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Inferences For Accelerated Life Tests Considering A Bivariate Exponential Distribution

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SUMMARY

In this paper, we present some considerations on inferences about the use of asymptotical normality of the maximum likelihood estimators of the bivariate exponential distribution (BVED) of Block and Basu (1974) considering accelerated life tests with a power rule model. We show in one numerical example, the effect of a different parametrization to improve the accuracy of the obtained inferences on the parameters of the model. We also consider Bayesian methods to get inferences on the parameters of the BVED applied to accelerated life tests.

Key Words: Bivariate exponential distribution, accelerated life tests, reparametrization, Bayesian methods.

1 Introduction

Let us assume that we have two failure times associated to each observational unit in an accelerated life test problem. That is, consider a two-component life times X and Y, J stress levels $V_1, V_2, \ldots, V_j$ and life tests are conducted at constant application of the selected stresses. Using this information, we get inferences about the component life times under normal stress condition given by V_0.

At a normal stress level V_0 assume that (X, Y) has the bivariate exponential distribution (BVED) of Block and Basu (1974) with parameters $\lambda_{10}, \lambda_{20}$ and λ_{30}. Also assume that under a stress level $V_j, j = 1, 2, \ldots, J, (X, Y)$ has the BVED with parameters $\lambda_{1j}, \lambda_{2j}$ and $\lambda_{3j}, j = 1, 2, \ldots, J$ and joint probability density function given by

$$f(x,y) = \begin{cases} \frac{\lambda_{1j}\lambda_j(\lambda_{2j}+\lambda_{3j})}{\lambda_{1j}+\lambda_{2j}} \quad exp \quad \{-\lambda_{1j}x - (\lambda_{2j}+\lambda_{3j})y\} & \text{if } x < y \\ \frac{\lambda_{2j}\lambda_j(\lambda_{1j}+\lambda_{3j})}{\lambda_{1j}+\lambda_{2j}} \quad exp \quad \{-(\lambda_{1j}+\lambda_{3j})x - \lambda_{2j}y\} & \text{if } x \geq y \end{cases} \tag{1}$$

where $\lambda_{1j} = \lambda_{1j} + \lambda_{2j} + \lambda_{3j}, j = 0, 1, 2, \ldots, J.$

The correlation between X and Y under the stress level V_j is given by,

$$\begin{aligned} \rho_{x,y} &= \lambda_{3j}\left[\left(\lambda_{1j}^2 + \lambda_{2j}^2\right)\lambda_j + \lambda_{1j}\lambda_{2j}\lambda_{3j}\right] \times \\ &\times \left[(\lambda_{1j} + \lambda_{2j})^2 \lambda_j^2 + \lambda_{2j}\lambda_{3j}(2\lambda_{1j}\lambda_j + \lambda_{3j}\lambda_{2j})\right]^{-1/2} \\ &\times \left[(\lambda_{1j} + \lambda_{2j})^2 \lambda_j^2 + \lambda_{1j}\lambda_{3j}(2\lambda_{2j}\lambda_j + \lambda_{3j}\lambda_{1j})\right]^{-1/2} \end{aligned} \tag{2}$$

where $\rho_{x,y}$ is independent of $j, j = 1, 2, \ldots, J$, that is, the correlation between X and Y should remain constant from stress to stress.

Also, consider the power rule model (see for example, Mann, Schafer and Singpurwalla, 1974), given by

$$\lambda_{ij} = c_i V_j^{\mathbb{P}} \tag{3}$$

where $i = 1, 2, 3; j = 0, 1, \ldots, J; c_1, c_2, c_3$ and $\mathbb{P}$ are constants. The model (3) is also considered by Basu and Ebrahimi (1987).

Usually, the inferences on c_1, c_2, c_3 and $\mathbb{P}$ or even functions of these parameters considering the BVED with density (1) and the power rule model (3) are obtained by using the asymptotical normality of the maximum likelihood estimators of c_1, c_2, c_3 and $\mathbb{P}$. A practical problem of great interest to statisticians is related to the accuracy of these asymptotical results considering small or moderate sample sizes.

In this paper, we present the standard classical approach for inferences on the parameters of the BVED with density (1) and the power rule model (3) based on the asymptotical normality of the maximum likelihood estimators and we show in a numerical example, the effect of a different parametrization to improve the accuracy of the obtained inferences on the parameters of the model. We also present a Bayesian analysis of the model based on a noninformative prior density for the parameters and we find the marginal posterior for $\mathbb{P}$ using the Laplace's method for approximation of integrals (see for example, Kass, Tierney and Kadane, 1990).

2 The Likelihood Function For c_1, c_2, c_3 And $\mathbb{P}$

Let (X, Y) be a nonnegative bivariate random vector representing the life times of each two-components unit in an accelerated life test problem with a BVED density (1) and the power rule model (3) for the parameter $\lambda_{ij}, i = 1, 2, 3$ under the stress variable $V_j, j = 0, 1, \ldots, J$.

Considering n_j units $(X_{1j}, Y_{1j}), (X_{2j}, Y_{2j}), \ldots\ldots, \left(X_{n_j j}, Y_{n_j j}\right)$ at the beginning of each test with stress V_j, the likelihood function for c_1, c_2, c_3 and $\mathbb{P}$ is given by

$$L_j(c_1, c_2, c_3, \mathbb{P}) = \prod_{i=1}^{n_j} f_1^{\delta_{ij}}(X_{ij}, Y_{ij}) \, f_2^{1-\delta_{ij}}(X_{ij}, Y_{ij}) \tag{4}$$

where $\delta_{ij} = 1$ if $X_{ij} < Y_{ij}, \delta_{ij} = 0$ if $X_{ij} \geq Y_{ij}$, and

$$\begin{aligned} f_1(X_{ij}, Y_{ij}) &= \tfrac{c_1 c_{23} c_{123}}{c_{12}} V_j^{2\mathbb{P}} exp\left\{-\left[c_1 X_{ij} + c_{23} Y_{ij}\right] V_j^{\mathbb{P}}\right\} , \\ f_2(X_{ij}, Y_{ij}) &= \tfrac{c_2 c_{13} c_{123}}{c_{12}} V_j^{2\mathbb{P}} exp\left\{-\left[c_{13} X_{ij} + c_2 Y_{ij}\right] V_j^{\mathbb{P}}\right\} , \\ c_{12} &= c_1 + c_2, c_{13} = c_1 + c_3, c_{23} = c_2 + c_3 \text{ and } c_{123} = c_1 + c_2 + c_3 . \end{aligned}$$

That is,

$$\begin{aligned} L_j(c_1, c_2, c_3, \mathbb{P}) &= \frac{c_{123}^{nj} c_1^{rj} c_{23}^{rj} c_2^{nj-rj} c_{13}^{nj-rj}}{c_{12}^{nj}} \left(V_j^{2\mathbb{P}}\right)^{nj} \\ &\times \; exp\left\{-\left[c_1 n_j \overline{X}_j + c_2 n_j \overline{Y}_j + c_3 R_j\right] V_j^{\mathbb{P}}\right\} \end{aligned} \tag{5}$$

where $n_j \overline{X}_j = \sum_{i=1}^{nj} X_{ij}, n_j \overline{Y}_j = \sum_{i=1}^{nj} Y_{ij}, r_j = \sum_{i=1}^{nj} \delta_{ij}$ and $R_j = \sum_{i=1}^{nj} \left[Y_{ij}\delta_{ij} + (1 - \delta_{ij}) X_{ij}\right]$.

Considering the data of J stress levels $V_1, V_2, \ldots, V_J$ taken at random, the likelihood function for c_1, c_2, c_3 and $\mathbb{P}$ is given by

$$L_n(c_1, c_2, c_3, \mathbb{P}) = \prod_{j=1}^{J} L_j(c_1, c_2, c_3, \mathbb{P}) \tag{6}$$

That is,

$$L_n(c_1,c_2,c_3,\mathbb{P}) = \frac{c_1^r c_{23}^r c_2^{n-r} c_{13}^{n-r} c_{123}^n}{c_{12}^n} \times \times \left\{\prod_{j=1}^{J}\left(V_j^{2\mathbb{P}}\right)^{nj}\right\} exp\{-[c_1 S_x(\mathbb{P}) + c_2 S_y(\mathbb{P}) + c_3 T(\mathbb{P})]\} \tag{7}$$

where $r = \sum_{j=1}^{J} rj, n = \sum_{j=1}^{J} n_j, S_x(\mathbb{P}) = \sum_{j=1}^{J} nj\overline{X}_j V_j^{\mathbb{P}}, S_y(\mathbb{P}) = \sum_{j=1}^{J} n_j \overline{Y}_j V_j^{\mathbb{P}}, T(\mathbb{P}) = \sum_{j=1}^{J} R_j V_j^{\mathbb{P}}; c_{12}, c_{13}, c_{23}$ and c_{123} are given in (4).

3 Maximum Likelihood Estimators For c_1, c_2, c_3 And $\mathbb{P}$

The logarithm of the likelihood function (7) is given by

$$\begin{aligned} ln(c_1,c_2,c_3,\mathbb{P}) &= rlnc_1 + rlnc_{23} + (n-r)lnc_2 + \\ &+ (n-r)lnc_{13} + nlnc_{123} - nlnc_{12} + \\ &+ 2\mathbb{P}\sum_{j=1}^{J} n_j ln V_j - c_1 S_x(\mathbb{P}) - c_2 S_y(\mathbb{P}) - c_3 T(\mathbb{P}) \end{aligned} \tag{8}$$

From $\partial ln/\partial c_1 = 0, \partial ln/\partial c_2 = 0, \partial ln/\partial c_3 = 0$ and $\partial ln/\partial \mathbb{P} = 0$, we find the maximum likelihood estimators $\hat{c}_1, \hat{c}_2, \hat{c}_3$ and $\hat{\mathbb{P}}$ by solving the likelihood equations,

$$\begin{aligned} \frac{r}{\hat{c}_1} &+ \frac{(n-r)}{(\hat{c}_1+\hat{c}_3)} + \frac{n}{(\hat{c}_1+\hat{c}_2+\hat{c}_3)} - \frac{n}{(\hat{c}_1+\hat{c}_2)} = \sum_{j=1}^{J} n_j \overline{X}_j V_j^{\hat{\mathbb{P}}} \\ \frac{r}{(\hat{c}_2+\hat{c}_3)} &+ \frac{(n-r)}{\hat{c}_2} + \frac{n}{(\hat{c}_1+\hat{c}_2+\hat{c}_3)} - \frac{n}{(\hat{c}_1+\hat{c}_2)} = \sum_{j=1}^{J} n_j \overline{Y}_j V_j^{\hat{\mathbb{P}}} \\ \frac{r}{(\hat{c}_2+\hat{c}_3)} &+ \frac{(n-r)}{(\hat{c}_1+\hat{c}_3)} + \frac{n}{(\hat{c}_1+\hat{c}_2+\hat{c}_3)} = \sum_{j=1}^{J} R_j V_j^{\hat{\mathbb{P}}} \end{aligned} \tag{9}$$

and $\hat{c}_1 \sum_{j=1}^{J} n_j \overline{X}_j V_j^{\hat{\mathbb{P}}} (lnV_j) + \hat{c}_2 \sum_{j=1}^{J} n_j \overline{Y}_j V_j^{\hat{\mathbb{P}}} (lnV_j) + \hat{c}_3 \sum_{j=1}^{J} R_j V_j^{\hat{\mathbb{P}}} (lnV_j) = 2\sum_{j=1}^{J} n_j lnV_j$.

For inferences on $\underset{\sim}{\Psi} = (c_1, c_2, c_3, \mathbb{P})$ or even functions of the parameters, we usually use the asymptotical normality of the maximum likelihood estimators, given by

$$\hat{\underset{\sim}{\Psi}} = \left(\hat{c}_1, \hat{c}_2, \hat{c}_3, \hat{\mathbb{P}}\right) \overset{a}{\sim} N\left\{\underset{\sim}{\Psi}\, ;\, I_0^{-1}\right\} \tag{10}$$

where I_o is the observed information matrix given by

$$I_o = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ & b_{22} & b_{23} & b_{24} \\ & & b_{33} & b_{34} \\ symmetric & & & b_{44} \end{pmatrix} \tag{11}$$

where

$$b_{11} = -\frac{\partial^2 ln\left(\underset{\sim}{\hat{\Psi}}\right)}{\partial c_1^2} = \frac{r}{\hat{c}_1} + \frac{(n-r)}{\left(\hat{c}_1+\hat{c}_3\right)^2} + \frac{n}{\left(\hat{c}_1+\hat{c}_2+\hat{c}_3\right)^2} - \frac{n}{\left(\hat{c}_1+\hat{c}_2\right)^2}$$

$$b_{12} = -\frac{\partial^2 ln\left(\underset{\sim}{\hat{\Psi}}\right)}{\partial c_1 \partial c_2} = \frac{n}{\left(\hat{c}_1+\hat{c}_2+\hat{c}_3\right)^2} - \frac{n}{\left(\hat{c}_1+\hat{c}_2\right)^2}$$

$$b_{13} = -\frac{\partial^2 ln\left(\underset{\sim}{\hat{\Psi}}\right)}{\partial c_1 \partial c_3} = \frac{(n-r)}{\left(\hat{c}_1+\hat{c}_3\right)^2} + \frac{n}{\left(\hat{c}_1+\hat{c}_2+\hat{c}_3\right)^2}$$

$$b_{14} = -\frac{\partial^2 ln\left(\underset{\sim}{\hat{\Psi}}\right)}{\partial c_1 \partial \mathbb{P}} = \sum_{j=1}^{J} n_j \overline{X}_j V_j^{\hat{\mathbb{P}}} \left(lnV_j\right)$$

$$b_{22} = -\frac{\partial^2 ln\left(\underset{\sim}{\hat{\Psi}}\right)}{\partial c_2^2} = \frac{r}{\left(\hat{c}_2+\hat{c}_3\right)^2} + \frac{(n-r)}{\hat{c}_2^2} + \frac{n}{\left(\hat{c}_1+\hat{c}_2+\hat{c}_3\right)^2} - \frac{n}{\left(\hat{c}_1+\hat{c}_2\right)^2}$$

$$b_{23} = -\frac{\partial^2 ln\left(\underset{\sim}{\hat{\Psi}}\right)}{\partial c_2 \partial c_3} = \frac{r}{\left(\hat{c}_c+\hat{c}_3\right)^2} + \frac{n}{\left(\hat{c}_1+\hat{c}_2+\hat{c}_3\right)^2}$$

$$b_{24} = -\frac{\partial^2 \ln\left(\hat{\underset{\sim}{\Psi}}\right)}{\partial c_2 \partial \mathbb{P}} = \sum_{j=1}^{J} n_j \bar{y}_j V_j^{\hat{\mathbb{P}}} (\ln V_j)$$

$$b_{33} = -\frac{\partial^2 \ln\left(\hat{\underset{\sim}{\Psi}}\right)}{\partial c_3^2} = \frac{r}{\left(\hat{c}_2+\hat{c}_3\right)^2} + \frac{(n-r)}{\left(\hat{c}_1+\hat{c}_3\right)^2} + \frac{n}{\left(\hat{c}_1+\hat{c}_2+\hat{c}_3\right)^2}$$

$$b_{34} = -\frac{\partial^2 \ln\left(\hat{\underset{\sim}{\Psi}}\right)}{\partial c_3 \partial \mathbb{P}} = \sum_{j=1}^{J} R_j V_j^{\hat{\mathbb{P}}} (\ln V_j)$$

$$b_{44} = -\frac{\partial^2 \ln\left(\hat{\underset{\sim}{\Psi}}\right)}{\partial \mathbb{P}^2} = \hat{c}_1 \sum_{j=1}^{j} n_j \overline{X}_j V_j^{\hat{\mathbb{P}}} (\ln V_j)^2 +$$

$$+ \quad \hat{c}_2 \sum_{j=1}^{J} n_j \overline{Y}_j V_j^{\hat{\mathbb{P}}} (\ln V_j)^2 + \hat{c}_3 \sum_{j=1}^{J} R_j V_j^{\hat{\mathbb{P}}} (\ln V_j)^2$$

We also could use the Fisher information matrix $I(\hat{\underset{\sim}{\Psi}})$ in place of the observed information matrix I_0 given in the asymptotical normal distribution (10). The approximation (10) may not be very good in small-or-moderate size samples. It can in fact be shown that these approximations are rather poor unless the sample size is fairly large. Fortunatelly, there are alternate approximate procedures that can be recommended, even with small samples. One of these alternate procedures is to consider a reparametrization that improves the normal approximation to the likelihood function for small or moderate sample sizes. We can explore the method proposed by Anscombe (1964) and by Sprott (1973,1980) considering a reparametrization such that the third derivatives of the logarithm of the likelihood function locally in the maximum likelihood estimators are close to zero to get good approximate normal likelihoods.

4 A Bayesian Approach For Inferences On The Paramaters Of The Model

Assuming that (X, Y) has a BVED with density (1) and the power rule model (3) for the parameter $\lambda_{ij}, i = 1,2,3$, under the stress variable $V_j, j = 0,1,2,\ldots,J$, the prior density for c_1, c_2, c_3 and $\mathbb{P}$ can be written in the form $\pi(c_1, c_2, c_3, \mathbb{P}) = \pi(c_1, c_2, c_3|\mathbb{P})\, \pi_0(\mathbb{P})$. Using the Jeffreys multiparameter rule (see for example, Box and Tiao, 1973), that is, $\pi(c_1, c_2, c_3|\mathbb{P}) \propto \{det I_{\mathbb{P}}(c_1, c_2, c_3)\}^{1/2}$, where $I_{\mathbb{P}}(c_1, c_2, c_3)$ is the Fisher information matrix given $\mathbb{P}$, and a locally uniform prior for $\mathbb{P}$ ($\pi_0(\mathbb{P}) \propto$ constant, $-\infty < \mathbb{P} < \infty$), we consider the noninformative prior for c_1, c_2, c_3 and $\mathbb{P}$ given by

$$\pi(c_1, c_2, c_3, \mathbb{P}) \propto \{det I_{\mathbb{P}}(c_1, c_2, c_3)\}^{1/2} \tag{12}$$

where $c_1, c_2, c_3 > 0, -\infty < \mathbb{P} < \infty$ and

$$I_{\mathbb{P}}(c_1, c_2, c_3) = \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ & d_{22} & d_{23} \\ symmetric & & d_{33} \end{pmatrix} \tag{13}$$

where

$$\begin{aligned}
d_{11} &= E\left(-\frac{\partial^2 l_n}{\partial c_1^2}\right) = n\left\{\frac{1}{c_1 c_{12}} + \frac{c_2}{c_{12} c_{13}^2} + \frac{1}{c_{123}^2} - \frac{1}{c_{12}^2}\right\} \\
d_{12} &= E\left(-\frac{\partial^2 l_n}{\partial c_1 \partial c_2}\right) = n\left\{\frac{1}{c_{123}^2} - \frac{1}{c_{12}^2}\right\} \\
d_{13} &= E\left(-\frac{\partial^2 l_n}{\partial c_1 \partial c_3}\right) = n\left\{\frac{c_2}{c_{12} c_{13}^2} + \frac{1}{c_{123}^2}\right\} \\
d_{22} &= E\left(-\frac{\partial^2 l_n}{\partial c_2^2}\right) = n\left\{\frac{c_1}{c_{12} c_{23}^2} + \frac{1}{c_2 c_{12}} + \frac{1}{c_{123}^2} - \frac{1}{c_{12}^2}\right\} \\
d_{23} &= E\left(-\frac{\partial^2 l_n}{\partial c_2 \partial c_3}\right) = n\left\{\frac{c_1}{c_{12} c_{23}^2} + \frac{1}{c_{123}^2}\right\} \\
d_{33} &= E\left(-\frac{\partial^2 l_n}{\partial c_3^2}\right) = n\left\{\frac{c_1}{c_{12} c_{23}^2} + \frac{c_2}{c_{12} c_{13}^2} + \frac{1}{c_{123}^2}\right\}
\end{aligned}$$

(see for example Basu and Ebrahimi, 1987).

That is,

$$\pi(c_1, c_2, c_3, \mathbb{P}) \propto a(c_1, c_2, c_3) \tag{14}$$

where

$$a(c_1, c_2, c_3) = \left\{d_{11} d_{22} d_{33} + 2 d_{12} d_{23} d_{13} - d_{22} d_{13}^2 - d_{11} d_{23}^2 - d_{33} d_{12}^2\right\}^{1/2}.$$

Considering the prior (14), the joint posterior density for c_1, c_2, c_3 and $\mathbb{P}$, is given by

$$\begin{aligned}
\pi(c_1, c_2, c_3, \mathbb{P} | \text{DATA}) &\propto \frac{a(c_1, c_2, c_3) c_1^r c_{23}^r c_2^{n-r} c_{13}^{n-r} c_{123}^n}{c_{12}^n} \times \\
&\times \left\{\prod_{j=1}^J V_j^{2\mathbb{P} n_j}\right\} exp\{-[c_1 S_X(\mathbb{P}) + c_2 S_y(\mathbb{P}) + c_3(T(P))]\}
\end{aligned} \tag{15}$$

where $c_1, c_2, c_3 > 0$ and $-\infty < \mathbb{P} < \infty$.

4.1 The Marginal Posterior Density For $\mathbb{P}$

The marginal posterior density for $\mathbb{P}$ is given (from (15)) by

$$\pi(\mathbb{P}|\text{DATA}) \propto \int_0^\infty \int_0^\infty \int_0^\infty a(c_1, c_2, c_3) e^{-nh(c_1,c_2,c_3|\mathbb{P})} dc_1 dc_2 dc_3 \tag{16}$$

where $a(c_1, c_2, c_3)$ is given in (14) and $-nh(c_1, c_2, c_3|\mathbb{P}) = ln(c_1, c_2, c_3, \mathbb{P})$ (see (8)).

An approximate marginal posterior density for $\mathbb{P}$, using the Laplace's method for approximation of integrals (see for example, Tierney and Kadane, 1986; or, Kass, Tierney and Kadane, 1990), is given by

$$\begin{aligned} \pi(\mathbb{P}|\text{DATA}) \propto & \left\{det\left(nD^2h\left(\hat{c}_1, \hat{c}_2, \hat{c}_3 \,|\mathbb{P}\right)\right)\right\}^{-1/2} \times \\ & \times \; a\left(\hat{c}_1, \hat{c}_2, \hat{c}_3\right) e^{-nh\left(\hat{c}_1, \hat{c}_2, \hat{c}_3|\mathbb{P}\right)} \end{aligned} \tag{17}$$

where $\hat{c}_1, \hat{c}_2$ and $\hat{c}_3$ maximize $-nh(c_1, c_2, c_3|\mathbb{P})$ for each value of $\mathbb{P}$ and $D^2h(c_1, c_2, c_3|\mathbb{P})$ is the Hessian matrix calculated in $\left(\hat{c}_1, \hat{c}_2, \hat{c}_3\right)$ given by

$$D^2h\left(\hat{c}_1, \hat{c}_2, \hat{c}_3 \,|\mathbb{P}\right) = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ & b_{22} & b_{23} \\ symmetric & & b_{33} \end{pmatrix} \tag{18}$$

where $b_{ij}, i = 1,2,3; j = 1,2,3$ are given in (11).

That is

$$\begin{aligned} \pi(\mathbb{P}|\text{DATA}) \propto & \frac{a\left(\hat{c}_1, \hat{c}_2, \hat{c}_3\right) \hat{c}_1^{r} \hat{c}_{23}^{r} \hat{c}_2^{n-r} \hat{c}_{13}^{n-r} \hat{c}_{123}^{n}}{\hat{c}_{12}^{n} b\left(\hat{c}_1, \hat{c}_2, \hat{c}_3\right)} \\ & \times \; \left\{\prod_{j=1}^{J} V_j^{2\mathbb{P}n_j}\right\} exp\left\{-\left[\hat{c}_1 \, S_X(\mathbb{P}) + \hat{c}_2 \, S_y(\mathbb{P}) + \hat{c}_3 \, T(\mathbb{P})\right]\right\} \end{aligned} \tag{19}$$

where $-\infty < \mathbb{P} < \infty$, $a(c_1, c_2, c_3)$ is given in (14), $\hat{c}_{12} = \hat{c}_1 + \hat{c}_2$, $\hat{c}_{13} = \hat{c}_1 + \hat{c}_3$, $\hat{c}_{23} = \hat{c}_2 + \hat{c}_3$, $\hat{c}_{123} = \hat{c}_1 + \hat{c}_2 + \hat{c}_3$ and

$$b\left(\hat{c}_1, \hat{c}_2, \hat{c}_3\right) = \left\{b_{11}b_{22}b_{33} + 2b_{12}b_{23}b_{13} - b_{22}b_{13}^2 - b_{11}b_{23}^2 - b_{33}b_{12}^2\right\}^{1/2}.$$

We also could use the Laplace's method to find the marginal posterior densities for c_1, c_2, c_3 or even functions of the parameters.

5 A Reparametrization For The Model

To get accurate normality of the likelihood function for c_1, c_2, c_3 and $\mathbb{P}$, usually the statistician considers different parametrizations dependent upon the data to obtain third derivatives of the logarithm of the likelihood function locally in the maximum likelihood estimators close to zero. As a special case, we could explore the parametrization $\Theta_i = lnc_i, i = 1,2,3$ and $\mathbb{P}$, by comparing the third derivatives of the log-likelihood function for $\underset{\sim}{\Theta} = (\Theta_1, \Theta_2, \Theta_3, \mathbb{P})$ with the third derivatives of the log-likelihood function for $\underset{\sim}{\Psi} = (c_1, c_2, c_3, \mathbb{P})$.

The logarithm of the likelihood function for $\Theta_1, \Theta_2, \Theta_3$, and $\mathbb{P}$ is given by

$$\begin{aligned} ln\,(\Theta_1, \Theta_2, \Theta_3, \mathbb{P}) &= r\Theta_1 + rln\left(e^{\Theta_2} + e^{\Theta_3}\right) + (n-r)\Theta_2 - nln\left(e^{\Theta_1} + e^{\Theta_2}\right) + \\ &+ (n-r)ln\left(e^{\Theta_1} + e^{\Theta_3}\right) + nln\left(e^{\Theta_1} + e^{\Theta_2} + e^{\Theta_3}\right) + \\ &+ 2\mathbb{P}\textstyle\sum_{j=1}^{J} n_j ln V_j - e^{\Theta_1} S_X(\mathbb{P}) - e^{\Theta_2} S_y(\mathbb{P}) - e^{\Theta_3} T(\mathbb{P}) \end{aligned} \tag{20}$$

where $r, n, S_X(\mathbb{P}), S_y(\mathbb{P})$ and $T(\mathbb{P})$ are given in (7).

The second derivatives of the log-likelihood function (20) are given by

$$\frac{\partial^2 ln}{\partial \Theta_1^2} = e^{\Theta_1}\left\{\frac{(n-r)e^{\Theta_3}}{\left(e^{\Theta_1}+e^{\Theta_3}\right)^2} - \frac{ne^{\Theta_2}}{\left(e^{\Theta_1}+e^{\Theta_2}\right)^2} + \frac{n\left(e^{\Theta_2}+e^{\Theta_3}\right)}{\left(e^{\Theta_1}+e^{\Theta_2}+e^{\Theta_3}\right)^2} - S_X(\mathbb{P})\right\}$$

$$\frac{\partial^2 ln}{\partial \Theta_2^2} = e^{\Theta_2}\left\{\frac{re^{\Theta_3}}{\left(e^{\Theta_2}+e^{\Theta_3}\right)^2} - \frac{ne^{\Theta_1}}{\left(e^{\Theta_1}+e^{\Theta_2}\right)^2} + \frac{n\left(e^{\Theta_1}+e^{\Theta_3}\right)}{\left(e^{\Theta_1}+e^{\Theta_2}+e^{\Theta_3}\right)^2} - S_y(\mathbb{P})\right\}$$

$$\frac{\partial^2 ln}{\partial \Theta_3^2} = e^{\Theta_3}\left\{\frac{re^{\Theta_2}}{\left(e^{\Theta_2}+e^{\Theta_3}\right)^2} + \frac{(n-r)e^{\Theta_1}}{\left(e^{\Theta_1}+e^{\Theta_3}\right)^2} + \frac{n\left(e^{\Theta_1}+e^{\Theta_2}\right)}{\left(e^{\Theta_1}+e^{\Theta_2}+e^{\Theta_3}\right)^2} - T(\mathbb{P})\right\}$$

$$\frac{\partial^2 ln}{\partial \Theta_1 \partial \Theta_2} = ne^{\Theta_1+\Theta_2}\left\{\frac{1}{\left(e^{\Theta_1}+e^{\Theta_2}\right)^2} - \frac{1}{\left(e^{\Theta_1}+e^{\Theta_2}+e^{\Theta_3}\right)^2}\right\} \tag{21}$$

$$\frac{\partial^2 ln}{\partial \Theta_1 \partial \Theta_3} = -e^{\Theta_1+\Theta_3}\left\{\frac{(n-r)}{\left(e^{\Theta_1}+e^{\Theta_3}\right)^2} + \frac{n}{\left(e^{\Theta_1}+e^{\Theta_2}+e^{\Theta_3}\right)^2}\right\}$$

$$\frac{\partial^2 ln}{\partial \Theta_2 \partial \Theta_3} = -e^{\Theta_2+\Theta_3}\left\{\frac{r}{\left(e^{\Theta_2}+e^{\Theta_3}\right)^2} + \frac{n}{\left(e^{\Theta_1}+e^{\Theta_2}+e^{\Theta_3}\right)^2}\right\}$$

$$\begin{aligned} \frac{\partial^2 ln}{\partial \mathbb{P}^2} &= -e^{\Theta_1}\textstyle\sum_{j=1}^{J} n_j \overline{X}_j V_j^{\mathbb{P}} (lnV_j)^2 - e^{\Theta_2}\sum_{j=1}^{J} n_j \overline{Y}_j V_j^{\mathbb{P}} (lnV_j)^2 - \\ &- e^{\Theta_3}\textstyle\sum_{j=1}^{J} R_j V_j^{\mathbb{P}} (lnV_j)^2 \end{aligned}$$

$$\frac{\partial^2 ln}{\partial \mathbb{P} \partial \Theta_1} = -e^{\Theta_1} \sum_{j=1}^{J} n_j \overline{X}_j V_j^{\mathbb{P}} (lnV_j)$$

$$\frac{\partial^2 ln}{\partial \mathbb{P} \partial \Theta_2} = -e^{\Theta_2} \sum_{j=1}^{J} n_j \overline{Y}_j V_j^{\mathbb{P}} (lnV_j)$$

$$\frac{\partial^2 ln}{\partial \mathbb{P} \partial \Theta_3} = -e^{\Theta_3} \sum_{j=1}^{J} R_j V_j^{\mathbb{P}} (lnV_j)$$

In the same way as it was considered in the parametrization $\underset{\sim}{\psi} = (c_1, c_2, c_3, \mathbb{P})$, we usually get inferences on the parameters $\Theta_1, \Theta_2, \Theta_3$ and $\mathbb{P}$ based on the asymptotical normality of the maximum likelihood estimators $\underset{\sim}{\hat{\Theta}} = \left(\hat{\Theta}_1, \hat{\Theta}_2, \hat{\Theta}_3, \hat{\mathbb{P}}\right)$.

6 An Example

In this example, we show in a simulated data set the importance of parametrization to get good inferences for the parameters of the BVED in accelerated life tests. In table 1, we have the data of an accelerated life test with three stress levels $V_1 = 1, V_2 = 2$ and $V_3 = 3$. At each stress level, 15 bivariate observations (X, Y) were generated from a BVED with density (1) and the power rule model (3) with $\mathbb{P} = 2$. From table 1, we get $n_1 = n_2 = n_3 = 15, n = 45, r_1 = 4, r_2 = 3, r_3 = 2$ and $r = 9$.

The likelihood equations for c_1, c_2, c_3 and $\mathbb{P}$ (see (9)) are given by

$$\begin{aligned} \frac{9}{c_1} + \frac{36}{(c_1+c_3)} &+ \frac{45}{(c_1+c_2+c_3)} - \frac{45}{(c_1+c_2)} = \\ &= 148.41(1)^{\mathbb{P}} + 35.44(2)^{\mathbb{P}} + 16.02(3)^{\mathbb{P}} \\ \frac{9}{(c_2+c_3)} + \frac{36}{c_2} &+ \frac{45}{(c_1+c_2+c_3)} - \frac{45}{(c_1+c_2)} = \\ &= 48.55(1)^{\mathbb{P}} + 10.53(2)^{\mathbb{P}} + 5.41(3)^{\mathbb{P}} \end{aligned} \tag{22}$$

$$\begin{aligned} \frac{9}{(c_2+c_3)} + \frac{36}{(c_1+c_3)} &+ \frac{45}{(c_1+c_2+c_3)} = \\ &= 152.28(1)^{\mathbb{P}} + 36.66(2)^{\mathbb{P}} + 16.14(3)^{\mathbb{P}} \end{aligned}$$

and $c_1 \left(24.56(2)^{\mathbb{P}} + 17.60(3)^{\mathbb{P}}\right) + c_2 \left(7.30(2)^{\mathbb{P}} + 5.94(3)^{\mathbb{P}}\right) + c_3 \left(25.41(2)^{\mathbb{P}} + 17.73(3)^{\mathbb{P}}\right) = 53.75$

To simplify the solution of the likelihood equations (22), we consider the maximized likelihood function for $\mathbb{P}$, that is, we use the Newton-Raphson method to compute the maximum likelihood estimator $)\hat{c}_1 (\mathbb{P}), \hat{c}_2 (\mathbb{P})$ and $\hat{c}_3 (\mathbb{P})$ for different values of $\mathbb{P}$ (see table 2). The maximized likelihood function for $\mathbb{P}$ is then $L_{max}(\mathbb{P}) = \mathrm{L}(\hat{c}_1 (\mathbb{P}), \hat{c}_2 (\mathbb{P}), \hat{c}_3 (\mathbb{P})/\mathbb{P})$. From table 2, we observe that the maximum likelihood estimators are given by $\hat{c}_1 = 0.0571, \hat{c}_2 = 0.2643, \hat{c}_3 = 0.0602$ and $\hat{\mathbb{P}} = 2.03$.

$V_1 = 1$ (X_{1i}, Y_{1i})	$V_2 = 2$ (X_{2i}, Y_{2i})	$V_3 = 3$ (X_{3i}, Y_{3i})
(7.65;2.18)	(2.29;0.02)	(0.34;0.20)
(16.67;9.26)	(0.10,0.38)	(1.50;1.30)
(39.30;6.72)	(0.88;0.27)	(0.63;0.69)
(1.30;3.22)	(0.45;0.04)	(0.68;0.12)
(9.04;2.23)	(1.66;1.60)	(3.22;0.09)
(5.15;0.41)	(0.74;1.67)	(1.91;0.91)
(5.20;5.91)	(2.50;0.37)	(0.52;0.58)
(5.00;0.84)	(3.50;0.03)	(0.30;0.01)
(5.66;0.42)	(8.45;0.71)	(1.30;0.02)
(11.80;0.15)	(4.60;0.83)	(0.52;0.10)
(17.08;10.37)	(2.66;1.06)	(2.08;0.30)
(17.92;0.76)	(1.46;1.04)	(0.95;0.91)
(1.62;2.73)	(1.03;0.41)	(0.43;0.02)
(1.42;1.85)	(4.36;1.34)	(0.25;0.08)
(3.60;1.50)	(0.76;0.77)	(1.39;0.08)

Table 1: Generated Bivariate Life Time Data

$\mathbb{P}$	$\hat{c}_1(\mathbb{P})$	$\hat{c}_2(\mathbb{P})$	$\hat{c}_3(\mathbb{P})$	$l(\hat{c}_1, \hat{c}_2, \hat{c}_3 \mid \mathbb{P})$
1.20	0.0883	0.4051	0.0901	-140.712
1.30	0.0844	0.3880	0.0864	-139.211
1.40	0.0806	0.3706	0.0827	-137.901
1.50	0.0768	0.3534	0.0791	-136.783
1.60	0.0729	0.3361	0.0754	-135.859
1.70	0.0692	0.3190	0.0718	-135.127
1.80	0.0655	0.3021	0.0682	-134.591
1.90	0.0618	0.2854	0.0647	-134.248
1.93	0.0607	0.2805	0.0636	-134.182
1.95	0.0600	0.2772	0.0629	-134.148
1.96	0.0596	0.2756	0.0626	-134.134
1.97	0.0593	0.2740	0.0622	-134.121
1.98	0.0589	0.2723	0.0619	-134.110
2.00	0.0582	0.2691	0.0612	-134.095
2.02	0.0575	0.2659	0.0605	-134.088
2.03	0.0571	0.2643	0.0602	-134.087
2.04	0.0568	0.2626	0.0599	-134.088
2.05	0.0564	0.2610	0.0595	-134.091
2.07	0.0557	0.2579	0.0588	-134.103
2.09	0.0550	0.2547	0.0582	-134.121
2.10	0.0547	0.2531	0.0578	-134.133
2.12	0.0540	0.2499	0.0572	-134.163
2.15	0.0530	0.2453	0.0562	-134.222
2.20	0.0513	0.2376	0.0545	-134.358
2.30	0.0480	0.2225	0.0513	-134.767
2.40	0.0448	0.2080	0.0482	-135.354
2.50	0.0417	0.1940	0.0452	-136.118
2.60	0.0388	0.1806	0.0424	-137.053
2.70	0.0360	0.1678	0.0396	-138.153
2.80	0.0334	0.1556	0.0369	-139.414

Table 2: Maximized Log-Likelihood Function $l\left(\hat{c}_1(\mathbb{P}), \hat{c}_2(\mathbb{P}), \hat{c}_3(\mathbb{P}) \mid \mathbb{P}\right)$ For Some Values Of $\mathbb{P}$.

j	V_j	r_j	n_j	$\overline{X}_j$	$\overline{Y}_j$	R_j
1	1	4	15	9.8940	3.2366	152.58
2	2	3	15	2.3627	0.7020	36.66
3	3	2	15	1.0680	0.3607	16.14

Table 3: Summary Statistics For Data In Table 1

In figure 1, we have the graph of the maximized likelihood function for $\mathbb{P}$.

From the information of table 3, we obtain the observed information matrix (see (11)) given by

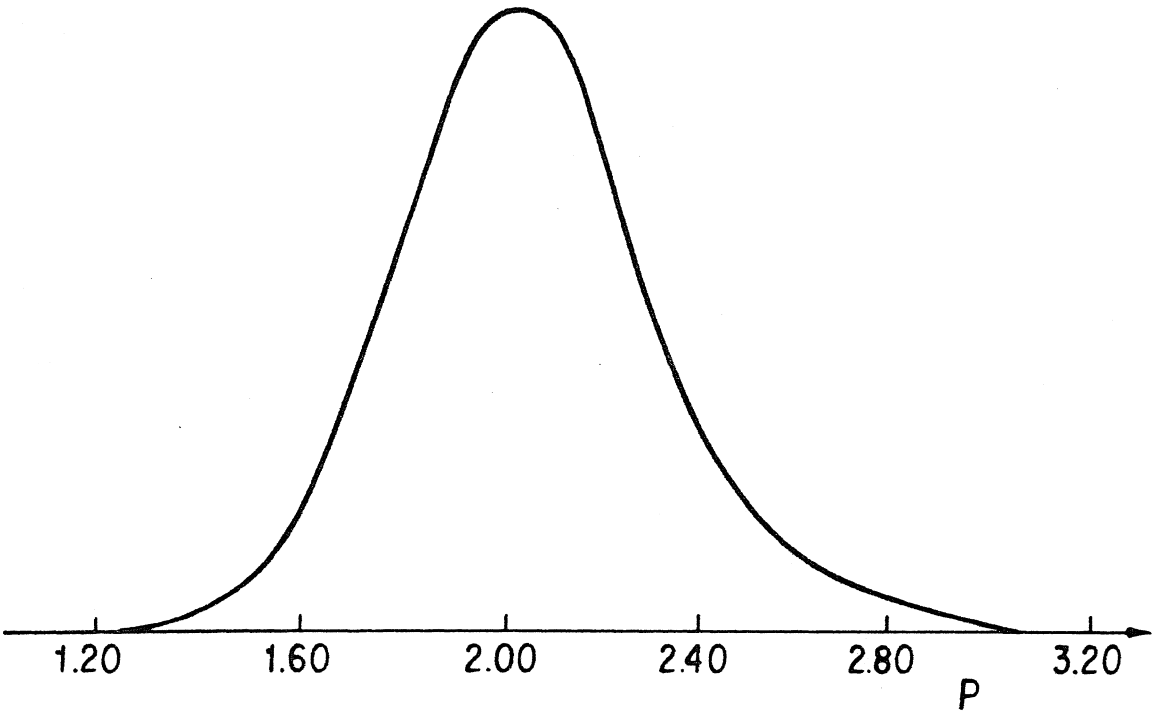

Figure 1: Maximized Likelihood Function for $\mathbb{P}$

$$I_0 = \begin{pmatrix} 5250.20 & -126.61 & 2925.44 & 264.03 \\ & 474.22 & 394.50 & 85.10 \\ & & 3010.91 & 268.71 \\ & & & 50.99 \\ symmetric & & & \end{pmatrix}$$

The inverse of the observed information matrix I_0 is given by

$$I_0^{-1} = \begin{pmatrix} 0.0006236 & 0.0009727 & -0.0005669 & -0.0018647 \\ & 0.0045440 & -0.0007820 & -0.0084989 \\ & & 0.0011459 & -0.0017977 \\ & & & 0.0529212 \\ symmetric & & & \end{pmatrix}$$

The approximate variances of $\hat{c}_1, \hat{c}_2, \hat{c}_3$ and $\hat{\mathbb{P}}$ (see (10)) are given by $\widehat{var}(\hat{c}_1) \cong 0.000624, \widehat{var}(\hat{c}_2$ $\cong 0.004544, \widehat{var}(\hat{c}_3) \cong 0.001146$ and $\widehat{var}(\hat{\mathbb{P}}) \cong 0.052921$. The 95% approximate confidence intervals for $\hat{c}_1, \hat{c}_2, \hat{c}_3$ and $\mathbb{P}$ are given by (0.00814;0.10606), (0.13218;0.39642),(−0.00615;0.12655) and (1.57911;2.48089), respectively.

Under the normal stress condition, we assume that $V_0 = 0.5$. Therefore, the maximum likelihood estimators for the parameters of the bivariate exponential distribution under normal condition are given by $\hat{\lambda}_{10}= 0.01398, \hat{\lambda}_{20}= 0.06471$ and $\hat{\lambda}_{30}= 0.01474$.

The mean life times of the two components under the stress level j are given by

$$\begin{aligned} \mu_{1j} = E(X) &= \frac{(c_1+c_2+c_3)(c_1+c_2)+c_2c_3}{(c_1+c_2+c_3)(c_1+c_2)(c_1+c_3)} V_j^{-\mathbb{P}} \\ \mu_{2j} = E(Y) &= \frac{(c_1+c_2+c_3)(c_1+c_2)+c_1c_3}{(c_1+c_2+c_3)(c_1+c_2)(c_2+c_3)} V_j^{-\mathbb{P}} \end{aligned} \tag{23}$$

Thus, the maximum likelihood estimators for the mean life times under the normal condition are given by $\hat{\mu}_{10}= 39.3340$ and $\hat{\mu}_{20}= 12.9384$.

In figure 2, we have the graph of the Laplace's approximate marginal posterior density for $\mathbb{P}$ given in (19) considering a noninformative prior density (14) for the parameters. We observe close inference results considering the maximized likelihood function for $\mathbb{P}$ (see figure 1) and the approximate marginal for $\mathbb{P}$ (see figure 2).

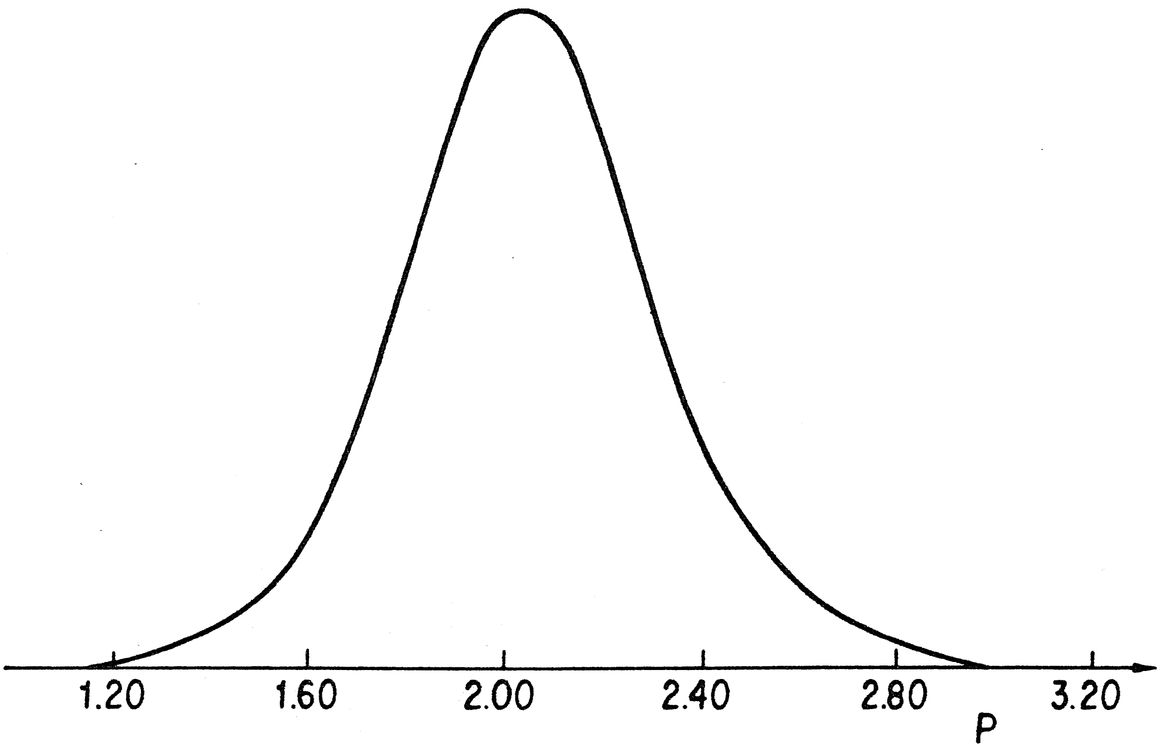

Figure 2: Marginal Posterior Density for $\mathbb{P}$

In table 4, we have some values for the maximized log-likelihood function $ln\left(\hat{c}_1, \hat{c}_2, \hat{c}_3 \mid \mathbb{P}\right)$ and for $\left\{a\left(\hat{c}_1, \hat{c}_2, \hat{c}_3\right)/b\left(\hat{c}_1, \hat{c}_2, \hat{c}_3\right)\right\} ln\left(\hat{c}_1, \hat{c}_2, \hat{c}_3 \mid \mathbb{P}\right)$ (see (19), the approximate marginal posterior density for $\mathbb{P}$).

$\mathbb{P}$	$ln(\hat{c}_1, \hat{c}_2, \hat{c}_3 \mid \mathbb{P})$	$\{a(\hat{c}_1, \hat{c}_2, \hat{c}_3)/b(\hat{c}_1, \hat{c}_2, \hat{c}_3)\}ln(\hat{c}_1, \hat{c}_2, \hat{c}_3 \mid \mathbb{P})$
1.20	-140.712	-140.775
1.30	-139.211	-139.275
1.40	-137.901	-137.965
1.50	-136.783	-136.848
1.60	-135.859	-135.924
1.70	-135.127	-135.192
1.80	-134.591	-134.657
1.90	-134.248	-134.314
2.00	-134.095	-134.182
2.10	-134.133	-134.200
2.20	-134.358	-134.425
2.30	-134.767	-134.835
2.40	-135.354	-135.422
2.50	-136.118	-136.187
2.60	-137.053	-137.122
2.70	-138.153	-138.223
2.80	-139.414	-139.484

Table 4: Some Values For $ln(\hat{c}_1, \hat{c}_2, \hat{c}_3 \mid \mathbb{P})$ And $\{a(\hat{c}_1, \hat{c}_2, \hat{c}_3)/b(\hat{c}_1, \hat{c}_2, \hat{c}_3)\}ln(\hat{c}_1, \hat{c}_2, \hat{c}_3 \mid \mathbb{P})$

In the parametrization $\Theta_1 = lnc_1, \Theta_2 = lnc_2, \Theta_3 = lnc_3$ and $\mathbb{P}$, we have the maximum likelihood estimators $\hat{\Theta}_1 = -2.86295, \hat{\Theta}_2 = -1.33067, \hat{\Theta}_3 = -2.81008$ and $\hat{\mathbb{P}} = 2.03$.

The observed information matrix in the parametrization $\underset{\sim}{\Theta} = (\Theta_1, \Theta_2, \Theta_3, \mathbb{P})$ is given by

$$I_0 = \begin{pmatrix} 17.1022 & -1.9107 & 10.0560 & 15.0762 \\ & 33.1328 & 6.2768 & 22.4913 \\ & & 10.9034 & 16.1764 \\ symmetric & & & 50.9934 \end{pmatrix}$$

The inverse of the observed information matrix I_0 in the parametrization $\underset{\sim}{\Theta} = (\Theta_1, \Theta_2, \Theta_3, \mathbb{P})$ is given by

$$I_0^{-1} = \begin{pmatrix} 0.19204 & 0.06469 & -0.16584 & -0.032270 \\ & 0.06511 & -0.04942 & -0.03216 \\ & & 0.31743 & -0.02987 \\ symmetric & & & 0.05294 \end{pmatrix}$$

The approximate variances for the maximum likelihood estimators $\hat{\Theta}_1, \hat{\Theta}_2, \hat{\Theta}_3$ and $\hat{\mathbb{P}}$ are given by $\widehat{var}\left(\hat{\Theta}_1\right) \cong 0.19204, \widehat{var}\left(\hat{\Theta}_2\right) \cong 0.06511, \widehat{var}\left(\hat{\Theta}_3\right) \cong 0.31743$ and $\widehat{var}(\mathbb{P}) \cong 0.05294$.

Considering the asymptotical normality of the maximum likelihood estimators $\hat{\Theta}_1, \hat{\Theta}_2, \hat{\Theta}_3$ and $\hat{\mathbb{P}}$, we obtaim 95% approximate confidence intervals for $\Theta_1, \Theta_2, \Theta_3$, and $\mathbb{P}$ given by (-3.72187;-2.00403), (-1.83080;-0.83054),(-3.91436;-1.70580) and (1.57903;2.48097), respectively. From these intervals, we get approximate 95% confidence intervals for c_1, c_2 and c_3 given by (0.02419;0.13479), (0.16028;0.43581), and (0.01995;0.18163), respectively. We observe different confidence intervals for c_1, c_2 and c_3 considering the two parametrizations $\underset{\sim}{\psi} = (c_1, c_2, c_3, \mathbb{P})$ and $\underset{\sim}{\Theta} = (\Theta_1, \Theta_2, \Theta_3, \mathbb{P})$.

In fact, the third derivatives of the logarithm of the likelihood function $ln(c_1, c_2, c_3, \mathbb{P})$ (see appendix), locally in the maximum likelihood estimator $\hat{\underset{\sim}{\Psi}} = \left(\hat{\Theta}_1, \hat{\Theta}_2, \hat{\Theta}_3, \hat{\mathbb{P}}\right)$ are given by $\partial^3 ln\left(\hat{\underset{\sim}{\Psi}}\right)/\partial c_1^3 = 140206.00$,

$$\partial^3 ln\left(\hat{\underset{\sim}{\Psi}}\right)/\partial c_1^2 \partial c_2 = -1091.21, \partial^3 ln\left(\hat{\underset{\sim}{\Psi}}\right)/\partial c_1^2 \partial c_3 = 46230.30\ ,$$

$$\partial^3 ln\left(\hat{\underset{\sim}{\Psi}}\right)/\partial c_1^2 \partial \mathbb{P} = 0,\ \partial^3 ln\left(\hat{\underset{\sim}{\Psi}}\right)/\partial c_2^3 = 3335.36,$$

$$\partial^3 ln\left(\hat{\underset{\sim}{\Psi}}\right)/\partial c_2^2 \partial c_1 = -1091.21, \partial^3 ln\left(\hat{\underset{\sim}{\Psi}}\right)/\partial c_2^2 \partial c_3 = 2146.42,$$

$$\partial^3 ln\left(\hat{\underset{\sim}{\Psi}}\right)/\partial c_2^2 \partial \mathbb{P} = 0, \partial^3 ln\left(\hat{\underset{\sim}{\Psi}}\right)/\partial c_3^3 = 46757.10$$

$$\partial^3 ln\left(\hat{\underset{\sim}{\Psi}}\right)/\partial c_3^2 \partial c_1 = 46230.30, \partial^3 ln\left(\hat{\underset{\sim}{\Psi}}\right)/\partial c_3^2 \partial \mathbb{P} = 0,$$

$$\partial^3 ln\left(\hat{\underset{\sim}{\Psi}}\right)/\partial c_3^2 \partial c_2 = 2146.42, \partial^3 ln\left(\hat{\underset{\sim}{\Psi}}\right)/\partial \mathbb{P}^3 = -50.44,$$

$$\partial^3 ln\left(\hat{\underset{\sim}{\Psi}}\right)/\partial \mathbb{P}^2 \partial c_1 = -249.39, \partial^3 ln\left(\hat{\underset{\sim}{\Psi}}\right)/\partial \mathbb{P}^2 \partial c_2 = -81.40$$

$\partial^3 ln\left(\hat{\underset{\sim}{\Psi}}\right)/\partial \mathbb{P}^2 \partial c_3 = -253.13, \partial^3 ln\left(\hat{\underset{\sim}{\Psi}}\right)/\partial c_1 \partial c_2 \partial c_3 = 1619.64,$

$\partial^3 ln\left(\hat{\underset{\sim}{\Psi}}\right)/\partial c_1 \partial c_2 \partial \mathbb{P} = 0, \partial^3 ln\left(\hat{\underset{\sim}{\Psi}}\right)/\partial c_2 \partial c_3 \partial \mathbb{P} = 0$

and $\partial^3 ln\left(\hat{\underset{\sim}{\Psi}}\right)/\partial c_1 \partial c_3 \partial \mathbb{P} = 0.$

We observe large values of the third derivatives of $l(c_1, c_2, c_3, \mathbb{P})$ locally in $\hat{c}_1, \hat{c}_2, \hat{c}_3$ and $\hat{\mathbb{P}}$, specially for the derivatives related to the parameters c_1, c_2 and c_3, indicating that the normality of the likelihood function for c_1, c_2, c_3 and $\mathbb{P}$ is not good considering the data of table 1.

In the parametrization $\Theta_1 = lnc_1, \Theta_2 = lnc_2, \Theta_3 = lnc_3$ and $\mathbb{P}$, the third derivatives of the logarithm of the likelihood function $ln(\Theta_1, \Theta_2, \Theta_3, \mathbb{P})$ (see appendix), locally in the maximum likelihood estimator $\hat{\underset{\sim}{\Theta}} = \left(\hat{\Theta}_1, \hat{\Theta}_2, \hat{\Theta}_2, \hat{\mathbb{P}}\right)$ are given by $\partial^3 ln\left(\hat{\underset{\sim}{\Theta}}\right)/\partial\Theta_1^3 = -25.23,$

$\partial^3 ln\left(\hat{\underset{\sim}{\Theta}}\right)/\partial\Theta_1^2 \partial\Theta_2 = 0.97, \partial^3 ln\left(\hat{\underset{\sim}{\Theta}}\right)/\partial\Theta_1^2 \partial\Theta_3 = -0.98,$

$\partial^3 ln\left(\hat{\underset{\sim}{\Theta}}\right)/\partial\Theta_1^2 \partial\mathbb{P} = -15.07, \partial^3 \mathrm{ln}\left(\hat{\underset{\sim}{\Theta}}\right)/\partial\Theta_2^3 = -37.80,$

$\partial^3 ln\left(\hat{\underset{\sim}{\Theta}}\right)/\partial\Theta_2^2 \partial\Theta_1 = -2.44, \partial^3 ln\left(\hat{\underset{\sim}{\Theta}}\right)/\partial\Theta_2^2 \partial\Theta_3 = +2.75,$

$\partial^3 ln\left(\hat{\underset{\sim}{\Theta}}\right)/\partial\Theta_2^2 \partial\mathbb{P} = -22.49, \partial^3 \mathrm{ln}\left(\hat{\underset{\sim}{\Theta}}\right)/\partial\Theta_3^3 = -22.33,$

$\partial^3 ln\left(\hat{\underset{\sim}{\Theta}}\right)/\partial\Theta_3^2 \partial\Theta_1 = -0.49, \partial^3 ln\left(\hat{\underset{\sim}{\Theta}}\right)/\partial\Theta_3^2 \partial\Theta_2 = -2.25,$

$\partial^3 ln\left(\hat{\underset{\sim}{\Theta}}\right)/\partial\Theta_3^2 \partial\mathbb{P} = -16.18, \partial^3 \mathrm{ln}\left(\hat{\underset{\sim}{\Theta}}\right)/\partial\Theta_1 \partial\Theta_2 \partial\Theta_3 = 1.47,$

$\partial^3 ln\left(\hat{\underset{\sim}{\Theta}}\right)/\partial\Theta_1 \partial\Theta_2 \partial\mathbb{P} = 0, \partial^3 \mathrm{ln}\left(\hat{\underset{\sim}{\Theta}}\right)/\partial\Theta_1 \partial\Theta_3 \partial\mathbb{P} = 0,$

$\partial^3 ln\left(\hat{\underset{\sim}{\Theta}}\right)/\partial\Theta_2 \partial\Theta_3 \partial\mathbb{P} = 0, \partial^3 \mathrm{ln}\left(\hat{\underset{\sim}{\Theta}}\right)/\partial\mathbb{P}^3 = -50.44,$

$\partial^3 ln\left(\hat{\underset{\sim}{\Theta}}\right)/\partial\mathbb{P}^2 \partial\Theta_1 = -14.24, \partial^3 \mathrm{ln}\left(\hat{\underset{\sim}{\Theta}}\right)/\partial\mathbb{P}^2 \partial\Theta_2 = -21.51,$

and $\partial^3 ln\left(\hat{\underset{\sim}{\Theta}}\right)/\partial\mathbb{P}^2 \partial\Theta_3 = -15.24$. In this parametrization, we observe small values for the third derivatives of the logarithm of the likelihood function for $\underset{\sim}{\Theta} = (\Theta_1, \Theta_2, \Theta_3, \mathbb{P})$ locally in $\hat{\underset{\sim}{\Theta}}$, indicating better normality of the likelihood function (see Anscombe, 1964).

7 Overall Conclusions

The results of this paper could be of great practical interest, since the choice of good parametrization is essential to get good inferences for the parameters of the BVED considering accelerated life tests. Considering a data set of bivariate observations introduced in the example of section 5, we observed better normality of the likelihood function in the parametrization $\underset{\sim}{\theta} = (\Theta_1, \Theta_2, \Theta_3, \mathbb{P})$, where $\Theta_i = lnc_i, i = 1,2,3$. However, we should check, for each special application of the BVED with the power rule model (3), the normality of the likelihood function. This parametrization also could be useful for a Bayesian Analysis of the proposed accelerated life model, since we usually use some numerical or approximation method (for example, the Laplace's method for approximation of integrals) to get posterior densities of interest dependent on a parametrization that depends upon both the data and the choice of prior to obtain accurate results (see for example, Achcar and Smith, 1990).

Appendix

1 Third Derivatives Of $l\,(c_1, c_2, c_3, \mathbb{P})$

In the parametrization $\underset{\sim}{\psi} = (c_1, c_2, c_3, \mathbb{P})$, the third derivatives of the logarithm of the likelihood function (8) are given by,

$$\frac{\partial^3 ln}{\partial c_1^3} = \frac{2r}{c_1^3} + \frac{2(n-r)}{(c_1+c_3)^3} + \frac{2n}{(c_1+c_2+c_3)^3} - \frac{2n}{(c_1+c_2)^3}$$

$$\frac{\partial^3 ln}{\partial c_1^2 \partial c_2} = \frac{2n}{(c_1+c_2c_3)^3} - \frac{2n}{(c_1+c_2)^3}$$

$$\frac{\partial^3 ln}{\partial c_1^2 \partial c_3} = \frac{2(n-r)}{(c_1+c_3)^3} + \frac{2n}{(c_1+c_2+c_3)^3}$$

$$\frac{\partial^3 ln}{\partial c_1^2 \partial \mathbb{P}} = 0$$

$$\frac{\partial^3 ln}{\partial c_2^3} = \frac{2r}{(c_2+c_3)^3} + \frac{2(n-r)}{c_2^3} + \frac{2n}{(c_1+c_2+c_3)^3} - \frac{2n}{(c_1+c_2)^3}$$

$$\frac{\partial^3 ln}{\partial c_2^2 \partial c_1} = \frac{2n}{(c_1+c_2+c_3)^3} - \frac{2n}{(c_1+c_2)^3}$$

$$\frac{\partial^3 ln}{\partial c_2^2 \partial c_3} = \frac{2r}{(c_2+c_3)^3} + \frac{2n}{(c_1+c_2+c_3)^3}$$

$$\frac{\partial^3 ln}{\partial c_2^2 \partial \mathbb{P}} = 0, \frac{\partial^3 ln}{\partial c_3^3} = \frac{2r}{(c_2+c_3)^3} + \frac{2(n-r)}{(c_1+c_3)^3} + \frac{2n}{(c_1+c_2+c_3)^3}$$

$$\frac{\partial^3 ln}{\partial c_3^2 \partial c_1} = \frac{2(n-r)}{(c_1+c_3)^3} + \frac{2n}{(c_1+c_2+c_3)^3}$$

$$\frac{\partial^3 ln}{\partial c_3^2 \partial c_2} = \frac{2r}{(c_2+c_3)^3} + \frac{2n}{(c_1+c_2+c_3)^3}$$

$$\frac{\partial^3 ln}{\partial c_3^2 \partial \mathbb{P}} = 0$$

$$\frac{\partial^3 ln}{\partial \mathbb{P}^3} = -c_1 \sum_{j=1}^{J} n_j \overline{X}_j V_j^{\mathbb{P}} (lnV_j)^3 - $$
$$- \quad c_2 \sum_{j=1}^{J} n_j \overline{Y}_j V_j^{\mathbb{P}} (lnV_j)^3 - c_3 \sum_{j=1}^{J} R_j V_j^{\mathbb{P}} (lnV_j)^3$$

$$\frac{\partial^3 ln}{\partial \mathbb{P}^2 \partial c_1} = -\sum_{j=1}^{J} n_j \overline{X}_j V_j^{\mathbb{P}} (lnV_j)^2$$

$$\frac{\partial^3 ln}{\partial \mathbb{P}^2 \partial c_2} = -\sum_{j=1}^{J} n_j \overline{Y}_j V_j^{\mathbb{P}} (lnV_j)^2$$

$$\frac{\partial^3 ln}{\partial \mathbb{P}^2 \partial c_3} = -\sum_{j=1}^{J} R_j V_j^{\mathbb{P}} (lnV_j)^2$$

$$\frac{\partial^3 ln}{\partial c_1 \partial c_2 \partial c_3} = \frac{2n}{(c_1+c_2+c_3)^3}$$

$$\frac{\partial^3 ln}{\partial c_1 \partial c_2 \partial \mathbb{P}} = 0 \; ; \; \frac{\partial^3 ln}{\partial c_2 \partial c_3 \partial \mathbb{P}} = 0$$

$$\frac{\partial^3 ln}{\partial c_1 \partial c_3 \partial \mathbb{P}} = 0$$

2 Third Derivatives Of $l\,(\Theta_1, \Theta_2, \Theta_3, \mathbb{P})$

In the parametrization $\underset{\sim}{\Theta} = (\Theta_1, \Theta_2, \Theta_3, \mathbb{P})$, where $\Theta_i = lnc_i, i = 1,2,3$, the third derivatives of the logarithm of the likelihood function (12) are given by

$$\frac{\partial^3 ln}{\partial \Theta_1^3} = \frac{\partial^2 ln}{\partial \Theta_1^2} + 2e^{2\Theta_1} \left\{ \frac{ne^{\Theta_2}}{\left(e^{\Theta_1}+e^{\Theta_2}\right)^3} - \frac{(n-r)e^{\Theta_3}}{\left(e^{\Theta_1}+e^{\Theta_3}\right)^3} - \right.$$
$$\left. - \quad \frac{n\left(e^{\Theta_2}+e^{\Theta_3}\right)}{\left(e^{\Theta_1}+e^{\Theta_2}+e^{\Theta_3}\right)^3} \right\}$$

$$\frac{\partial^3 ln}{\partial \Theta_1^2 \partial \Theta_2} = e^{\Theta_1+\Theta_2} \left\{ -\frac{n}{\left(e^{\Theta_1}+e^{\Theta_2}\right)^2} + \frac{2ne^{\Theta_2}}{\left(e^{\Theta_1}+e^{\Theta_2}\right)^3} + \right.$$
$$\left. + \quad \frac{n}{\left(e^{\Theta_1}+e^{\Theta_2}+e^{\Theta_3}\right)^2} - \frac{2n\left(e^{\Theta_2}+e^{\Theta_3}\right)}{\left(e^{\Theta_1}+e^{\Theta_2}+e^{\Theta_3}\right)^3} \right\}$$

$$\frac{\partial^3 ln}{\partial\Theta_1^2\partial\Theta_3} = e^{\Theta_1+\Theta_3}\left\{\frac{(n-r)}{\left(e^{\Theta_1}+e^{\Theta_3}\right)^2} - \frac{2(n-r)e^{\Theta_3}}{\left(e^{\Theta_1}+e^{\Theta_3}\right)^3} + \right.$$

$$\left. + \frac{n}{\left(e^{\Theta_1}+e^{\Theta_2}+e^{\Theta_3}\right)^2} - \frac{2n\left(e^{\Theta_2}+e^{\Theta_3}\right)}{\left(e^{\Theta_1}+e^{\Theta_2}+e^{\Theta_3}\right)^3}\right\}$$

$$\frac{\partial^3 ln}{\partial\Theta_1^2\partial\mathbf{P}} = -e^{\Theta_1}\sum_{j=1}^{J} n_j\overline{X}_j V_j^{\mathbf{P}}\,(lnV_j)$$

$$\frac{\partial^3 ln}{\partial\Theta_2^3} = \frac{\partial^2 ln}{\partial\Theta_2^2} + 2e^{2\Theta_2}\left\{\frac{ne^{\Theta_1}}{\left(e^{\Theta_1}+e^{\Theta_2}\right)^3} - \frac{re^{\Theta_3}}{\left(e^{\Theta_2}+e^{\Theta_3}\right)^3} - \right.$$

$$\left. - \frac{n\left(e^{\Theta_1}+e^{\Theta_3}\right)}{\left(e^{\Theta_1}+e^{\Theta_2}+e^{\Theta_3}\right)^3}\right\}$$

$$\frac{\partial^3 ln}{\partial\Theta_2^2\partial\Theta_1} = e^{\Theta_2+\Theta_1}\left\{-\frac{n}{\left(e^{\Theta_1}+e^{\Theta_2}\right)^2} + \frac{2ne^{\Theta_1}}{\left(e^{\Theta_1}+e^{\Theta_2}\right)^3} + \right.$$

$$\left. + \frac{n}{\left(e^{\Theta_1}+e^{\Theta_2}+e^{\Theta_3}\right)^2} - \frac{2n\left(e^{\Theta_1}+e^{\Theta_3}\right)}{\left(e^{\Theta_1}+e^{\Theta_2}+e^{\Theta_3}\right)^3}\right\}$$

$$\frac{\partial^3 ln}{\partial\Theta_2^2\partial\Theta_3} = e^{\Theta_2+\Theta_3}\left\{\frac{r}{\left(e^{\Theta_2}+e^{\Theta_3}\right)^2} - \frac{2re^{\Theta_3}}{\left(e^{\Theta_2}+e^{\Theta_3}\right)^3} + \right.$$

$$\left. + \frac{n}{\left(e^{\Theta_1}+e^{\Theta_2}+e^{\Theta_3}\right)^2} - \frac{2n\left(e^{\Theta_1}+e^{\Theta_3}\right)}{\left(e^{\Theta_1}+e^{\Theta_2}+e^{\Theta_3}\right)^3}\right\}$$

$$\frac{\partial^3 ln}{\partial\Theta_2^2\partial\mathbf{P}} = -e^{\Theta_2}\sum_{j=1}^{J} n_j\overline{Y}_j V_j^{\mathbf{P}}\,(lnV_j)$$

$$\frac{\partial^3 ln}{\partial\Theta_3^3} = \frac{\partial^2 ln}{\partial\Theta_3^2} + 2e^{2\Theta_3}\left\{-\frac{re^{\Theta_2}}{\left(e^{tetad}+e^{\Theta_3}\right)^3} - \right.$$

$$\left. - \frac{(n-r)e^{\Theta_1}}{\left(e^{\Theta_1}+e^{\Theta_3}\right)^3} - \frac{n\left(e^{\Theta_1}+e^{\Theta_2}\right)}{\left(e^{\Theta_1}+e^{\Theta_2}+e^{\Theta_3}\right)^3}\right\}$$

$$\frac{\partial^3 ln}{\partial\Theta_3^2\partial\Theta_1} = e^{\Theta_3+\Theta_1}\left\{\frac{(n-r)}{\left(e^{\Theta_1}+e^{\Theta_3}\right)^2} - \frac{2(n-r)e^{\Theta_1}}{\left(e^{\Theta_1}+e^{\Theta_3}\right)^3} + \right.$$

$$\left. + \frac{n}{\left(e^{\Theta_1}+e^{\Theta_2}+e^{\Theta_3}\right)^2} - \frac{2n\left(c^{\Theta_1}+c^{\Theta_2}\right)}{\left(e^{\Theta_1}+e^{\Theta_2}+e^{\Theta_3}\right)^3}\right\}$$

$$\frac{\partial^3 ln}{\partial\Theta_3^2\partial\Theta_2} = e^{\Theta_3+\Theta_2}\left\{\frac{r}{\left(e^{\Theta_2}+e^{\Theta_3}\right)^2} - \frac{2e^{\Theta_2}r}{\left(e^{\Theta_2}+e^{\Theta_3}\right)^3} + \right.$$

$$\left. + \frac{n}{\left(e^{\Theta_1}+e^{\Theta_2}+e^{\Theta_3}\right)^2} - \frac{2n\left(e^{\Theta_1}+e^{\Theta_2}\right)}{\left(e^{\Theta_1}+e^{\Theta_2}+e^{\Theta_3}\right)^3}\right\}$$

$$\frac{\partial^3 ln}{\partial\Theta_3^2\partial\mathbb{P}} = -e^{\Theta_3}\sum_{j=1}^{J} R_j V_j^{\mathbb{P}} lnV_j$$

$$\frac{\partial^3 ln}{\partial\Theta_1\partial\Theta_2\partial\Theta_3} = \frac{2ne^{\Theta_1+\Theta_2+\Theta_3}}{\left(e^{\Theta_1}+e^{\Theta_2}+e^{\Theta_3}\right)^3}$$

$$\frac{\partial^3 ln}{\partial\Theta_1\partial\Theta_2\partial\mathbb{P}} = 0 \; ; \; \frac{\partial^3 ln}{\partial\Theta_1\partial\Theta_3\partial\mathbb{P}} = 0 \; ; \; \frac{\partial^3 ln}{\partial\Theta_2\partial\Theta_3\partial\mathbb{P}} = 0;$$

$$\frac{\partial^3 ln}{\partial\mathbb{P}^3} = -e^{\Theta_1}\sum_{j=1}^{J} n_j\overline{X}_j V_j^{\mathbb{P}}\left(lnV_j\right)^3 -$$

$$- e^{\Theta_2}\sum_{j=1}^{J} n_j\overline{Y}_j V_j^{\mathbb{P}}\left(lnV_j\right)^3 - e^{\Theta_3}\sum_{j=1}^{J} R_j V_j^{\mathbb{P}}\left(lnV_j\right)^3$$

$$\frac{\partial^3 ln}{\partial\mathbb{P}^2\partial\Theta_1} = -e^{\Theta_1}\sum_j^J n_j\overline{X}_j V_j^{\mathbb{P}}\left(lnV_j\right)^2$$

$$\frac{\partial^3 ln}{\partial\mathbb{P}^2\partial\Theta_2} = -e^{\Theta_2}\sum_j^J n_j\overline{Y}_j V_j^{\mathbb{P}}\left(lnV_j\right)^2$$

$$\frac{\partial^3 ln}{\partial\mathbb{P}^2\partial\Theta_3} = -e^{\Theta_3}\sum_j^J R_j V_j^{\mathbb{P}}\left(lnV_j\right)^2$$

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Nº 105/92 - ACHCAR, J.A.; ESPINOSA, M.M. - Use of Bayesian in accelerated life tests considering a log-linear model for the Birnbaum-Saunders distribution

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