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CARLOS BIASI; WALTER MOTTA; OSAMU SAEKI

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A note on separation properties of codimension-1 immersions with normal crossings

Carlos Biasi
Departamento de Matemática, ICMSC-USP, Caixa Postal 668, 13560 São Carlos, SP, Brasil

Walter Motta*
Departamento de Matemática, CCET-UFMS, Caixa Postal 649, 79069 Campo Grande, MS, Brasil

Osamu Saeki
Department of Mathematics, Yamagata University, Yamagata 990, Japan

Abstract

Let $f : M^{n-1} \rightarrow N^n$ be an immersion with normal crossings between closed connected manifolds. The article is concerned with the problem of separation of N by $f(M)$. The main result of this paper is a converse of the Jordan-Brouwer theorem, under the hypothesis that M is oriented and $H_1(N, Z_2) = 0$. More precisely, with the above hypothesis, f is an embedding if and only if $N - f(M)$ has two connected components.

Keywords: self-intersection set, characterization of embeddings, immersion with normal crossings, separation of connected components.

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1 Introduction

In this paper our aim is to answer the following questions:

Problem 1.1 Let M^{n-1} and N^n be closed connected manifolds of dimensions $n-1$ and n respectively, and let $f : M^{n-1} \to N^n$ be an immersion. Under what conditions is it possible to assure that $f(M^{n-1})$ separates N^n ?

Problem 1.2 Let M^{n-1} be a closed connected $(n-1)$-manifold and let $f : M^{n-1} \to S^n$ be an embedding (M^{n-1} is clearly orientable). Then, by the well known Jordan-Brouwer separation theorem, $f(M^{n-1})$ separates S^n into exactly two connected components. Under what conditions is it possible to state a converse of this result?

In [4], Feighn has considered Problem 1.1, obtaining a positive answer for separation when f is a C^2-immersion and $H_1(N^n; \mathbb{Z}_2) = 0$. In [2], Ballesteros has worked with 1-generic and quasi-regular f, assuming also that $H_1(N^n; \mathbb{Z}_2) = 0$. Under these conditions, $f(M^{n-1})$ also separates N^n ([2, Theorem 3.8]). We observe that immersions with normal crossings are a particular case of 1-generic and quasi-regular maps. Under the assumption that $f : X \to Y$ is proper closed continuous map, with X is connected n-manifold, Y is connected $(n+1)$-manifold with $H_1(Y, Z_2) = 0$, and assuming that the set of self-intersection of f is not dense, Ballesteros-Fuster in [1] prove that $Y - f(X)$ is disconnected. In [3], Biasi and Fuster have considered Problem 1.2, working with an immersion with normal crossings $f : M^{n-1} \to N^n$ between closed connected manifolds such that $H_1(M^{n-1}; \mathbb{Z}_2) = 0$ for $n > 2$ and $H_1(N^n; \mathbb{Z}_2) = 0$. Under these assumptions, they have proved that f is an embedding if and only if $f(M^{n-1})$ separates N^n into exactly two connected components.

Our results of this paper are the following theorems which answer the above problems when f is an immersion with normal crossings.

Theorem 1.3 *Let $f : M^{n-1} \to N^n$ be an immersion with normal crossings between closed connected manifolds.*

(1) If $f_*([M^{n-1}]) = 0$ in $H_{n-1}(N^n; \mathbb{Z}_2)$, then $f(M^{n-1})$ separates N^n, where $[M^{n-1}] \in H_{n-1}(M^{n-1}; \mathbb{Z}_2)$ denotes the fundamental class of M^{n-1}.

(2) *If* $\dim\ker((f \mid A)_* : H_{n-2}(A;\mathbb{Z}_2) \to H_{n-2}(B;\mathbb{Z}_2)) \geq \beta_1(M^{n-1}) + \beta_1(N^n)$, *then* $f(M^{n-1})$ *separates* N^n, *where* $A(\subset M^{n-1})$ *is the self-intersection set of* $f, B = f(A)$, *and* $\beta_i(X) = \dim H_i(X;\mathbb{Z}_2)$ *for a topological space* X.

(3) *Assume that* $A \neq \emptyset, \beta_1(N^n) \leq 1$, *that* $f_* : H_1(M^{n-1};\mathbb{Z}_2) \to H_1(N^n;\mathbb{Z}_2)$ *vanishes and that the normal bundle* $\nu(f)$ *of the immersion* f *is trivial. Then* $f(M^{n-1})$ *separates* N^n.

Theorem 1.4 *Let* $f : M^{n-1} \to N^n$ *be an immersion with normal crossings between closed connected manifolds such that its self-intersection set is nonempty.*

(1) *If* M^{n-1} *is orientable and* $H_1(N^n;\mathbb{Z}_2) = 0$, *then* $\beta_0(N^n - f(M^{n-1})) \geq 3$.

(2) *If* $i_* : H_{n-1}(f(M^{n-1});\mathbb{Z}_2) \to H_{n-1}(N^n;\mathbb{Z}_2)$ *vanishes and if the normal bundle* $\nu(f)$ *of the immersion* f *is trivial, then* $\beta_0(N^n - f(M^{n-1})) \geq 3$, *where* $i : f(M^{n-1}) \to N^n$ *is the inclusion map.*

2 Basic tools

In all that follows, M^{n-1} and N^n are closed connected manifolds of dimensions $n-1$ and n respectively, $f : M \to N$ is an immersion with normal crossings and we will be working with homology with coefficients in $\mathbb{Z}_2$.

Lemma 2.1 *The normal bundle* $\nu(f)$ *of the immersion* f *is trivial if* M *is orientable and* $H_1(N) = 0$.

Proof. First note that the first Stiefel-Whitney class $w_1(TN)$ of TN vanishes since $H^1(N) = 0$ by our hypothesis. Since, by definition, $f^*(TN) \cong TM \oplus \nu(f)$, it follws that $0 = f^*(w_1(TN)) = w_1(TM) + w_1(\nu(f))$. Since M is orientable, $w_1(TM) = 0$. Thus we have $w_1(\nu(f)) = 0$, which implies that the line bundle $\nu(f)$ is trivial. □

Now, let $A(\subset M)$ be the self-intersection set of f (that is, the set of the multiple points of f), which is stratified by the submanifolds $S_m : A = \bigcup_{m=2}^n S_m$ (codim$S_m = m-1$), where S_m is the set of the points of M whose f-images have multiplicity m. We also use the notation $A_m = \bigcup_{k\geq m} S_k$. Let us denote $f(A) = B$, the set of the multiple values of f, which is also stratified by $f(S_m)$: $B = \bigcup_{m=2}^n f(S_m)$ (codim$f(S_m) = m-2$). Let $[A] \in H_{n-2}(A)$ be the fundamental class modulo 2 carried by the 2-fold multiple point locus $A_2 = A$. Note that its existence is assured by [5] and that $[A]$ does not vanish in $H_{n-2}(A)$.

Lemma 2.2 *We have* $(f \mid A)_*([A]) = 0$, *where* $(f \mid A)_* : H_{n-2}(A) \to H_{n-2}(B)$.

Proof. This follows from the fact that $f \mid S_2 : S_2 \to f(S_2)$ is a 2-fold covering (see [5]). □

Lemma 2.3 *Assume that* M *is orientable and* $H_1(N) = 0$. *Then* $j_*([A]) = 0$, *where* $j : A \to M$ *is the inclusion map.*

Proof. By the Poincaré duality, we have only to show that

$$(j_*([A])) \cdot \gamma = 0 \quad \text{for} \quad {}^\forall \gamma \in H_1(M),$$

where $(j_*([A])) \cdot \gamma$ denotes the intersection number modulo 2 of $j_*([A])$ and γ in M. Let c be a smooth closed curve in M representing γ. We may assume that c does not intersect A_3 and that c intersects S_2 transversely. Since $\nu(f)$ is trivial by Lemma 2.1, we have an immersion $F : M \times \mathbb{R} \to N$ such that $F|_{M \times \{0\}} = f$. Set $\tilde{c} = F(c \times \{\varepsilon\})$, where $\varepsilon > 0$ is sufficiently small. Then it is easy to see that the intersection number of A_2 and c in M is equal to that of $f(M)$ and $\tilde{c}$ in N. Hence we have, $(j_*([A])) \cdot \gamma = [f(M)] \cdot [\tilde{c}]$. Since $H_1(N) = 0$, it follows that $[\tilde{c}] = 0$ and $(j_*([A])) \cdot \gamma = 0$. This completes the proof. □

Examining the above proof, we see easily that the following more general result holds.

Lemma 2.4 *Assume that the normal bundle* $\nu(f)$ *of the immersion* f *is trivial and that* $f_*([M]) = 0$ *in* $H_{n-1}(N)$ *or* $f_* : H_1(M) \to H_1(N)$ *vanishes. Then* $j_*([A]) = 0$ *in* $H_1(M)$.

3 Proofs of the theorems

Proof of Theorem 1.3 *(1).* Let us assume that $N - f(M)$ is connected and let $x \in f(M)$ be a point in $f(M - A)$. Note that such a point exists since f is an immersion with normal crossings. Let $\varphi : [-1, 1] \to N$ be an embedding transverse to f such that $\varphi([-1,1]) \cap f(M) = \{x\}$. Since $N - f(M)$ is connected, we have an embedding $\tilde{\varphi} : S^1 \to N$ which extends φ such that $\tilde{\varphi}(S^1) \cap f(M) = \{x\}$. Since $x \in f(M - A)$, it is easy to see that the intersection number modulo two of $f(M)$ and $\tilde{\varphi}(S^1)$ is not zero, which implies that $(f_*([M])) \cdot [\tilde{\varphi}(S^1)] \neq 0$. Since $f_*([M]) = 0$ by our hypothesis, this is a contradiction. □

Before we proceed to the proofs of the other theorems, we recall some of the results from [3].

Lemma 3.1

(1) $\beta_0(N - f(M)) = 1 + \dim\ker(i_* : H_{n-1}(f(M)) \to H_{n-1}(N))$, *where* $i : f(M) \to N$ *is the inclusion map.*

(2) $\beta_{n-1}(f(M)) = 1 + \dim\ker(\alpha : H_{n-2}(A) \to H_{n-2}(B) \oplus H_{n-2}(M))$, *where* $\alpha = (f \mid A)_* \oplus j_*$.

We can prove Lemma 3.1 using the arguments of [3].

Proof of Theorem 1.3 *(2)*. By Lemma 3.1 (1), we have

$$\begin{aligned} \beta_0(N - f(M)) &\geq 1 + \beta_{n-1}(f(M)) - \beta_{n-1}(N) \\ &= 1 + \beta_{n-1}(f(M)) - \beta_1(N). \end{aligned}$$

On the other hand, by Lemma 3.1 (2), we have

$$\begin{aligned} \beta_{n-1}(f(M)) &\geq 1 + \dim\ker(f \mid A)_* - \beta_{n-2}(M) \\ &= 1 + \dim\ker(f \mid A)_* - \beta_1(M). \end{aligned}$$

Thus we have

$$\beta_0(N - f(M)) \geq 2 + \dim\ker(f \mid A)_* - \beta_1(M) - \beta_1(N) \geq 2 .$$

This completes the proof. □

Proof of Theorem 1.4 *(1)*. Since $H_{n-1}(N) \cong H_1(N) = 0$, $\beta_0(N - f(M)) = 1 + \beta_{n-1}(f(M))$ by Lemma 3.1 (1). On the other hand, by Lemmas 2.2 and 2.3, we see that $[A] \in H_{n-2}(A)$ is a non-zero element with $\alpha([A]) = 0$. Hence, by Lemma 3.1 (2), $\beta_{n-1}(f(M)) \geq 2$. Thus we have $\beta_0(N - f(M)) \geq 3$. □

Proof of Theorem 1.4 *(2)*. Since $i_* : H_{n-1}(f(M)) \to H_{n-1}(N)$ vanishes, $\beta_0(N - f(M)) = 1 + \beta_{n-1}(f(M))$ by Lemma 3.1 (1). On the other hand, by Lemmas 2.2 and 2.4, $[A] \in H_{n-2}(A)$ is a non-zero element with $\alpha([A]) = 0$. Note that $f_*([M]) = 0$ in $H_{n-1}(N)$ since $f_*([M]) = i_*([f(M)]) = 0$. Thus, by Lemma 3.1 (2), we have $\beta_{n-1}(f(M)) \geq 2$ and hence $\beta_0(N - f(M)) \geq 3$. □

Proof of Theorem 1.3 *(3)*. Since $\beta_{n-1}(N) = \beta_1(N) \leq 1$, we have $\beta_0(N - f(M)) \geq \beta_{n-1}(f(M))$ by Lemma 3.1 (1). Furthermore, by Lemmas 2.2, 2.4 and 3.1 (2), we see that $\beta_{n-1}(f(M)) \geq 2$, using an argument similar to the above. □

Remark 3.2

(1) Consider the immersion $f : S^1 \to T^2$ with normal crossings as in Figure 1. It is easily seen that its normal bundle $\nu(f)$ is trivial, $f_*([M]) = 0$ and that $\beta_0(T^2 - f(S^1)) = 2$. Hence, this example shows that the condition on i_* is essential in Theorem 1.4 (2).

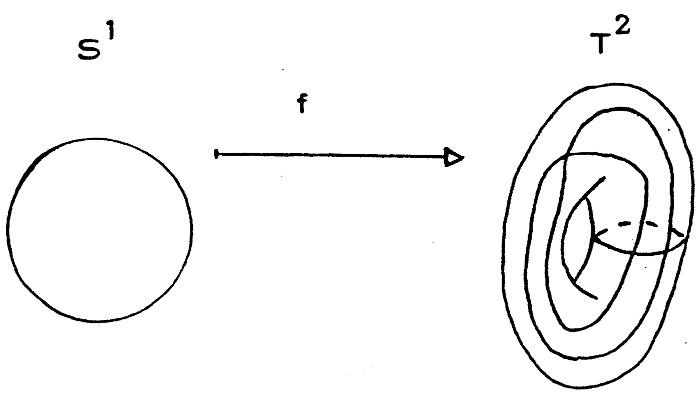

Figure 1

(2) One can also see that the triviality of $\nu(f)$ is an essential condition in Theorem 1.4 (2), by immersing the Klein bottle in $\mathbb{R}^3$ "standardly" with normal crossings.

4 Consequences

As an immediate corollary of Theorem 1.4 (1), we obtain the following, a converse of the Jordan-Brouwer Theorem, which answers Problem 1.2 in the introduction.

Corollary 4.1 *Let M be a closed orientable connected $(n-1)$-manifold and let $f : M \to S^n$ be an immersion with normal crossings. If $S^n - f(M)$ has exactly two connected components, then f is an embedding.*

More generally, we get the following, which characterizes embeddings among codimension-1 immersions with normal crossings.

Corollary 4.2 *Let M and N be closed connected manifolds of dimensions $n-1$ and n respectively. Suppose that M is orientable and that $H_1(N) = 0$. Then an immersion $f : M \to N$ with normal crossings is an embedding if and only if $f(M)$ separates N into exactly two connected components.*

Note that, when $H_1(M) = 0$, Corollary 4.2 has been obtained by Biasi and Fuster [3].

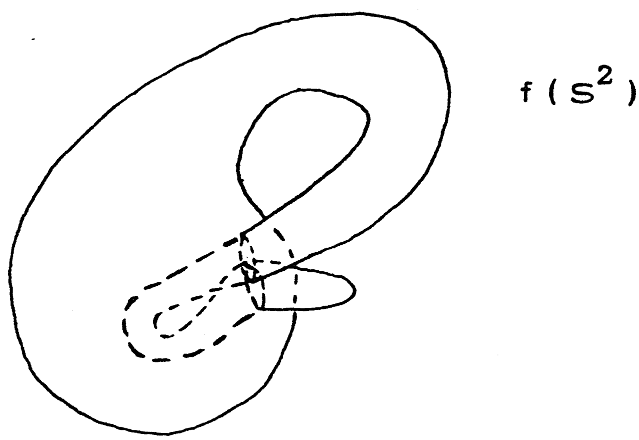

Figure 2

Remark 4.3 In Corollary 2 of [3], we do not need the assumption that the self-intersection set of f be reduced to the set of double points S_2. (We can

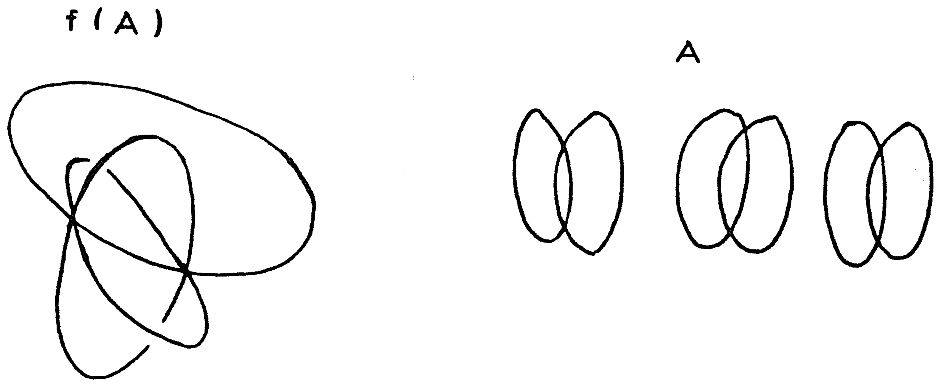

Figure 3

also replace the hypothesis that $H_1(M) \cong H_1(N) = 0$ by the nullity of $i_* : H_{n-1}(f(M)) \to H_{n-1}(N)$ and $j_* : H_{n-2}(A) \to H_{n-2}(M)$.)

When $A = S_2$, that is when the self-intersection set of f is reduced to the set of double points, we see easily that $\dim \ker(f \mid A)_* = \beta_{n-2}(B) = \beta_0(B)$. Hence, in Theorem 1.4, if $A = S_2$ and $H_1(M) = 0$, then $\beta_0(N - f(M)) = 2 + \beta_0(B)$. Note that if $A \neq S_2$, then A is not a submanifold of M and that we can have situations where $\dim \ker(f \mid A)_* \neq \beta_{n-2}(B)$. For example consider the immersion $f : S^2 \to S^3 = \mathbb{R}^3 \cap \{\infty\}$ with normal crossings as in Figure 2. Then the self-intersection set A and its f-image $f(A) = B$ is as in Figure 3. Thus, $\dim \ker(f \mid A)_* = 4$ and $\beta_{n-2}(B) = 5$. Note also that $\beta_0(S^3 - f(S^2)) = 2 + \dim \ker(f \mid A)_* = 6$.

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