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A REMARK ON THE SEPARATION BY IMMERSIONS IN CODIMENSION-1

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Abstract. Let $f : M^{n-1} \to N^n$ be an immersion with normal crossings from a closed connected $(n-1)$-manifold M into a connected, open or closed n-manifold N. In this paper, we give a necessary and sufficient condition for f to be an embedding using the number of connected components of $N - f(M)$.

1. Introduction

Let M^{n-1} and N^n be *closed* connected manifolds and $f : M \to N$ an immersion with normal crossings. In [BMS] it is obtained a characterization of f as an embedding, which is a converse to the Jordan-Brouwer theorem, under certain conditions. Note that the same characterization had been obtained in [BF] under a more restricted condition.

In this paper, we give some results which correspond to those of [BMS], when N^n is an arbitrary connected n-manifold, open or closed. More specifically our main results are as follows:

THEOREM 1.1. *Let $f : M^{n-1} \to N^n$ be an immersion with normal crossings between connected manifolds such that M is closed and that the self-intersection set of f is non-empty.*

(1) If M is orientable and $H_1(N; \mathbf{Z}_2) = 0$ or $H_{n-1}(N; \mathbf{Z}_2) = 0$, then $\beta_0(N - f(M)) \geq 3$, where $\beta_0(N - f(M))$ denotes the number of connected components of $N - f(M)$.

(2) If $i_* : H_{n-1}(f(M); \mathbf{Z}_2) \to H_{n-1}(N;\ \mathbf{Z}_2)$ vanishes and if the normal bundle of the immersion f is trivial, then $\beta_0(N - f(M)) \geq 3$, where $i : f(M) \to N$ is the inclusion map.

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2. Proof of Theorem 1.1

In all that follows, M^{n-1} and N^n are connected manifolds of dimensions $n-1$ and n respectively, M is closed, $f : M \to N$ is an immersion with normal crossings and the homologies are with coefficients in $\mathbf{Z}_2$. Let $A(\subset M)$ be the self-intersection set of f; i.e., $A = \{p \in M; f^{-1}(f(p)) \neq \{p\}\}$. We denote by $B = f(A)$ the set of multiple values of f. Since f is an immersion with normal crossings, there exists a fundamental class $[A] \in H_{n-2}(M)$, which does not vanish ([**H**]).

LEMMA 2.1. *If* $H_1(N) = 0$, *then* $\beta_0(N - f(M)) = 1 + \beta_{n-1}(f(M))$, *where* $\beta_i(X) = \dim H_i(X)$ *for a topological space* X.

Proof. If N is closed, then the result follows from Lemma 2 in [**BF**]. Here we give a proof which works even when N is open. Consider the exact sequence:

$$\tilde{H}^0(N) \to \tilde{H}^0(N - f(M)) \to H^1(N, N - f(M)) \to H^1(N).$$

Since $\tilde{H}^0(N) = H^1(N) = 0$, it follows that $\beta_0(N - f(M)) = 1 + \dim H^1(N, N - f(M))$. On the other hand, by excision and Poincaré-Lefschetz duality, we see that $H^1(N, N - f(M)) \cong H_{n-1}(f(M))$. Thus we have $\beta_0(N - f(M)) = 1 + \beta_{n-1}(f(M))$. □

LEMMA 2.2. (1) $\beta_0(N - f(M)) \geq 1 + \dim \ker(i_* : H_{n-1}(f(M)) \to H_{n-1}(N))$, *where* $i : f(M) \to N$ *is the inclusion map.*

(2) $\beta_{n-1}(f(M)) = 1 + \dim \ker(\alpha : H_{n-2}(A) \to H_{n-2}(B) \oplus H_{n-2}(M))$, *where* $\alpha = (f|A)_* \oplus j_*$ *and* $j : A \to M$ *is the inclusion map.*

Proof. If N is closed, the results follow from Lemma 3.1 in [**BMS**]. In fact the inequality in (1) is an equality in this case. Hence we assume that N is open.

(1) Set $K = \ker i_* \subset H_{n-1}(f(M))$. Since $f(M)$ has the homotopy type of a finite CW complex, there are a finite number of generators $\mu_k = [c_k]$ $(1 \leq k \leq m)$ of K, where c_k are cycles in $f(M)$. Let C_k $(1 \leq k \leq m)$ be n-chains in N with boundary c_k. Take a connected compact codimension-0 submanifold W of N such that $f(M) \cup \bigcup_{k=1}^{m} C_k \subset \mathrm{Int}W$. Note that $\beta_0(N - f(M)) \leq \beta_0(W - f(M))$ in general. However, since $\beta_0(N - f(M))$ is finite, we may assume that $\beta_0(N - f(M)) = \beta_0(W - f(M))$. Then it is easy to see that $K = \ker(i_* : H_{n-1}(f(M)) \to H_{n-1}(W))$. On the other hand, consider the double N' of W. Since W is a retract of N', we see that $i'_* : H_{n-1}(W) \to H_{n-1}(N)$ is injective, where $i' : W \to N'$ is the canonical inclusion. Therefore, $K = \ker(\bar{i}_* : H_{n-1}(f(M)) \to H_{n-1}(N'))$, where $\bar{i}$ is the composite of the inclusion maps $f(M) \subset W \subset N'$. Furthermore, it is easy to see that $\beta_0(N - f(M)) = \beta_0(W - f(M)) \geq \beta_0(N' - \bar{i}(f(M)))$. Since N' is closed, we see that $\beta_0(N - f(M)) \geq \beta_0(N' - f(M)) = 1 + \dim K$ by Lemma 3.1 (1) of [**BMS**]. This completes the proof of (1).

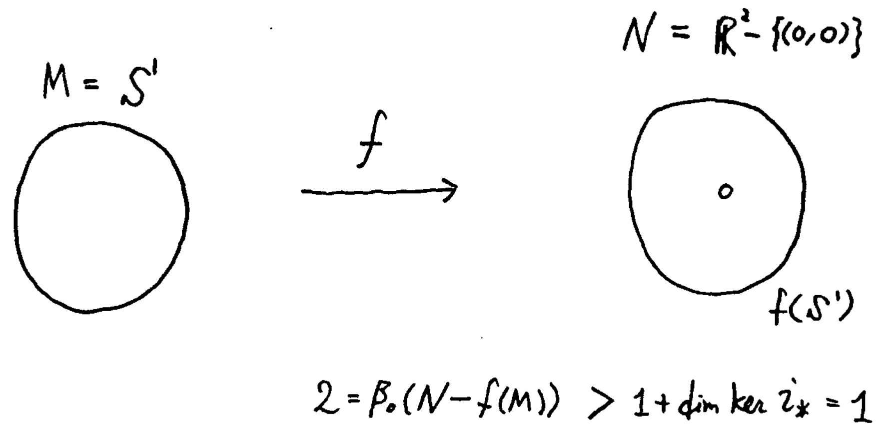

Figure 1

(2) The proof follows from the same argument which was used in Lemma 3.1 (2) in [**BMS**]. □

REMARK 2.3. Note that, in Lemma 2.2 (1), the equality does not hold in general (see Figure 1). However, if $i_* : H_{n-1}(f(M)) \to H_{n-1}(N)$ vanishes, then we always have the equality. The proof goes as follows. Let E be a codimension-0 submanifold of N with boundary such that $E \cap f(M) = \emptyset$ and $N - \mathrm{Int}E$ is a compact connected manifold with boundary. For example, let E be the exterior of a regular neighborhood of $f(M)$ in N. Then we have the exact sequence as follows:

$$H_n(E \cup f(M), E) \to H_n(N, E) \to H_n(N, E \cup f(M)) \to$$

$$H_{n-1}(E \cup f(M), E) \xrightarrow{\gamma} H_{n-1}(N, E).$$

Note that $H_n(E \cup f(M), E) = 0$ and that $H_n(N, E) \cong \mathbf{Z}_2$. Now, since $i_* : H_{n-1}(f(M)) \to H_{n-1}(N)$ vanishes, we see that $\gamma = 0$. Thus, $H_n(N, E \cup f(M)) \cong \mathbf{Z}_2 \oplus H_{n-1}(f(M))$, since $H_{n-1}(E \cup f(M), E) \cong H_{n-1}(f(M))$. On the other hand, we have $H_n(N, E \cup f(M)) \cong H^0(N - (E \cup f(M)))$ by duality and it is easy to see that $\beta_0((N - E) - f(M)) \geq \beta_0(N - f(M))$. Thus we have $\beta_0(N - f(M)) \leq \dim H_n(N, E \cup f(M)) = 1 + \beta_{n-1}(f(M))$. Combining this inequality with the inequality of Lemma 2.2 (1), we obtain $\beta_0(N - f(M)) = 1 + \dim \ker(i_* : H_{n-1}(f(M)) \to H_{n-1}(N))$.

When N is closed, Theorem 1.1 follows from Theorem 1.4 of [**BMS**]. Hence, in the following, we assume that N is open.

Proof of Theorem 1.1 (1). First suppose that $H_1(N) = 0$. Then we see that $\beta_0(N - f(M)) = 1 + \beta_{n-1}(f(M))$ by Lemma 2.1. On the other hand, $\beta_{n-1}(f(M)) = 1 + \dim \ker \alpha$ by Lemma 2.2 (2). Since $[A] \in H_{n-2}(M)$ satisfies $[A] \neq 0$ and $\alpha([A]) = 0$ (see Lemma 2.3 of [**BMS**]), we see that $\beta_{n-1}(f(M)) \geq 2$. Thus we have $\beta(N - f(M)) \geq 3$.

Now assume $H_{n-1}(N) = 0$. By Lemma 2.2 (1), we see that $\beta_0(N - f(M)) \geq 1 + \beta_{n-1}(f(M))$ (in fact, this is an equality by Remark 2.3). Then the same argument as above shows that $\beta_0(N - f(M)) \geq 3$ (cf. Lemma 2.4 of [BMS]). This completes the proof. □

Proof of Theorem 1.1 (2). Since $i_* : H_{n-1}(f(M)) \to H_{n-1}(N)$ vanishes, we see that $\beta_0(N - f(M)) \geq 1 + \beta_{n-1}(f(M))$ by Lemma 2.2 (1). Then by Lemma 2.2 (2) and Lemma 2.4 of [BMS], we see that $\beta_{n-1}(f(M)) \geq 2$. Thus we have $\beta_0(N - f(M)) \geq 3$. This completes the proof. □

REMARK 2.4. If both M and N are orientable, Theorem 1.1 is proved in a more general context in [S]. Note that there exist (open) n-manifolds N with $H_{n-1}(N) = 0$ which is non-orientable. In this case, the result of [S] cannot be applied.

3. Applications

As a consequence of Theorem 1.1, one gets a characterization of embeddings among codimension-1 immersions with normal crossings.

COROLLARY 3.1. *Let M and N be connected manifolds of dimensions $n-1$ and n respectively such that M is closed. Suppose that M is orientable and $H_1(N) = 0$ or $H_{n-1}(N) = 0$. Then an immersion with normal crossings $f : M \to N$ is an embedding if and only if $f(M)$ separates N into exactly two connected components.*

REMARK 3.2. If N is closed, Corollary 3.1 has been obtained by Biasi-Motta-Saeki in [BMS]. Note also that, under the hypothesis that N be closed and $H_1(N) = H_1(M) = 0$, Corollary 3.1 has been obtained by Biasi-Fuster in [BF].

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