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SUMMARY

In this paper, we present a Bayesian Analysis of an important bivariate life model: the bivariate exponential distribution (BVED) of Block and Basu (1974) considering accelerated life tests with a power rule model. We consider a Jeffreys noninformative prior for the parameters and we use the Laplace's method for approximation of integrals to get marginal posterior densities of interest. We illustrate the proposed method with a generated data set.

Key Words: Bivariate exponential distribution, accelerated life tests, Jeffreys prior, Laplace's method.

1 Introduction

Let us assume that we have two failure times associated to each observational unit in an accelerated life test problem. That is, consider a two-component life times X and Y, J stress levels $V_1, V_2, \ldots, V_J$ and life tests are conducted at constant application of the selected stresses.

At a normal stress level V_0 assume that (X, Y) has the bivariate exponential distribution (BVED) of Block and Basu (1974) with parameters $\lambda_{10}, \lambda_{20}$ and λ_{30}. Also assume that under a stress level $V_j, j = 1, 2, \ldots, J, (X, Y)$ has the BVED with parameters $\lambda_{1j}, \lambda_{2j}$ and $\lambda_{3j}, j = 1, 2, \ldots, J$ and joint probability density function given by

$$f(x,y) = \begin{cases} \frac{\lambda_{1j}\lambda_j(\lambda_{2j}+\lambda_{3j})}{\lambda_{1j}+\lambda_{2j}} \; exp \; \{-\lambda_{1j}x - (\lambda_{2j}+\lambda_{3j})y\} & \text{if } x < y \\ \frac{\lambda_{2j}\lambda_j(\lambda_{1j}+\lambda_{3j})}{\lambda_{1j}+\lambda_{2j}} \; exp \; \{-(\lambda_{1j}+\lambda_{3j})x - \lambda_{2j}y\} & \text{if } x \geq y \end{cases} \tag{1}$$

where $\lambda_j = \lambda_{1j} + \lambda_{2j} + \lambda_{3j}, j = 0, 1, 2, \ldots, J$.

Also, consider the power rule model (see for example, Mann, Schafer and Singpurwalla, 1974), given by

$$\lambda_{ij} = c_i V_j^{\mathbb{P}} \tag{2}$$

where $i = 1, 2, 3; j = 0, 1, \ldots, J; c_1, c_2, c_3$ and $\mathbb{P}$ are unknown parameters. The model (2) was also considered by Basu and Ebrahimi (1987).

In this paper, we present a Bayesian analysis of the BVED with density (1) and the power rule model (2) considering noninformative prior densities for the parameters and the Laplace's method for approximation of integrals (see for example, Kass, Tierney and Kadane, 1990) to get the marginal posterior densities for the parameters of interest.

2 The Likelihood Function For c_1, c_2, c_3 And $\mathbb{P}$

Let (X, Y) be a nonnegative bivariate random vector representing the life times of each two-components unit in an accelerated life test problem with a BVED density (1), and the power rule model (2) for the parameter $\lambda_{ij}, i = 1, 2, 3$ under the stress variable $V_j, j = 0, 1, \ldots, J$.

Considering n_j units $(X_{1j}, Y_{1j}), (X_{2j}, Y_{2j}), \ldots, \left(X_{n_j j}, Y_{n_j j}\right)$ in the stress level V_j, the likelihood function for c_1, c_2, c_3 and $\mathbb{P}$ is given by

$$L_j(c_1, c_2, c_3, \mathbb{P}) = \prod_{i=1}^{n_j} f_1^{\delta_{ij}}(X_{ij}, Y_{ij}) f_2^{1-\delta_{ij}}(X_{ij}, Y_{ij}) \tag{3}$$

where $\delta_{ij} = 1$ if $X_{ij} < Y_{ij}, \delta_{ij} = 0$ if $X_{ij} \geq Y_{ij}$, and

$$f_1(X_{ij},Y_{ij}) = \frac{c_1c_{23}c_{123}}{c_{12}} V_j^{2\mathbb{P}} exp\left\{-[c_1X_{ij}+c_{23}Y_{ij}]V_j^{\mathbb{P}}\right\},$$

$$f_2(X_{ij},Y_{ij}) = \frac{c_2c_{13}c_{123}}{c_{12}} V_j^{2\mathbb{P}} exp\left\{-[c_{13}X_{ij}+c_2Y_{ij}]V_j^{\mathbb{P}}\right\},$$

$$c_{12} = c_1+c_2, c_{13}=c_1+c_3, c_{23}=c_2+c_3 \text{ and } c_{123}=c_1+c_2+c_3 .$$

That is,

$$L_j(c_1,c_2,c_3,\mathbb{P}) = \frac{c_{123}^{nj}c_1^{rj}c_{23}^{rj}c_2^{nj-rj}c_{13}^{nj-rj}}{c_{12}^{nj}}\left(V_j^{2\mathbb{P}}\right)^{nj} \times$$
$$\times \; exp\left\{-\left[c_1n_j\overline{X}_j + c_2n_j\overline{Y}_j + c_3R_j\right]V_j^{\mathbb{P}}\right\} \tag{4}$$

where $n_j\overline{X}_j = \sum_{i=1}^{nj} X_{ij}, n_j\overline{Y}_j = \sum_{i=1}^{nj} Y_{ij}, r_j = \sum_{i=1}^{nj}\delta_{ij}$ and $R_j = \sum_{i=1}^{nj}[Y_{ij}\delta_{ij} + (1-\delta_{ij})X_{ij}]$.

Considering the data of J stress levels $V_1, V_2, \ldots, V_J$ taken at random, the likelihood function for c_1, c_2, c_3 and $\mathbb{P}$ is given by

$$L_n(c_1,c_2,c_3,\mathbb{P}) = \prod_{j=1}^{J} L_j(c_1,c_2,c_3,\mathbb{P}) . \tag{5}$$

That is,

$$L_n(c_1,c_2,c_3,\mathbb{P}) = \frac{c_1^rc_{23}^rc_2^{n-r}c_{13}^{n-r}c_{123}^n}{c_{12}^n} \times$$
$$\times \left\{\prod_{j=1}^{J} V_j^{2\mathbb{P}n_j}\right\} exp\left\{-[c_1S_x(\mathbb{P}) + c_2S_y(\mathbb{P}) + c_3T(\mathbb{P})]\right\} \tag{6}$$

where $r = \sum_{j=1}^{J} rj, n = \sum_{j=1}^{J} n_j, S_x(\mathbb{P}) = \sum_{j=1}^{J} nj\overline{X}_jV_j^{\mathbb{P}}, S_y(\mathbb{P}) = \sum_{j=1}^{J} n_j\overline{Y}_jV_j^{\mathbb{P}}, T(\mathbb{P}) = \sum_{j=1}^{J} R_jV_j^{\mathbb{P}}$; c_{12}, c_{13}, c_{23} and c_{123} are given in (3).

The logarithm of the likelihood function (6) is given by

$$ln(c_1,c_2,c_3,\mathbb{P}) = rlnc_1 + rlnc_{23} + (n-r)lnc_2 +$$
$$+ (n-r)lnc_{13} + nlnc_{123} - nlnc_{12} + \tag{7}$$
$$+ 2\mathbb{P}\sum_{j=1}^{J} n_j lnV_j - c_1S_x(\mathbb{P}) - c_2S_y(\mathbb{P}) - c_3T(\mathbb{P}) .$$

The maximum likelihood estimators $\tilde{c}_1, \tilde{c}_2, \tilde{c}_3$ and $\hat{\mathbb{P}}$ are given by solving the likelihood equations,

$$\frac{r}{\hat{c}_1} + \frac{(n-r)}{\left(\hat{c}_1+\hat{c}_3\right)} + \frac{n}{\left(\hat{c}_1+\hat{c}_2+\hat{c}_3\right)} - \frac{n}{\left(\hat{c}_1+\hat{c}_2\right)} = \sum_{j=1}^{J} n_j \overline{X}_j V_j^{\hat{\mathbb{P}}}$$

$$\frac{r}{\left(\hat{c}_2+\hat{c}_3\right)} + \frac{(n-r)}{\hat{c}_2} + \frac{n}{\left(\hat{c}_1+\hat{c}_2+\hat{c}_3\right)} - \frac{n}{\left(\hat{c}_1+\hat{c}_2\right)} = \sum_{j=1}^{J} n_j \overline{Y}_j V_j^{\hat{\mathbb{P}}}$$

$$\frac{r}{\left(\hat{c}_2+\hat{c}_3\right)} + \frac{(n-r)}{\left(\hat{c}_1+\hat{c}_3\right)} + \frac{n}{\left(\hat{c}_1+\hat{c}_2+\hat{c}_3\right)} = \sum_{j=1}^{J} R_j V_j^{\hat{\mathbb{P}}} \tag{8}$$

and $\hat{c}_1 \sum_{j=1}^{J} n_j \overline{X}_j V_j^{\hat{\mathbb{P}}} (lnV_j) + \hat{c}_2 \sum_{j=1}^{J} n_j \overline{Y}_j V_j^{\hat{\mathbb{P}}} (lnV_j) + \hat{c}_3 \sum_{j=1}^{J} R_j V_j^{\hat{\mathbb{P}}} (lnV_j)$ $= 2 \sum_{j=1}^{J} n_j lnV_j$.

For inferences on $\underset{\sim}{\Psi} = (c_1, c_2, c_3, \mathbb{P})$ or even functions of the parameters, we usually use the asymptotical normality of the maximum likelihood estimators, given by

$$\hat{\underset{\sim}{\Psi}} = \left(\hat{c}_1, \hat{c}_2, \hat{c}_3, \hat{\mathbb{P}}\right) \overset{a}{\sim} N\left\{\underset{\sim}{\Psi}\,;\, I_0^{-1}\right\} \tag{9}$$

where I_o is the observed information matrix given by

$$I_0 = \left[(b_{kl})\right], \tag{10}$$

$(k = 1,2,3,4; l = 1,2,3,4)$, where,

$$b_{11} = -\frac{\partial^2 ln\left(\hat{\underset{\sim}{\Psi}}\right)}{\partial c_1^2} = \frac{r}{\tilde{c}_1^2} + \frac{(n-r)}{\left(\tilde{c}_1+\tilde{c}_3\right)^2} + \frac{n}{\left(\tilde{c}_1+\tilde{c}_2+\tilde{c}_3\right)^2} - \frac{n}{\left(\tilde{c}_1+\tilde{c}_2\right)^2}$$

$$b_{12} = -\frac{\partial^2 ln\left(\hat{\underset{\sim}{\Psi}}\right)}{\partial c_1 \partial c_2} = \frac{n}{\left(\tilde{c}_1+\tilde{c}_2+\tilde{c}_3\right)^2} - \frac{n}{\left(\tilde{c}_1+\tilde{c}_2\right)^2}$$

$$b_{13} = -\frac{\partial^2 ln\left(\hat{\underset{\sim}{\Psi}}\right)}{\partial c_1 \partial c_3} = \frac{(n-r)}{\left(\tilde{c}_1+\tilde{c}_3\right)^2} + \frac{n}{\left(\tilde{c}_1+\tilde{c}_2+\tilde{c}_3\right)^2}$$

$$b_{14} = -\frac{\partial^2 ln\left(\hat{\underset{\sim}{\Psi}}\right)}{\partial c_1 \partial \mathbb{P}} = \sum_{j=1}^{J} n_j \overline{X}_j V_j^{\hat{\mathbb{P}}} (lnV_j)$$

$$b_{22} = -\frac{\partial^2 \ln\left(\hat{\underset{\sim}{\ell}}\right)}{\partial c_2^2} = \frac{r}{\left(\tilde{c}_2+\tilde{c}_3\right)^2} + \frac{(n-r)}{\tilde{c}_2^2} + \frac{n}{\left(\tilde{c}_1+\tilde{c}_2+\tilde{c}_3\right)^2} - \frac{n}{\left(\tilde{c}_1+\tilde{c}_2\right)^2}$$

$$b_{23} = -\frac{\partial^2 \ln\left(\hat{\underset{\sim}{\ell}}\right)}{\partial c_2 \partial c_3} = \frac{r}{\left(\tilde{c}_2+\tilde{c}_3\right)^2} + \frac{n}{\left(\tilde{c}_1+\tilde{c}_2+\tilde{c}_3\right)^2}$$

$$b_{24} = -\frac{\partial^2 \ln\left(\hat{\underset{\sim}{\ell}}\right)}{\partial c_2 \partial \mathbb{P}} = \sum_{j=1}^{J} n_j \bar{y}_j V_j^{\hat{\mathbb{P}}} \left(\ln V_j\right)$$

$$b_{33} = -\frac{\partial^2 \ln\left(\hat{\underset{\sim}{\ell}}\right)}{\partial c_3^2} = \frac{r}{\left(\tilde{c}_2+\tilde{c}_3\right)^2} + \frac{(n-r)}{\left(\tilde{c}_1+\tilde{c}_3\right)^2} + \frac{n}{\left(\tilde{c}_1+\tilde{c}_2+\tilde{c}_3\right)^2}$$

$$b_{34} = -\frac{\partial^2 \ln\left(\hat{\underset{\sim}{\ell}}\right)}{\partial c_3 \partial \mathbb{P}} = \sum_{j=1}^{J} R_j V_j^{\hat{\mathbb{P}}} \left(\ln V_j\right)$$

$$b_{44} = -\frac{\partial^2 \ln\left(\hat{\underset{\sim}{\ell}}\right)}{\partial \mathbb{P}^2} = \tilde{c}_1 \sum_{j=1}^{J} n_j \bar{X}_j V_j^{\hat{\mathbb{P}}} \left(\ln V_j\right)^2 +$$

$$+ \quad \tilde{c}_2 \sum_{j=1}^{J} n_j \bar{Y}_j V_j^{\hat{\mathbb{P}}} \left(\ln V_j\right)^2 + \tilde{c}_3 \sum_{j=1}^{J} R_j V_j^{\hat{\mathbb{P}}} \left(\ln V_j\right)^2$$

3 A Bayesian Analysis Assuming c_1, c_2, c_3 and $\mathbb{P}$ Unknown

Assuming that (X, Y) has a BVED with density (1) and the power rule model (2) for the parameter $\lambda_{ij}, i = 1, 2, 3$, under the stress variable $V_j, j = 0, 1, 2, \ldots, J$, the prior density for c_1, c_2, c_3 and $\mathbb{P}$ can be written in the form

$$\pi\left(c_1, c_2, c_3, \mathbb{P}\right) = \pi\left(c_1, c_2, c_3 | \mathbb{P}\right) \pi_0(\mathbb{P}). \tag{11}$$

Using the Jeffreys multiparameter rule (see for example, Box and Tiao, 1973), that is, $\pi\left(c_1, c_2, c_3 | \mathbb{P}\right) \propto \left\{det I_{\mathbb{P}}\left(c_1, c_2, c_3\right)\right\}^{1/2}$, where $I_{\mathbb{P}}\left(c_1, c_2, c_3\right)$ is the Fisher information matrix given $\mathbb{P}$, and a locally uniform prior for $\mathbb{P}$, we consider the noninformative prior for c_1, c_2, c_3 and $\mathbb{P}$ given by

$$\pi(c_1, c_2, c_3, \mathbb{P}) \propto \{det I_{\mathbb{P}}(c_1, c_2, c_3)\}^{1/2} \tag{12}$$

where $c_1, c_2, c_3 > 0, -\infty < \mathbb{P} < \infty$ and

$$I_{\mathbb{P}}(c_1, c_2, c_3) = [(d_{kl})] \ , \tag{13}$$

$(k = 1,2,3; l = 1,2,3)$, where,

$$\begin{aligned}
d_{11} &= E\left(-\frac{\partial^2 l n}{\partial c_1^2}\right) = n\left\{\frac{1}{c_1 c_{12}} + \frac{c_2}{c_{12} c_{13}^2} + \frac{1}{c_{123}^2} - \frac{1}{c_{12}^2}\right\} \\
d_{12} &= E\left(-\frac{\partial^2 l n}{\partial c_1 \partial c_2}\right) = n\left\{\frac{1}{c_{123}^2} - \frac{1}{c_{12}^2}\right\} \\
d_{13} &= E\left(-\frac{\partial^2 l n}{\partial c_1 \partial c_3}\right) = n\left\{\frac{c_2}{c_{12} c_{13}^2} + \frac{1}{c_{123}^2}\right\} \\
d_{22} &= E\left(-\frac{\partial^2 l n}{\partial c_2^2}\right) = n\left\{\frac{c_1}{c_{12} c_{23}^2} + \frac{1}{c_2 c_{12}} + \frac{1}{c_{123}^2} - \frac{1}{c_{12}^2}\right\} \\
d_{23} &= E\left(-\frac{\partial^2 l n}{\partial c_2 \partial c_3}\right) = n\left\{\frac{c_1}{c_{12} c_{23}^2} + \frac{1}{c_{123}^2}\right\} \\
d_{33} &= E\left(-\frac{\partial^2 l n}{\partial c_3^2}\right) = n\left\{\frac{c_1}{c_{12} c_{23}^2} + \frac{c_2}{c_{12} c_{13}^2} + \frac{1}{c_{123}^2}\right\}
\end{aligned}$$

(see for example Basu and Ebrahimi, 1987).

That is,

$$\pi(c_1, c_2, c_3, \mathbb{P}) \propto a(c_1, c_2, c_3) \tag{14}$$

where

$$a(c_1, c_2, c_3) = \left\{d_{11} d_{22} d_{33} + 2 d_{12} d_{23} d_{13} - d_{22} d_{13}^2 - d_{11} d_{23}^2 - d_{33} d_{12}^2\right\}^{1/2} .$$

Considering the prior (14), the joint posterior density for c_1, c_2, c_3 and $\mathbb{P}$, is given by

$$\begin{aligned}
\pi(c_1, c_2, c_3, \mathbb{P}|\text{DATA}) &\propto \frac{a(c_1, c_2, c_3) c_1^r c_{23}^r c_2^{n-r} c_{13}^{n-r} c_{123}^n}{c_{12}^n} \times \\
&\times \left\{\prod_{j=1}^J V_j^{2\mathbb{P} n_j}\right\} exp\{-[c_1 S_X(\mathbb{P}) + c_2 S_y(\mathbb{P}) + c_3 T(\mathbb{P})]\}
\end{aligned} \tag{15}$$

where $c_1, c_2, c_3 > 0$ and $-\infty < \mathbb{P} < \infty$.

3.1 The Marginal Posterior Density For $\mathbb{P}$

The marginal posterior density for $\mathbb{P}$ is given (from (15)) by

$$\pi(\mathbb{P}|\mathrm{DATA}) \propto \int_0^\infty \int_0^\infty \int_0^\infty a(c_1, c_2, c_3) e^{-nh_{\mathbb{P}}(c_1,c_2,c_3)} dc_1 dc_2 dc_3 \tag{16}$$

where $a(c_1, c_2, c_3)$ is given in (14) and $-nh_{\mathbb{P}}(c_1, c_2, c_3) = ln(c_1, c_2, c_3, \mathbb{P})$ (see (7)).

An approximate marginal posterior density for $\mathbb{P}$, using the Laplace's method for approximation of integrals (see for example, Tierney and Kadane, 1986; or, Kass, Tierney and Kadane, 1990), is given by

$$\pi(\mathbb{P}|\mathrm{DATA}) \propto \left\{\det\left(nD^2 h_{\mathbb{P}}\left(\tilde{c}_1, \tilde{c}_2, \tilde{c}_3\right)\right)\right\}^{-1/2} a\left(\tilde{c}_1, \tilde{c}_2, \tilde{c}_3\right) e^{-nh_{\mathbb{P}}\left(\tilde{c}_1,\tilde{c}_2,\tilde{c}_3\right)} \tag{17}$$

where $\tilde{c}_1, \tilde{c}_2$ and $\tilde{c}_3$ maximize $-nh_{\mathbb{P}}(c_1, c_2, c_3)$, for each value of $\mathbb{P}$ and $D^2 h_{\mathbb{P}}(c_1, c_2, c_3)$ is the Hessian matrix calculated at $\left(\tilde{c}_1, \tilde{c}_2, \tilde{c}_3\right)$ given by

$$D^2 h_{\mathbb{P}}\left(\tilde{c}_1, \tilde{c}_2, \tilde{c}_3\right) = [(b_{kl})] \tag{18}$$

where $b_{kl}, k = 1,2,3; l = 1,2,3$ are given in (10), with $\tilde{c}_1, \tilde{c}_2, \tilde{c}_3$ in place of the maximum likelihood estimators $\tilde{c}_1, \tilde{c}_2$ and $\tilde{c}_3$.

That is

$$\pi(\mathbb{P}|\mathrm{DATA}) \propto \frac{a\left(\tilde{c}_1, \tilde{c}_2, \tilde{c}_3\right) \tilde{c}_1^{\,r} \tilde{c}_{23}^{\,r} \tilde{c}_2^{\,n-r} \tilde{c}_{13}^{\,n-r} \tilde{c}_{123}^{\,n}}{\tilde{c}_{12}^{\,n} b\left(\tilde{c}_1, \tilde{c}_2, \tilde{c}_3\right)} \times$$
$$\times \left\{\prod_{j=1}^{J} V_j^{2\mathbb{P}n_j}\right\} exp\left\{-\left[\tilde{c}_1\, S_X(\mathbb{P}) + \tilde{c}_2\, S_y(\mathbb{P}) + \tilde{c}_3\, T(\mathbb{P})\right]\right\} \tag{19}$$

where $-\infty < \mathbb{P} < \infty, a(c_1, c_2, c_3)$ is given in (14), $\tilde{c}_{12} = \tilde{c}_1 + \tilde{c}_2, \tilde{c}_{13} = \tilde{c}_1 + \tilde{c}_3, \tilde{c}_{23} = \tilde{c}_2 + \tilde{c}_3$ $, \tilde{c}_{123} = \tilde{c}_1 + \tilde{c}_2 + \tilde{c}_3$ and

$$b\left(\tilde{c}_1, \tilde{c}_2, \tilde{c}_3\right) = \left\{b_{11}b_{22}b_{33} + 2b_{12}b_{23}b_{13} - b_{22}b_{13}^2 - b_{11}b_{23}^2 - b_{33}b_{12}^2\right\}^{1/2}.$$

In the same way, we also could use the Laplace's method to find the marginal posterior densities for c_1, c_2, c_3 or even functions of the parameters.

4 A Bayesian Analysis Assuming $\mathbb{P}$ Known

Assuming $\mathbb{P}$ Known, the likelihood function for c_1, c_2 and c_3 (see(6)) is given by

$$L_n(c_1, c_2, c_3) \propto \frac{c_1^r c_{23}^r c_2^{n-r} c_{13}^{n-r} c_{123}^n}{c_{12}^n} exp\{-[c_1 S_x(\mathbb{P}) + c_2 S_y(\mathbb{P}) + c_3 T(\mathbb{P})]\} . \tag{20}$$

The Jeffreys prior density for c_1, c_2 and c_3 is given by

$$\pi(c_1, c_2, c_3) \propto a(c_1, c_2, c_3) \tag{21}$$

where $c_1, c_2, c_3 > 0$ and $a(c_1, c_2 c_3)$ is given in (14).

Considering the noninformative prior density (21), the joint posterior density for c_1, c_2 and c_3 is given by

$$\pi(c_1, c_2, c_3|DATA) \propto$$
$$\propto \frac{a(c_1,c_2,c_3) c_1^r c_{23}^r c_2^{n-r} c_{13}^{n-r} c_{123}^n}{c_{12}^n} exp\{-[c_1 S_x(\mathbb{P}) + c_2 S_y(\mathbb{P}) + c_3 T(\mathbb{P})]\} \tag{22}$$

where $c_1, c_2, c_3 > 0$.

4.1 The Marginal Posterior Density For c_3

The marginal posterior density for c_3 is given (from (22)) by

$$\pi(c_3|DATA) \propto \int_0^\infty \int_0^\infty f_{c_3}(c_1, c_2) e^{-nh_{c_3}(c_1,c_2)} dc_1 dc_2 \tag{23}$$

where $f_{c_3}(c_1, c_2) = a(c_1, c_2, c_3)$ and $-nh_{c_3}(c_1, c_2) = l_n(c_1, c_2, c_3)$.

A Laplace's approximate marginal posterior density for c_3 is given by

$$\pi(c_3|DATA) \propto \frac{a\left(\tilde{c}_1, \tilde{c}_2, \tilde{c}_3\right) \tilde{c}_1^{\,r} \left(\tilde{c}_2 + c_3\right)^r \tilde{c}_2^{\,n-r} \left(\tilde{c}_1 + c_3\right)^{n-r}}{b\left(\tilde{c}_1, \tilde{c}_2\right)\left(\tilde{c}_1 + \tilde{c}_2\right)^n} \times$$
$$\times \left(\tilde{c}_1 + \tilde{c}_2 + c_3\right)^n exp\left\{-\left[\tilde{c}_1\, S_x(\mathbb{P}) + \tilde{c}_2\, S_y(\mathbb{P}) + \tilde{c}_3\, T(\mathbb{P})\right]\right\} \tag{24}$$

$c_3 > 0, a(c_1, c_2, c_3)$ is given (14), $b\left(\tilde{c}_1, \tilde{c}_2\right) = \{b_{11} b_{22} - b_{12}^2\}^{1/2}$,

$$b_{11} = \frac{r}{\tilde{c}_1^2} + \frac{(n-r)}{\left(\tilde{c}_1+c_3\right)^2} + \frac{n}{\left(\tilde{c}_1+\tilde{c}_2+c_3\right)^2} - \frac{n}{\left(\tilde{c}_1+\tilde{c}_2\right)^2}$$

$$b_{12} = \frac{n}{\left(\tilde{c}_1+\tilde{c}_2+c_3\right)^2} - \frac{n}{\left(\tilde{c}_1+\tilde{c}_2\right)^2}$$

$$b_{22} = \frac{r}{\left(\tilde{c}_2+c_3\right)^2} + \frac{(n-r)}{\tilde{c}_2^2} + \frac{n}{\left(\tilde{c}_1+\tilde{c}_2+c_3\right)^2} - \frac{n}{\left(\tilde{c}_1+\tilde{c}_2\right)^2}$$

and $\tilde{c}_1$ and $\tilde{c}_2$ are given by

$$\frac{r}{\tilde{c}_1} + \frac{(n-r)}{\left(\tilde{c}_1+\tilde{c}_3\right)} + \frac{n}{\left(\tilde{c}_1+\tilde{c}_2+c_3\right)} - \frac{n}{\left(\tilde{c}_1+\tilde{c}_2\right)} = \sum_{j=1}^{J} n_j \overline{x}_j V_j^{\mathbb{P}}$$

$$\frac{r}{\left(\tilde{c}_2+c_3\right)} + \frac{(n-r)}{\tilde{c}_2} + \frac{n}{\left(\tilde{c}_1+\tilde{c}_2+c_3\right)} - \frac{n}{\left(\tilde{c}_1+\tilde{c}_2\right)} = \sum_{j=1}^{J} n_j \overline{y}_j V_j^{\mathbb{P}}$$

We also could find approximate marginal posterior densities for c_1 and c_2 using the Laplace's method (see appendix).

4.2 The Joint Marginal Posterior Density For c_1 And c_2

The joint marginal posterior density for c_1 and c_2 is given (from (22)) by

$$\pi\left(c_1, c_2 | DATA\right) \propto \frac{c_1^r c_2^{n-r}}{c_{12}^n} e^{-\left[c_1 S_x(\mathbb{P})+c_2 S_y(\mathbb{P})\right]} \int_0^\infty f_{c_1,c_2}\left(c_3\right) e^{-nh_{c_1,c_2}(c_3)} dc_3 \qquad (25)$$

where $f_{c_1,c_2}\left(c_3\right) = a\left(c_1, c_2, c_3\right)$ and

$$-nh_{c_1,c_2}\left(c_3\right) = r ln c_{23} + (n-r) ln c_{13} + n ln c_{123} - c_3 T(\mathbb{P}) .$$

A Laplace's approximate joint marginal posterior density for c_1 and c_2 is given by

$$\pi\left(c_1, c_2 | DATA\right) \propto \frac{c_1^r c_2^{n-r}\left(c_2+\tilde{c}_3\right)^r\left(c_1+\tilde{c}_3\right)^{n-r}\left(c_1+c_2+\tilde{c}_3\right)^n}{\left(c_1+c_2\right)^n} \times$$

$$\times \frac{a\left(c_1,c_2,\tilde{c}_3\right) exp\left\{-\left[c_1 S_x(\mathbb{P})+c_2 S_y(\mathbb{P})+\tilde{c}_3 T(\mathbb{P})\right]\right\}}{\left\{\frac{r}{\left(c_2+\tilde{c}_3\right)^2}+\frac{(n-r)}{\left(c_1+\tilde{c}_3\right)^2}+\frac{n}{\left(c_1+c_2+\tilde{c}_3\right)^2}\right\}^{1/2}}$$

where $c_1, c_2 > 0$ and $\tilde{c}_3$ satisfies the equation

$$\frac{r}{\left(c_2+\tilde{c}_3\right)} + \frac{(n-r)}{\left(c_1+\tilde{c}_3\right)} + \frac{n}{\left(c_1+c_2+\tilde{c}_3\right)} = T(\mathbb{P}) .$$

5 An Example

In table 1, we have the data of an accelerated life test considering three stress levels $V_1 = 1, V_2 = 2$ and $V_3 = 3$. At each stress level, 15 bivariate observations (X, Y) were generated from a BVED with density (1) and the power rule model (2) with $\mathbb{P} = 2$. From table 1, we get $n_1 = n_2 = n_3 = 15, n = 45, r_1 = 4, r_2 = 3, r_3 = 2$ and $r = 9$.

$V_1 = 1$ (X_{1i}, Y_{1i})	$V_2 = 2$ (X_{2i}, Y_{2i})	$V_3 = 3$ (X_{3i}, Y_{3i})
(7.65;2.18)	(2.29;0.02)	(0.34;0.20)
(16.67;9.26)	(0.10,0.38)	(1.50;1.30)
(39.30;6.72)	(0.88;0.27)	(0.63;0.69)
(1.30;3.22)	(0.45;0.04)	(0.68;0.12)
(9.04;2.23)	(1.66;1.60)	(3.22;0.09)
(5.15;0.41)	(0.74;1.67)	(1.91;0.91)
(5.20;5.91)	(2.50;0.37)	(0.52;0.58)
(5.00;0.84)	(3.50;0.03)	(0.30;0.01)
(5.66;0.42)	(8.45;0.71)	(1.30;0.02)
(11.80;0.15)	(4.60;0.83)	(0.52;0.10)
(17.08;10.37)	(2.66;1.06)	(2.08;0.30)
(17.92;0.76)	(1.46;1.04)	(0.95;0.91)
(1.62;2.73)	(1.03;0.41)	(0.43;0.02)
(1.42;1.85)	(4.36;1.34)	(0.25;0.08)
(3.60;1.50)	(0.76;0.77)	(1.39;0.08)

Table 1: Bivariate Life Time Data

j	V_j	r_j	n_j	$\overline{X}_j$	$\overline{Y}_j$	R_j
1	1	4	15	9.8940	3.2366	152.58
2	2	3	15	2.3627	0.7020	36.66
3	3	2	15	1.0680	0.3607	16.14

Table 2: Summary Statistics For Data In Table 1

The likelihood equations for c_1, c_2, c_3 and $\mathbb{P}$ (see (8)) are given by

$$\begin{aligned} &\tfrac{9}{c_1} + \tfrac{36}{(c_1+c_3)} + \tfrac{45}{(c_1+c_2+c_3)} - \tfrac{45}{(c_1+c_2)} = \\ &= 148.41(1)^{\mathbb{P}} + 35.44(2)^{\mathbb{P}} + 16.02(3)^{\mathbb{P}}\,. \\ &\tfrac{9}{c_2+c_3} + \tfrac{36}{c_2} + \tfrac{45}{(c_1+c_2+c_3)} - \tfrac{45}{(c_1+c_2)} = \\ &= 48.55(1)^{\mathbb{P}} + 10.53(2)^{\mathbb{P}} + 5.41(3)^{\mathbb{P}} \end{aligned} \tag{26}$$

$$\frac{9}{c_2+c_3} + \frac{36}{(c_1+c_3)} + \frac{45}{(c_1+c_2+c_3)}$$

$$= 152.28(1)^{\mathbb{P}} + 36.66(2)^{\mathbb{P}} + 16.14(3)^{\mathbb{P}}$$

and $c_1\left(24.56(2)^{\mathbb{P}} + 17.60(3)^{\mathbb{P}}\right) + c_2\left(7.30(2)^{\mathbb{P}} + 5.94(3)^{\mathbb{P}}\right) + c_3\left(25.41(2)^{\mathbb{P}} + 17.73(3)^{\mathbb{P}}\right)$ $= 53.75$

To simplify the solutions of the likelihood equations (27), we consider the maximized likelihood function for $\mathbb{P}$, that is, we use the Newton-Raphson method to Compute the maximum likelihood estimators $\tilde{c}_1(\mathbb{P}), \tilde{c}_2(\mathbb{P})$ and $\tilde{c}_3(\mathbb{P})$ for different values of $\mathbb{P}$ (see table 3). The maximized likelihood function for $\mathbb{P}$ is then $L_{max}(\mathbb{P}) = \mathrm{L}\left(\tilde{c}_1(\mathbb{P}), \tilde{c}_2(\mathbb{P}), \tilde{c}_3(\mathbb{P}), \mathbb{P}\right)$. From table 3, we observe that the maximum likelihood estimators are given by $\tilde{c}_1 = 0.0571, \tilde{c}_2 = 0.2643, \tilde{c}_3 = 0.0602$ and $\hat{\mathbb{P}} = 2.03$.

From the information of table 2, we obtain the observed information matrix (see (10)) given by

$$I_0 = \begin{pmatrix} 5250.20 & -126.61 & 2925.44 & 264.03 \\ & 474.22 & 394.50 & 85.10 \\ & & 3010.91 & 268.71 \\ & & & 50.99 \\ \textit{symmetric} & & & \end{pmatrix}$$

The inverse of the observed information matrix I_0 is given by

$\mathbb{P}$	$\tilde{c}_1(\mathbb{P})$	$\tilde{c}_2(\mathbb{P})$	$\tilde{c}_3(\mathbb{P})$	$l_n(\tilde{c}_1, \tilde{c}_2, \tilde{c}_3 \mid \mathbb{P})$
1.20	0.0883	0.4051	0.0901	-140.712
1.30	0.0844	0.3880	0.0864	-139.211
1.40	0.0806	0.3706	0.0827	-137.901
1.50	0.0768	0.3534	0.0791	-136.783
1.60	0.0729	0.3361	0.0754	-135.859
1.70	0.0692	0.3190	0.0718	-135.127
1.80	0.0655	0.3021	0.0682	-134.591
1.90	0.0618	0.2854	0.0647	-134.248
1.93	0.0607	0.2805	0.0636	-134.182
1.95	0.0600	0.2772	0.0629	-134.148
1.96	0.0596	0.2756	0.0626	-134.134
1.97	0.0593	0.2740	0.0622	-134.121
1.98	0.0589	0.2723	0.0619	-134.110
2.00	0.0582	0.2691	0.0612	-134.095
2.02	0.0575	0.2659	0.0605	-134.088
2.03	0.0571	0.2643	0.0602	-134.087
2.04	0.0568	0.2626	0.0599	-134.088
2.05	0.0564	0.2610	0.0595	-134.091
2.07	0.0557	0.2579	0.0588	-134.103
2.09	0.0550	0.2547	0.0582	-134.121
2.10	0.0547	0.2531	0.0578	-134.133
2.12	0.0540	0.2499	0.0572	-134.163
2.15	0.0530	0.2453	0.0562	-134.222
2.20	0.0513	0.2376	0.0545	-134.358
2.30	0.0480	0.2225	0.0513	-134.767
2.40	0.0448	0.2080	0.0482	-135.354
2.50	0.0417	0.1940	0.0452	-136.118
2.60	0.0388	0.1806	0.0424	-137.053
2.70	0.0360	0.1678	0.0396	-138.153
2.80	0.0334	0.1556	0.0369	-139.414

Table 2: Maximized Log-Likelihood Function $l_n\left(\tilde{c}_1(\mathbb{P}), \tilde{c}_2(\mathbb{P}), \tilde{c}_3(\mathbb{P}) \mid \mathbb{P}\right)$ For Some Values Of $\mathbb{P}$.

$$I_0^{-1} = \begin{pmatrix} 0.0006236 & 0.0009727 & -0.0005669 & -0.0018647 \\ & 0.0045440 & -0.0007820 & -0.0084989 \\ & & 0.0011459 & -0.0017977 \\ \textit{symmetric} & & & 0.0529212 \end{pmatrix}$$

The approximate variances of $\tilde{c}_1, \tilde{c}_2, \tilde{c}_3$ and $\hat{\mathbb{P}}$ (see (9)) are given by

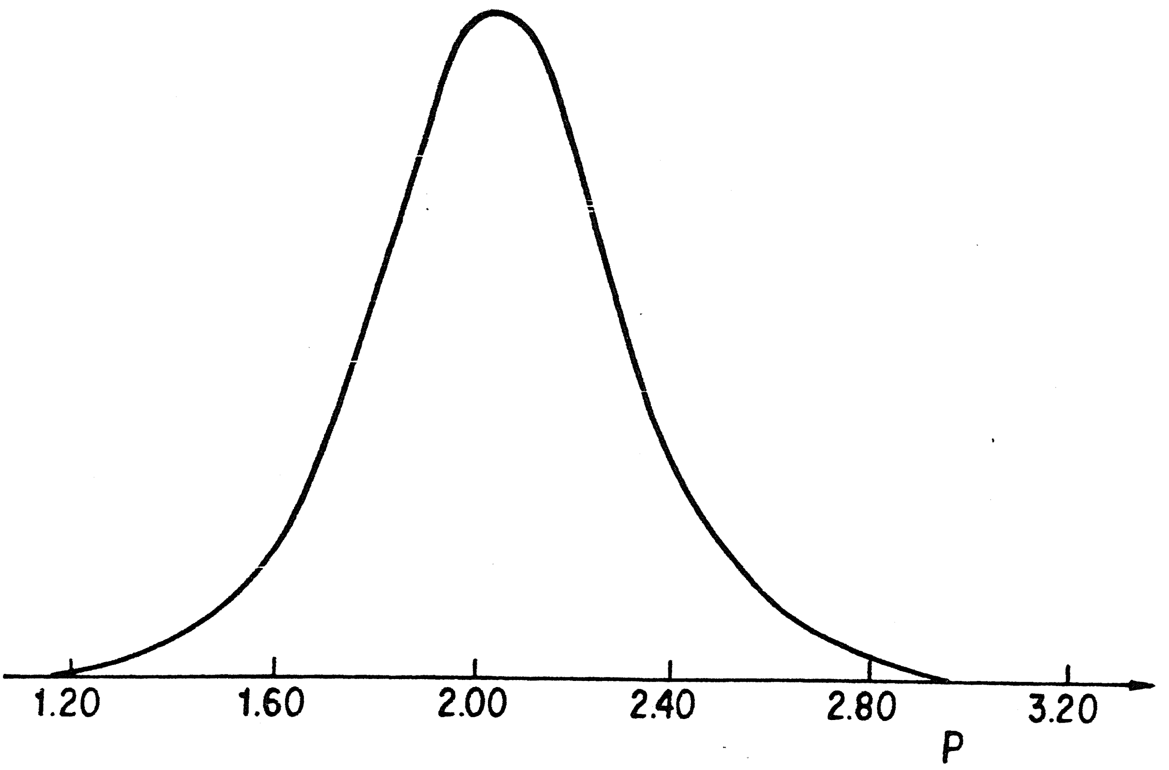

Figure 1: Marginal Posterior Density for $\mathbb{P}$

$\widehat{var}(\tilde{c}_1) \cong 0.000624, \widehat{var}(\tilde{c}_2) \cong 0.004544, \widehat{var}(\tilde{c}_3) \cong 0.001146$ and $\widehat{var}(\hat{\mathbb{P}}) \cong 0.052921$. The 95% approximate confidence intervals for c_1, c_2, c_3 and $\mathbb{P}$ are given by (0.00814,0.10606), (0.13218,0.39642),(−0.00615;0.12655) and (1.57911,2.48089), respectively.

Under the normal stress condition, we assume that $V_0 = 0.5$. Therefore, the maximum likelihood estimators for the parameters of the bivariate exponential distribution under normal condition are given by $\hat{\lambda}_{10}= 0.01398$, $\hat{\lambda}_{20}= 0.06471$ and $\hat{\lambda}_{30}= 0.01474$.

In figure 1, we have the graph of the Laplace's approximate marginal posterior density for $\mathbb{P}$ given in (19) considering a noninformative prior density (14) for the parameters c_1, c_2, c_3 and $\mathbb{P}$.

The mode of the approximate marginal posterior density for $\mathbb{P}$ is given by $\tilde{\mathbb{P}} \cong 2.05$, and we find an approximate 95% credible interval for $\mathbb{P}$ using a numerical method given by (1.55,2.45). We observe very close inference results considering the marginal posterior density for $\mathbb{P}$ (19) or considering the asymptotical normality of the maximum likelihood estimators $\tilde{c}_1, \tilde{c}_2, \tilde{c}_3$ and $\hat{\mathbb{P}}$.

In figure 2, we have the graph of the approximate marginal posterior density for c_3(24) assuming $\mathbb{P} = 2.03$ known. The mode of the marginal posterior density for c_3 is given by $\tilde{c}_3 \cong 0.060$, that is, very close to the maximum likelihood estimator for c_3.

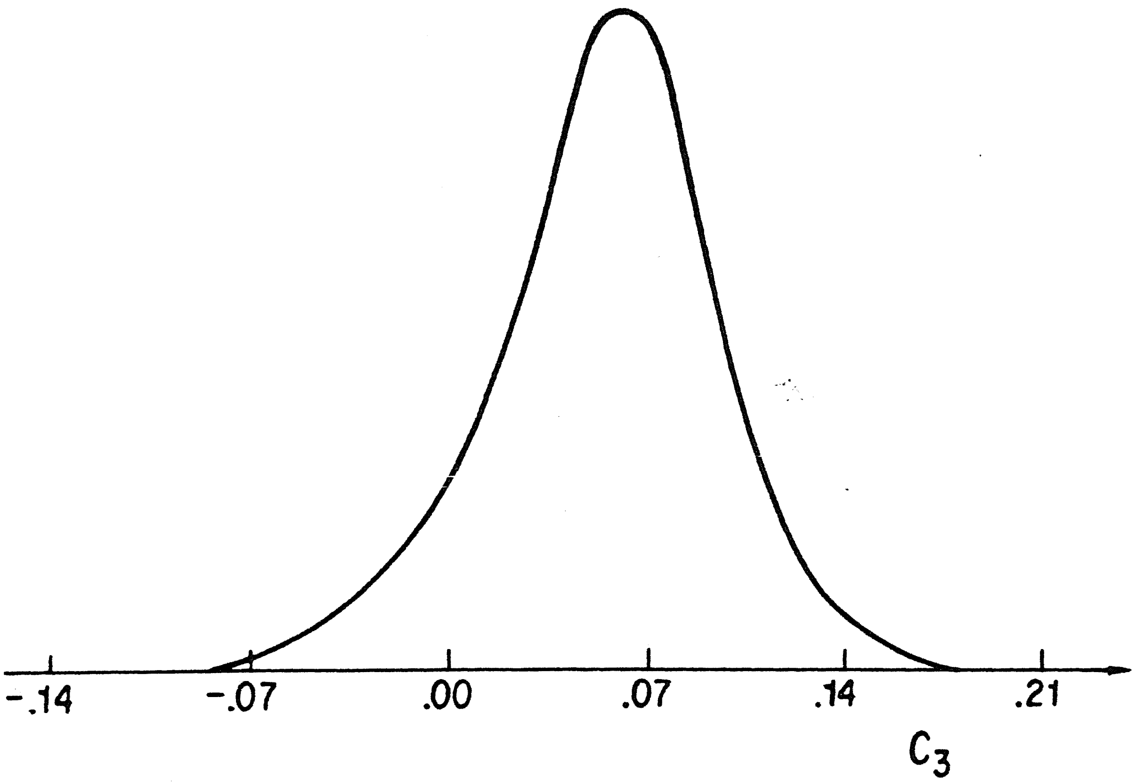

Figure 2: Marginal Posterior Density for c_3.

A 94% credible interval for c_3 obtained by using a numerical method is given by $(-0.015, 0.115)$.

In figure 3, we have some countour regions for the approximate joint marginal posterior density for c_1 and c_2 given in (26). The mode of the joint marginal posterior density for c_1 and c_2 is given by $\tilde{c}_1 \cong 0.05$ and $\tilde{c}_2 \cong 0.24$.

We could use numerical methods to obtain, from the joint marginal posterior density for c_1 and c_2, credible regions for the parameters c_1 and c_2.

6 Some Conclusions

In this paper, we presented a Bayesian analysis of the BVED with density (1) and a power rule model (2) assuming accelerated life tests. We considered a noinformative prior density for the parameters of the model and we used the Laplace's method of approximation for integrals to obtain the marginal posterior densities of interest, since we cannot find explicitly analytical solutions for the integrals. These results could be of great practical interest, since the proposed model is very popular in the life tests applications. We also could use other priors for the parameters obtaining similar results considering the Laplace's method. It is important to

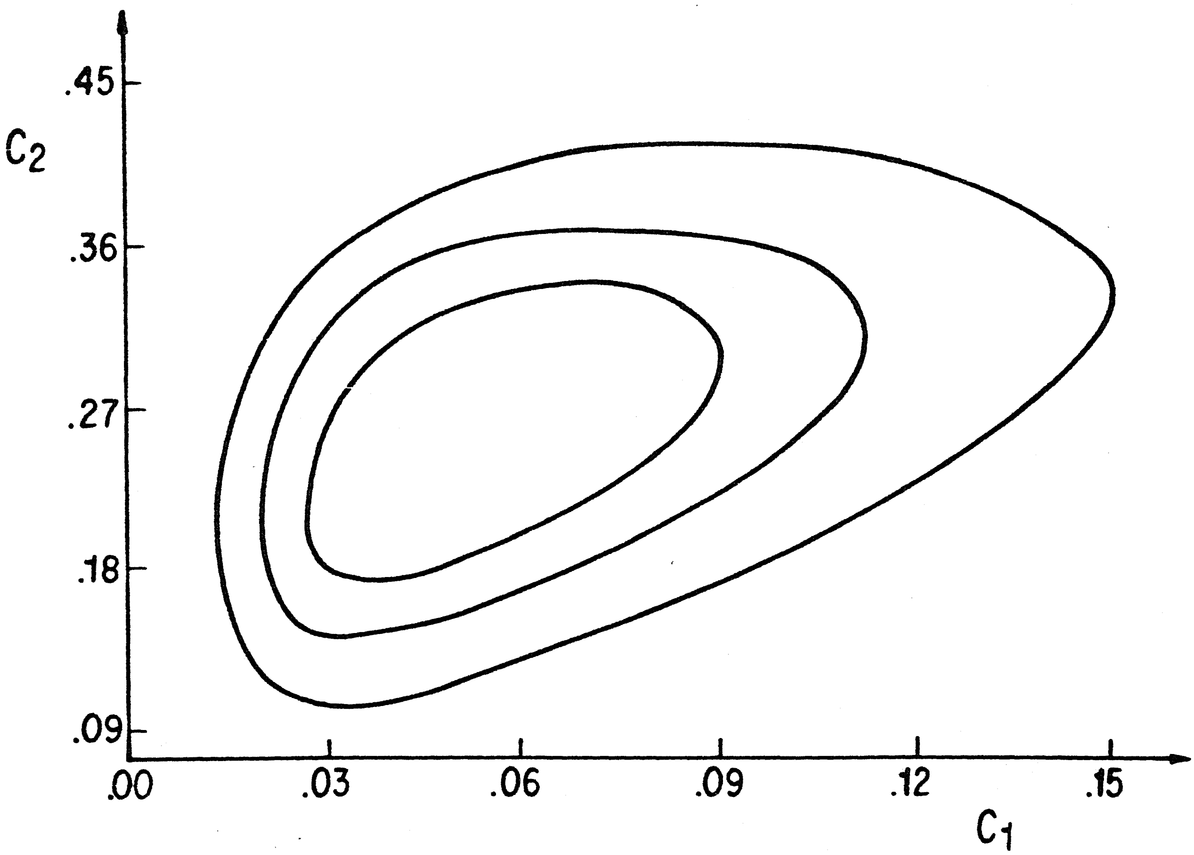

Figure 3: Contour Regions For The Joint Marginal Posterior Density for c_1 and c_2.

point out that the accuracy of the Laplace's method usually depends on a good parametrization dependent upon both the data and the choice of prior to get accurate approximate posterior moments or posterior densities (see for example, Achcar and Smith, 1990).

Appendix
The Laplace's Method

Assuming h is a smooth function of an m-dimensional parameter Θ with-h having a maximum at $\hat{\Theta}$, the Laplace's method approximates an integral of the form,

$$I = \int f(\Theta) exp[-nh(\Theta)]\, d\Theta \qquad (A.1)$$

by expanding h and f in a Taylor series about $\hat{\Theta}$ (see for example, Kass, Tierney and Kadane, 1990).

Considering first the case in which Θ is one-dimensional, the Laplace's method gives the approximation,

$$\hat{I} \cong \left(\frac{2\pi}{n}\right)^{1/2} \sigma f(\hat{\Theta}) exp\left[-nh(\hat{\Theta})\right] \qquad (A.2)$$

where $\sigma = \left\{h''(\hat{\Theta})\right\}^{-1/2}$.

In the multiparameter case, with $\Theta \in R^m$, we have,

$$\hat{I} \cong (2\pi)^{m/2} \left\{det\left(nD^2 h(\hat{\Theta})\right)\right\}^{-1/2} f(\hat{\Theta}) exp\left[-nh(\hat{\Theta})\right] \qquad (A.3)$$

where $\hat{\Theta}$ maximizes $-h(\hat{\Theta})$ and $D^2 h(\Theta)$ is the Hessian matrix of h evaluated at $\hat{\Theta}$.

The accuracy of these aproximations were studied by Kass, Tierney and Kadane (1990). A special case of the Laplace's method is given for integrals of the form $\int e^{nh(\Theta)} d\Theta$ (see Tierney and Kadane, 1986; Tierney, Kass and Kadane, 1989).

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