see, for some fixed λ, that:

$$\begin{aligned}
\frac{A\hat{\theta}^2 - 2B\hat{\theta} + C}{n\hat{\theta}^2} &= \lambda + \frac{A - n\lambda}{n\hat{\theta}^2}[\hat{\theta}^2 - 2\hat{\theta}\frac{B}{A - n\lambda} + \frac{C}{A - n\lambda}] = \\
&= \lambda + \frac{A - n\lambda}{n\hat{\theta}^2}(\theta - \theta_X)^2 \iff \\
\theta_X &= \frac{B}{A - n\lambda} \quad \text{and} \\
\lambda &= \frac{1}{n}(A - \frac{B^2}{C}).
\end{aligned}$$

A similar proof was used by Sprent (1966) in a more general context. The next result is a trivial application of Lemma 1.

Lemma 2:

Under the exponential model (1) we have that

1.

$$\frac{\sum_{i=1}^{n}(X_i-\hat{\theta})^2}{n\hat{\theta}^2}=\lambda+\frac{(1-\lambda)}{\hat{\theta}^2}(\hat{\theta}-\theta_S)^2 \tag{7}$$

where

$$\begin{aligned}
\theta_S &= \frac{\overline{X}}{1-\lambda} && (8)\\
&= [1-\frac{\sum_{i=1}^{n}(X_i-\overline{X})^2}{n\overline{X}^2}]\overline{X} = \text{James - Stein estimator, (1961)} \\
\lambda &= \frac{\sum_{i=1}^{n}(X_i-\overline{X})^2}{\sum_{i=1}^{n}X_i^2} \quad \text{and} && (9)\\
\overline{X} &= \frac{\sum_{i=1}^{n}X_i}{n}.
\end{aligned}$$

2.

$$\frac{(\hat{\theta}-\overline{\theta})^2+v}{\hat{\theta}^2}=\frac{1}{n-1}+\frac{n-2}{(n-1)\hat{\theta}^2}(\hat{\theta}-\theta_B)^2. \tag{10}$$

where $v = E(\theta-\overline{\theta})^2$ = posterior variance, $\overline{\theta}$ = posterior mean and $\theta_B = (\frac{n-1}{n-2})\overline{\theta}$.

Proof:

The results follow from Lemma 1 by identifying A, B and C.
Lemma 2 (part 1) motivates the loss function in (2) because having $\hat{\theta}$ close to θ_S is reasonable as well having $\hat{\theta}$ close to θ in a relative square error sense.

Sinha/Gutman (1976) state that a non-informative prior for θ is proportional to $1/\theta$. Thus, from Martz/Waller (1982), p.346, we have that:

$$\overline{\theta}=\frac{n\overline{X}}{n-1} \quad \text{and} \quad v=\frac{n^2\overline{X}^2}{(n-1)^2(n-2)}. \tag{11}$$

For this non-informative case we have that $\theta_B = [n/(n-2)]\overline{X}$. It is interesting to note that λ in Lemma 2 can be used as a measure to test the model (1). If $\hat{\lambda} = \frac{2n\lambda}{n-1}$ is close to one the model is accepted.

From Lemma 2, we can express (2) as folows:

$$L_B(\hat{\theta}, \theta) = w[\lambda + \frac{(1-\lambda)}{\hat{\theta}^2}(\hat{\theta} - \theta_S)^2] + (1-w)\frac{(\hat{\theta} - \theta)^2}{\hat{\theta}^2}. \quad (12)$$

Using the posterior probability function for θ, the posterior expected loss is:

$$EL_B(\hat{\theta}, \theta) = w[\lambda + \frac{1-\lambda}{\hat{\theta}^2}(\hat{\theta} - \theta_S)^2] + (1-w)[\frac{(\hat{\theta} - \overline{\theta})^2 + v}{\hat{\theta}^2}]. \quad (13)$$

From Lemma 2, Lemma 1 and a non-informative prior for θ, after tedious algebra, we can write (13) as follows:

$$EL_B(\hat{\theta}, \theta) = 1 - \alpha + \frac{\alpha}{\hat{\theta}^2}(\hat{\theta} - \theta_{**})^2, \quad (14)$$

where

$$\begin{aligned} \theta_{**} &= \frac{w\theta_S + (1-w)\frac{n}{(n-1)}\theta_B}{w + (1-w)\frac{n}{(n-1)}} \quad (15) \\ &= \Sigma\theta_S + (1-\Sigma)\theta_B, \\ \Sigma &= \frac{w}{w + (1-w)\frac{n}{(n-1)}} \quad \text{and} \\ \alpha &= \frac{[w + (1-w)\frac{n}{(n-1)}]^2}{\frac{w}{1-\lambda} + (1-w)\frac{n^2}{(n-2)(n-1)}}. \quad (16) \end{aligned}$$

From (14), it is clear that θ_{**} is the value of $\hat{\theta}$ that leads to minimal posterior expected loss and is thus the Bayesian estimate of θ relative to the WBLF in (2). Thus, conditional on the data and noninformative prior for θ, θ_{**} in (15) is optimal in the sense of providing minimal posterior weighted expected loss.

It is obvious from (15), that θ_{**} is a combined estimator of two non-Bayesian and Bayesian estimators for $w = 1$ and $w = 0$, respectively. A similar result was obtained by Zellner (1991) for a scalar mean and kown variance. If $w = 1$ in (15), $\theta_{**} = \theta_S$ while if $w = 0$, $\theta_{**} = \theta_B$.

From (14), posterior expected loss given $\hat{\theta} = \theta_{**}$ is:

$$EL_B(\theta_{**}, \theta) = 1 - \alpha. \tag{17}$$

It is of interest to compare (17) with posterior expected loss associated with θ_S and θ_B. We have from (14) and (17):

$$\begin{aligned} \Delta = EL_B(\theta_S, \theta) - EL_B(\theta_{**}) &= \alpha\frac{(\theta_S - \theta_{**})^2}{\hat{\theta}^2} \\ &= (1-\Sigma)^2(1-\frac{\theta_B}{\theta_S})^2[\frac{(w/\Sigma)^2}{u}]. \end{aligned} \tag{18}$$

where

$$u = w\frac{1}{1-\lambda} + (1-w)\frac{n^2}{(n-1)(n-2)}. \tag{19}$$

From (18), it is seen that the inflation of the posterior expected loss, associated with use of θ_S, depends on n, w, λ and θ_B/θ_S. For given n, w and λ, Δ grows with the difference between θ_B and θ_S. Similar conclusion is true for θ_B. Further,

$$\frac{\Delta}{EL_B(\theta_{**}, \theta)} = (1-\Sigma)^2 Z^2 (w/\Sigma)^2. \tag{20}$$

where

$$Z^2 = \frac{(1 - \frac{\theta_B}{\theta_S})^2}{u - (\frac{w}{\Sigma})^2}. \tag{21}$$

If, for example, $w = \Sigma = 1/2$ and $Z = 1$, (20) is equal to 0.25. That is, under these conditions expected loss is inflated by 25% by use of θ_S rather than the optimal estimate θ_{**} given in (18).

In Table 1, (20) has been evaluated for various values of Z^2 and $w = \Sigma$. Table 1 shows that relative loss (20) grows with Z^2 for given values of w.When $Z^2 = 0$, $\theta_S = \theta_B = \theta_{**}$. Also if $w = 1$, $\theta_{**} = \theta_S$ and thus the relative loss is equal to zero. Similar considerations relate to θ_B.

Table 1: Relative Loss in (20) for Various Values of w and Z^2

Z^2	w				
	0	0.25	0.50	0.75	1.00
0	0	0	0	0	0
0.50	0.50	0.28	0.12	0.03	0
1.00	1.00	0.56	0.25	0.06	0
2.00	2.00	1.12	0.5	0.12	0
4.00	4.00	2.25	1.0	0.25	0

An engeneer's judgement can be introduced into the analysis by using an informative prior for θ, say

$$\pi(\theta) \propto \frac{1}{\theta^{\nu+1}} \exp\{-\frac{a}{\theta}\}, \quad \nu, a > 0. \tag{22}$$

Thus, from Martz and Waller (1982), p.48, we have that

$$\overline{\theta} = \frac{n\overline{X} + a}{n + \nu - 1} \quad \text{and} \quad v = \frac{(n\overline{X} + a)^2}{(n + \nu - 1)^2 (n + a - 2)}. \tag{23}$$

Lemma 3:

Under the exponential model (1) and the informative prior in (22), we have that

$$\frac{(\hat{\theta} - \overline{\theta})^2 + v}{\hat{\theta}^2} = \frac{1}{n + a - 1} + \frac{n + a - 2}{(n + a - 1)\hat{\theta}^2} (\hat{\theta} - \theta_{BI})^2, \tag{24}$$

where $\theta_{BI} = [(n + a - 1)/(n + a - 2)]\overline{\theta}$.

Proof:

The result follows from Lemma 1 by identifying A, B and C.
Again, from Lemma 1 and 3 after tedious but trivial algebra, we can write (13) as follows:

$$EL_B(\hat{\theta}, \theta) = 1 - \beta + \frac{\beta}{\hat{\theta}^2} (\hat{\theta} - \theta_{**})^2, \tag{25}$$

where

$$\theta_{**} = \frac{w\overline{X}\theta_S + (1-w)\overline{\theta}\theta_{BI}}{w\overline{X} + (1-w)\overline{\theta}} \quad \text{and} \tag{26}$$

$$\beta = \frac{[w\overline{X} + (1-w)\overline{\theta}]^2}{\frac{w}{1-\lambda}\overline{X}^2 + (1-w)(\frac{n+a-1}{n+a-2})\overline{\theta}^2}. \tag{27}$$

Also, in the informative case,the posterior mean can be write as $\overline{\theta} = c\overline{X} + (1-c)\theta_0$, where c has a given value, $0 < c < 1$, and θ_0=prior mean. If $w = 1/2$ and $\theta_0 > \overline{X}$, then θ_{BI} has a greater weight in the expression for θ_{**}, than in that for θ_S. Similary, for $\theta_0 < \overline{X}$.

From (25), posterior expected loss given $\hat{\theta} = \theta_{**}$ is:

$$\begin{aligned} EL_B(\theta_{**}, \theta) &= 1 - \beta \\ &= 1 - \frac{w\overline{X} + (1-w)\overline{\theta}}{\theta_{**}}. \end{aligned} \tag{28}$$

3 Conclusions

The WBLF reflects goodness of fit and precision of estimation. Traditional loss functions such as the sum of weighted squared residuals and mean squared error of estimation can be obtained from WBLF by setting w equal to one or zero. However, there are many practical situations in which w between zero and one is more useful than the extreme values zero and one. For the exponential model, the optimal point estimator relative to WBLF is easy to derive, has a remarkably simple form and dominates other estimators. In conclusion, use of a WBLF creates a kind of balance between Bayesians and non-Bayesians, that is, it leads to a simple estimator that is a linear combination of the James - Stein estimator θ_S and the Bayes estimator θ_B (the non-informative case) or θ_{BI} (the informative case).

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