ISSN - 0103-2577

Comparing Several Accelerated Life Models

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Nº 112

NOTAS DO ICMSC

São Carlos (SP)
maio 1992

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Summary

This paper presents two approaches for comparing several exponential accelerated life models under the usual stress levels. The approaches are based on on the likelihood ratio statistics and on the posterior Bayes factor (Aitkin, 1991). These procedures can be useful in many practical situations. An exact distribution and a table of critical values of the likelihood ratio statistics are considered. A simulation study is also performed.
Key words: accelerated life models; posterior Bayes factor; exponential; Laplace's method; likelihood ratio statistics.

1 Introduction

Accelerated life testing of a product is often used to obtain information on its performance under usual conditions. Such testing involves conditions more severe than encountered in the everyday's life. This results in decreasing the item's mean life,leads to shorter test times and reduces experimental costs. The basic idea is to collect data at high stresses levels and use it to extrapolate to the usual stress level where testing is not possible. Several authors have considered this problem, see Nelson (1990) and Bhattacharyya and Soejoeti (1989) for a complete reference. However, most of the articles in the literature related to the accelerated data are concerned to the estimation of the parameters involved in the model. There are a few papers devoted to testing hypothesis, for example, see J. Yum and H. Kim (1990). At present, we are unware of statistical literature related to the the comparison of the mean time to failure of products under the usual stress level via the accelerated data.

In this paper, to compare the mean time to failure of a product under the usual stress level , we make use of the posterior Bayes factor (Aitkin, 1991) and also of the likelihood ratio statistics. As mentioned by Aitkin, the comparison of models via posterior Bayes factor may be applicable for a wide class of models and does not suffer from the Lindley's paradox. To compute the posterior Bayes factor, we have to compute the posterior mean of the likelihood function $L_h(\theta_h)$ of the observed data D and the parameter θ_h, which corresponds to model M_h, $h = 1, 2$, or,

$$\begin{aligned} \overline{L}_h^A &= \int L_h(\theta_h)\pi(\theta_h \,|)d\theta_h, \\ &= \frac{\int L_h^2(\theta_h)\pi(\theta_h)}{\int L_h(\theta_h)\pi(\theta_h)d\theta_h}, \end{aligned}$$

where $\pi(\theta_h)$ is the prior density assigned to the parameter θ_h. The posterior Bayes factor for model M_1 over model M_2 is then given by

$$A = \frac{\overline{L}_1^A}{\overline{L}_2^A}.$$

The posterior Bayes factor provides a measure of the weight of sample evidence in favor of M_1 over M_2. As pointed out by Aitkin, (1991), values of A less than 1/20, 1/100 or 1/1000 constitute strong, very strong and overwhelming sample evidence aginst M_1 in favour of M_2.

In Section 2, we formulate the exponential accelerated model by introducing the power rule model (see Mann et al., 1974) which puts some relationships between the parameters of the model and the invironmental conditions. Section 3 is devoted to justify the amendement of the power rule model given by Mann et al., 1974, p. 425, via the orthogonal parametrization (Cox and Reid, 1987), in order to simplify the numerical determination of the M.L.E.

involved in the model. In Section 4, we obtain the posterior Bayes factor and the likelihood ratio statistics. Section 5 provides the exact distribution of the proposed statistics and its critical points. Also, in Section 6, a numerical illustration of the repeated sampling property of the likelihood ratio statistics is considered. It is shown that, as the number of items on test grows, the significance level of the likelihood ratio statistics computed by considering that the extra parameter in the power rule is known is close to the significance level when the likelihood ratio statistics is computed by treating the extra parameter as estimated from the data.

2 Formulation of the Model

Suppose that p sets of independent accelerated data, using a type II censoring, are available and have been drawn from an exponential density under the stress V_i given by

$$f(t,\theta_{ij}) = \frac{1}{\theta_{ij}}\exp\{-\frac{t}{\theta_{ij}}\}, \qquad \theta_{ij} > 0 \tag{1}$$
$$i = 1,\ldots,k \quad \text{and} \quad j = 1,\ldots,p.$$

The unknown parameter θ_{ij} is the mean to failure of the component of type j under the stress V_i. The test terminates after a fixed number r_i of failures $t_{ij1},\ldots,t_{ijr_i}$ have occurred when n_i components of type j are put on test under the stress V_i. Let θ_{1j} be the mean time to failure of the component of type j under the usual stress V_1, $j = 1.\ldots,p$.

In this paper, we adopt the power rule model (see Mann et al., 1974) as the relationships between the mean time to failure and the stress V_i, that is,

$$\theta_{ij} = \frac{\alpha_j}{V_i^{\beta_j}} \tag{2}$$

where

$$\alpha_j \;>\; 0 \qquad -\infty < \beta_j < \infty,$$
$$i = 1,\ldots,k \qquad \text{and} \qquad j = 1,\ldots,p.$$

Let:

$$V^* \;=\; \Pi_{i=1}^k V_i^{\frac{r_i}{r}},$$

$$
\begin{aligned}
r &= \sum_{i=1}^{k} r_i, \\
\hat{\theta}_{ij} &= \frac{A_{ij}}{r_i} \qquad \text{and} \qquad A_{ij} = \sum_{l=1}^{r_i} t_{ijl} + (n_i - r_i) t_{ijr_i}.
\end{aligned}
$$

The likelihood function for (α_j, β_j), given the data set $\{V_i, \hat{\theta}_{ij}, r_i\}$, $i = 1, \ldots, k$, $j = 1, \ldots, p$, is

$$
L(\alpha_j, \beta_j) = \alpha_j^{-r} (V^*)^{r\beta_j} \exp\{-\frac{1}{\alpha_j} \sum_{i=1}^{k} A_{ij} V_i^{\beta_j}\}. \tag{3}
$$

One important problem is to test: $H_o : \theta_{1j} = \theta_1$, for $j = 1, \ldots, p$ against $H_1 : \theta_{1j} \neq \theta_1$, at least for one j. In this paper, to test H_o we make use of the posterior Bayes factor (Aitkin, 1991) and the likelihood ratio statistics. To make easy this comparison we use the orthogonal parametrization (Cox and Reid, 1987). To the best of our knowledge, we are unware of any statistical literature dealing with such problem.

3 The orthogonal parametrization

It is easy to verify that the Fisher information matrix for (α_j, β_j) is given by

$$
I(\alpha_j, \beta_j) = \begin{pmatrix} \frac{r}{\alpha_j^2} & -\frac{rV^*}{\alpha_j} \\ -\frac{rV^*}{\alpha_j} & \sum_{i=1}^{k} r_i ln^2 V_i \end{pmatrix}. \tag{4}
$$

To obtain an estimator of β_j which is stable with respect to α_j, we solve the differential equation (Cox and Reid, 1987)

$$
\frac{d\alpha_j}{d\beta_j} = ln(V^*), \tag{5}
$$

given the parametrization,

$$
\begin{aligned}
\alpha_j &= (V^*)^{\beta_j} \lambda_j \\
\beta_j &= \beta_j.
\end{aligned} \tag{6}
$$

The parameters λ_j and β_j are orthogonal in Cox and Reid's sense, that is, the Fisher information matrix in terms of λ_j and β_j is diagonal, or,

$$I(\lambda_j, \beta_j) = \begin{pmatrix} \frac{r}{\lambda_j^2} & 0 \\ 0 & \sum_{i=1}^{k} r_i ln^2(\frac{V_i}{V^*}) \end{pmatrix}. \tag{7}$$

The likelihood of (λ_j, β_j) is

$$L(\lambda_j, \beta_j) = \lambda_j^{-r} \exp\{-\frac{1}{\lambda_j} \sum_{i=1}^{k} A_{ij} (\frac{V_i}{V^*})^{\beta_j}\}. \tag{8}$$

From (6), we may write the power rule model (2) as

$$\begin{aligned} \theta_{ij} &= \frac{\lambda_j}{(\frac{V_i}{V^*})^{\beta_j}}, \\ j &= 1, \ldots, p \quad \text{and} \\ i &= 1, \ldots, k. \end{aligned} \tag{9}$$

This orthogonal paametrization and (9) give an important justification of the amendment of (2) suggested by Mann et al. (1974). It is easy to see from (8), since V_i is chosen at random from the k values $V_i, i = 1, \ldots, k$, that the maximun likelihood estimator (MLE) of β_j is the solution of the equation

$$\sum_{i=1}^{k} A_{ij} (\frac{V_i}{V^*})^{\hat{\beta}_j} ln(\frac{V_i}{V^*}) = 0. \tag{10}$$

As emphasized by Cox and Reid (1987), the estimator $\hat{\beta}_j$ is stable with respect to λ_j. The next result provides a justification of why taking $\beta_j = \hat{\beta}_j$ when testing H_o via the posterior Bayes factor or via the likelihood ratio statistics.

Let $\pi(\beta_j \mid \lambda_j)$ be the conditional prior of β_j given λ_j. Following Sweeting (see Cox and Reid, 1987), we introduce a new likelihood function from the Bayesian point of view:

Definition:

The integrated likelihood, $L(\lambda_j)$, is defined as

$$\begin{aligned} L(\lambda_j) &= \int L(\lambda_j, \beta_j) \pi(\beta_j \mid \lambda_j) d\beta_j, \\ \text{for} \quad j &= 1, \ldots, p. \end{aligned} \tag{11}$$

Lemma 1:

Under the accelerated life model (1) and $\pi(\beta_j \mid \lambda_j) \propto \frac{1}{\lambda_j^{1/2}}$ (the approximated data-translated prior, Box and Tiao, 1973) we have that

$$L(\lambda_j) = L(\lambda_j, \hat{\beta}_j), \tag{12}$$

$$\propto \lambda_j^{-r_j} \exp\{-\frac{1}{\lambda_j}\sum_{i=1}^{k} A_{ij}(\frac{V_i}{V^*})^{\hat{\beta}_j}\}, \tag{13}$$

where $L(\lambda_j, \hat{\beta}_j)$ is the profile likelihood and $\hat{\beta}_j$ is the MLE defined in (10).

Proof:

The result (12) follows from (8) and (10). The result (13) is a trivial application of the Laplace approximation (Tierney and Kadane, 1986) to the integral

$$L(\lambda_j) = \int \lambda_j^{-(r+1/2)} \exp\{-\frac{1}{\lambda_j}\sum_{i=1}^{k} A_{ij}(\frac{V_i}{V^*})^{\beta_j}\} d\beta_j.$$

When λ_j and β_j are taken to be a prior independent, the integrated likelihood is equal to the conditional profile likelihood (Cox and Reid, 1987). Motivated by the above Lemma, we take $\beta_j = \hat{\beta}_j$ and considering the parametrization

$$\theta_{1j} = \frac{\lambda_j}{(\frac{V_1}{V^*})^{\hat{\beta}_j}}, \tag{14}$$

we have the following "likelihood function " for θ_{1j}:

$$L(\theta_{1j}) \propto \theta_{1j}^{-r_j} \exp\{-\frac{1}{\theta_{1j}}\sum_{i=1}^{k} A_{ij}(\frac{V_i}{V_1})^{\hat{\beta}_j}\}. \tag{15}$$

Our purpose is to compute the posterior Bayes factor and the likelihood ratio statistics, using (15), for testing $H_o : \theta_{1j} = \theta_1$ for $j = 1, \ldots, p$ versus $H_1 : \theta_{1j} \neq \theta_1$,for some j.

4 Posterior Bayes Factor (Aitkin, 1991)

Let M_1 be a model under H_o and M_2 a model under H_1. The posterior Bayes factor for model M_1 over M_2 is given by

$$A = \frac{\overline{L}_1^A}{\overline{L}_2^A},$$

where

$$\begin{aligned} \overline{L}_h^A &= \int L_h(\theta_{(h)})\pi(\theta_{(h)} \mid data)d\theta_{(h)}, \quad h = 1, 2, \qquad (16) \\ \theta_{(1)} &= \theta_{1j} = \theta_1, \quad j = 1, \ldots, p \quad \text{and} \\ \theta_{(2)} &= (\theta_{11}, \theta_{12}, \ldots, \theta_{1p}). \end{aligned}$$

To simplify the computation of A, we can write $\overline{L}_h^A$ as

$$\overline{L}_h^A = \frac{l_2^{(h)}}{l_1^{(h)}},$$

$$l_u^{(h)} = \int L^u(\theta_{(h)})\pi(\theta_{(h)})d\theta_{(h)}, \quad u = 1, 2. \qquad (17)$$

Since, the Jeffrey's prior for (λ_j, β_j) is $\pi(\lambda_j, \beta_j) \propto 1/\lambda_j^2$ and $\pi(\beta_j \mid \lambda_j) \propto 1/\lambda_j^{1/2}$, then $\pi(\lambda_j) \propto 1/\lambda_j^{1/2}$. Thus, from (14) we have that

$$\begin{aligned} \pi(\theta_{(1)}) &\propto \frac{1}{\theta_1} \\ \pi(\theta_{(2)}) &\propto \frac{1}{\Pi_{j=1}^p \theta_{1j}}. \qquad (18) \end{aligned}$$

The main result of this section is stated next.

Theorem:

1. Under (15), the posterior Bayes factor for testing $H_o : \theta_{1j} = \theta_1$ for $j = 1, \ldots, p$, is

$$A = \frac{2^{\frac{1}{2}(1-p)} p^{-pr} \Gamma(2pr - 1/2)\Gamma^p(r - 1/2)}{\Gamma^p(2r - 1/2)\Gamma(pr - 1/2)} U, \quad \text{where}$$

$$
\begin{aligned}
U &= \frac{\Pi_{j=1}^{p}\{\sum_{i=1}^{k}\frac{r_i}{r}\hat{\theta}_{ij}(\frac{V_i}{V_1})^{\hat{\beta}_j}\}^{r}}{\{\frac{1}{p}(\sum_{j=1}^{p}\sum_{i=1}^{k}\frac{r_i}{r}\hat{\theta}_{ij}(\frac{V_i}{V_1})^{\hat{\beta}_j})\}^{pr}}, \\
&= L^{r}. \qquad (19)
\end{aligned}
$$

2. Under (15), given $\hat{\beta}_j$, U is likelihood ratio statistics.

Proof:

Combining the prior and the likelihood function under the model M_1 and M_2 and integranting out with respect to $\theta_{(1)}$ and $\theta_{(2)}$, respectively, we get

$$
l_u^{(1)} = [u\sum_{j=1}^{p}\sum_{i=1}^{k}A_{ij}(\frac{V_i}{V_1})^{\hat{\beta}_j}]^{-(pur-1/2)}\Gamma(pur-1/2)
$$

and

$$
l_u^{(2)} = u^{-pur+p/2}\Gamma^{p}(ur-1/2)\Pi_{j=1}^{p}[\sum_{i=1}^{k}A_{ij}(\frac{V_i}{V_1})^{\hat{\beta}_j}]^{-(ur-1/2)}. \qquad (20)
$$

Thus, from (17) and the above expressions we get the first part of the theorem. The second part of the theorem can be easily obtained from (15). A similar result with no censored data can be found in Nagarsenker 1980.

5 Repeated Sampling Properties of U

In this section, an exact distribution of U is obtained under the following assumption:

Assumption 1:

$$
\begin{aligned}
(i) &\quad - \quad \beta_j \text{ is known for } j = 1, \ldots p, \\
(ii) &\quad - \quad r_i = a, \quad \text{for} \quad i = 1, \ldots, k.
\end{aligned}
$$

To obtain the null distribution of U, under Assumption 1, in a closed computional form, we will use essentially the method given by Nagarsenker, (1980). The next Lemma will be useful to obtain the distribution of U in a closed form.

Lemma 2:

The h-th moment of L defined in (19), under the null hypothesis H_o and Assumption 1, is given by

$$E[L^h] = E[U^{h/ak}] = \{\frac{p^h\Gamma(ak+h)}{\Gamma(ak)}\}^p \frac{\Gamma(pak)}{\Gamma(pak+ph)}. \tag{21}$$

Proof:

To obtain the h-th moment of L defined in (21), under the null hypothesis and Assumption 1, we first write U defined in (19) as

$$U = L^{ka} = \frac{\Pi_{j=1}^{p} u_j^{ka}}{[\sum_{j=1}^{p} \frac{u_j}{p}]^{pka}}$$

where

$$u_j = \sum_{i=1}^{k} \hat{\theta}_{ij} (\frac{V_i}{V_1})^{\beta_j}. \tag{22}$$

It is well known that (Mann et al., 1974) $\hat{\theta}_{ij}$ has a gamma distribution with shape parameter a and a scale parameter $\frac{\theta_{ij}}{a}$. Thus, it is not hard to shown from (22) and (2) that u_j has a gamma distribution with a shape parameter ka and a scale parameter $\frac{\theta_{1j}}{a}$. Then, Lemma 2 follows directly from the Nagarsenker's (1980) result.

Using the method of Nagarsenker (1980), we obtain the null distribution of L in a closed computational form. This is done using the Mellin inverse transformation and Lemma 2, that is, the density function of L is

$$f(l) = K(p,a,k)\frac{1}{2\pi}\int_{-i\infty}^{i\infty} l^{-h-1}p^{ph}\{\Gamma(n+h)\}^p/\Gamma(pka+ph)dh$$

where $$K(p,a,k) = \{\Gamma(ka)\}^{-p}\Gamma(kap). \tag{23}$$

From (23), using the procedure given by Nagarsenker (1980), we have the exact distribution of L in the form

$$P[L \leq x] = (2\pi)^v p^{\frac{1}{2}-pak} \frac{\Gamma(akp)}{\{\Gamma(ak)\}^{p-1}} \sum_{r=0}^{\infty} \{D_r I_x(ak, v+r)\},$$

where

$$D_r = R_r / \Gamma(ak + v + r), \quad v = \frac{1}{2}(p-1), \tag{24}$$

$I_x(c, d)$ is the incomplete beta function and R_r is obtained by recurrent formulas defined in Nagarsenker (1980). Putting $p = 2$ in (21), it can easily be checked that the distribution of L is given by $P[L \leq x] = I_x(ak, \frac{1}{2})$.

Table 1 gives the 5% significance points of L for $k = 5$ and for various values of a and p. This table were obtained from Table 1 and 2 given by Nagarsenker 1980.

Table 1: 5% significance points of L with $k = 5$

a	$p = 2$	$p = 3$	$p = 4$
1	0.66824	0.5351	0.4433
2	0.82131	0.7363	0.6711
4	0.90732	0.8595	0.8209
6	0.93748	-	-
8	-	0.9275	0.9065

6 Numerical illustration

A simulation study presented in this numerical illustration is based on 1,000 samples generated according to the exponential model and the power rule model with $p = 2$, $k = 5$, $\beta_1 = \beta_2 = 0.7$, $\alpha_1 = \alpha_2 = 400$ and for each $n_i = a$, $i = 1, 2, 3, 4, 5$, given in Table 1 and the parameters and stresses given in Table 2.

Table 2: Parameters and the stresses of the generated data

i	V_i	$\lambda_{ij} = \frac{1}{\theta_{ij}} = \frac{1}{\theta_i}$
1	5	0.007713
2	10	0.012530
3	15	0.016642
4	20	0.020355
5	25	0.023796

For each generated sample, U (given in (17)) for β_i known and estimated by the first generated sample were computed and compared with the critical points for each a given in Table 1. Table 3 presents the number of times (out of 1,000) H_o (when true) was rejected, by using each one of the above statistics.

Table 3 : Number of rejections of H_o in 1,000 simulated samples

a	U:($\beta_1 = \beta_2 = 0.7$)	U: ($\beta_i = \hat{\beta}_i$)	$\hat{\beta}_j$ given by the 1rst generated sample
1	0.05	0.13	$\hat{\beta}_1 = 0.68$ $\hat{\beta}_2 = 1.26$
2	0.06	0.064	$\hat{\beta}_1 = 0.90$ $\hat{\beta}_2 = 0.96$
4	0.054	0.046	$\hat{\beta}_1 = 0.93$ $\hat{\beta}_2 = 0.87$
6	0.053	0.048	$\hat{\beta}_1 = 0.74$ $\hat{\beta}_2 = 0.83$

As seen from Table 3, the U statistics (when β_j:known) has a tendency of presenting the rejection rates closed to 5% . The U statistics (when β_j is estimated) has the rejection rate closed or not to 5% when the estimators of β_j are approximately equal to each other or not. The simulation studies seem to indicate that, if the number of items on test are reasonably large, then, the rejection rates that follow by using U (or L) when β_j is treated as estimated are close to the rejection rates when β_j is treated as fixed at the true value.

Acknowledgement: We are very grateful to Eric F. Rodrigues for helping us in the simulation study.

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