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The polyhedron of equisingularity of germs of hypersurfaces

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The Polyhedron of equisingularity of germs of hypersurfaces.

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Introduction.

Let $G : k^n \times k, 0 \to k, 0$ be a deformation of a germ $g : k^n, 0 \to k, 0$, which has an (algebraically) isolated singularity at 0. ($k = \mathbb{R}$ or $\mathbb{C}$, and the germs are correspondingly smooth, analytic or holomorphic).

We consider the problem of determining when such a deformation is topologically trivial, and, correspondingly, when the pair $\{G^{-1}(0), 0 \times k\}$ is Whitney equisingular. We show that the approach based on the Newton filtration by Damon and Gaffney [DG], the method of the gradient polygon of E. Yoshinaga [Y], and the algebraic approach of integral closure of ideals by Teissier [T1,T2] lead to the same convex subset $C(\overline{Jg})$ of the Newton polyhedron of the germ g.

Let $\mathcal{E}(g)$ be the polyhedron of equisingularity of the germ g, i. e, the convex hull of the set $\bigcup\{k + \mathbb{R}^n_+ \mid$ the pair $\{G^{-1}(0), o \times k\}$ is Whitney equisingular, where $G(x,t) = g(x) + tx^k, x^k = x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n}\}$.

We apply Teissier's results on equisingularity to show that $C(\overline{Jg})$ is contained in $\mathcal{E}(g)$.

For a special class of holomorphic curves in $\mathbb{C}^2$, we show that $C(\overline{Jg})$ characterizes precisely the polygon $\mathcal{E}(g) \bigcap \Gamma^+(g)$

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§0. Notation

We consider germs $g : k^n, 0 \to k, 0$ which are either C^∞ or real analytic if $k = \mathbb{R}$ or holomorphic if $k = \mathbb{C}$. We will use local coordinatas x for k^n and denote the ring of germs $k^n, 0 \to k, 0$, (in the appropriate category) by C_n. $\mathcal{M}_n$ denotes its maximal ideal.

Let $\Theta(g)$ denote the module of germs of vector fields $\xi : k^n, 0 \to Tk^p$ such that $\pi_p \circ \xi = g$ for $\pi_p : Tk^p \to k^p$.$\Theta(Id_{k^n})$ will be denoted by Θ_n, this is a free C_n- module generated by $\frac{\partial}{\partial x_i}$ $i = 1, \ldots, n$.

For a deformation $G : k^n \times k^r, 0 \to k, 0$ of a germ $g : k^n, 0 \to k, 0$ we let t denote local coordinates for k^r, the space of deformation parameters.

By a deformation G of g being topologically trivial, we mean that there is a germ of a homeomorphism $\psi : k^n \times k^r, 0 \rightarrow k^n \times k^r, 0$ of the form $\psi(x,t) = (\overline{\psi}(x,t), t)$ such that

$$G \circ \psi(x,t) = g(x)$$

We denote by X_G the variety in $k^n \times k^r, 0$ defined by $G^{-1}(0)$.

The pair $\{X_G, 0 \times k^r\}$ is Whitney - equisingular if $0 \times k^r$ is a stratum of a Whitney stratification of X_G in a neighbourhood of $0.[T_2]$

§1. The Newton filtration

The Newton polyhedron $\Gamma^+(g)$ of a germ $g : k^n, o \rightarrow k, o$ is the convex hull of the set

$$\bigcup \{k + \mathbb{R}^n_+ \mid a_k \neq o\} \text{ in } \mathbb{R}^n_+,$$

and $j^\infty g(x) = \sum_k a_k x^k$ is the Taylor expansion of the germ g.

The Newton boundary $\Gamma(g)$ is the union of the compact faces of $\Gamma^+(g)$

We denote by in(g) the terms of $j^\infty g$ belonging to $\Gamma(g)$.

A germ g is commode if for each j there is an $x_j^{m_j}$ with nonzero coefficient in the Taylor expansion of g.

From the Newton polyhedron, we construct the Newton filtration.

For each face $\Delta \subset \Gamma(g), C(\Delta)$ denotes the cone of half-rays emanating from 0 and passing through Δ. The collection of all $C(\Delta), \Delta \subset \Gamma(g)$ gives a polyhedral decomposition .

The Newton filtration is then defined via a piecewise-linear map $\phi : \mathbb{R}^n_+ \rightarrow \mathbb{R}$ satisfying:

(i) ϕ is linear on each $C(\overline{\Delta})$ ($\overline{\Delta}$ is the closure of Δ in $\Gamma^+(g)$)

(ii) ϕ takes positive integer values on the lattice points of $\mathbb{R}^n - \{0\}$

(iii) $\phi \mid_{\Gamma(g)} = m$ (for some positive integer m)

For any monomial x^α we define fil $(x^\alpha) = \phi(\alpha)$

This extends to a filtration on $k[x_n]$, $k[[x_n]]$, and C_n by defining.

$$fil(\sum C_\alpha \, x^\alpha) = min \, \{\phi(\alpha) : C_\alpha \neq 0\}$$

We denote by $A_l = \{g \in C_n \mid \text{fil } (g) \geq l\}$

Because of the identification of a monomial x^α with $\alpha \in \mathbb{R}^n_+$, we can speak of x^α belonging to $C(\Delta)$, when we mean that the associated α belongs to $C(\Delta)$.

Also, we can define the rings $k[\overline{\Delta}], k[[\overline{\Delta}]]$, etc, to consist of polynomials, power series, etc. with non-zero monomials in $C(\overline{\Delta})$.

A germ g is non-degenerate if g is commode and $\left\{in\left(x_i\frac{\partial g}{\partial x_i}\right) \mid \Delta\right\}$ generate an ideal of finite codimension in $k[[\overline{\Delta}]]$ for each closed face $\overline{\Delta} \subset \Gamma(g)$

All the concepts in this section extend to C_{n+r}, the ring of r-parameter families of germs in n- variables, by defining fil $(x^\alpha\, t^\beta) =$ fil (x^α).

§2 - The method of the Newton filtration of Damon and Gaffney

Damon and Gaffney describe in $[DG]$ a method based on the Newton filtration of the germ g to prove directly that G is topologically trivial. The general form of their result is given by a filtration condition involving the deformed terms $\frac{\partial G}{\partial t_i}$ and the deformation G. Their method consists in constructing controlled vector fields for the faces of the Newton polyhedron of g. When the filtration properties of the deformation behave well with respect to the local data, the local conditions "patch together" to give the "global" filtration properties. Then, topological triviality is proven by solving the localized equation for triviality using controlled vector fields.

Let $g : k^n, 0 \to k, 0$ be C^∞ or analytic when $k = \mathbb{R}$ and holomorphic when $k = \mathbb{C}$, and let ϕ denote its Newton filtration.

We can define a filtration in Θ_n associated to the filtration in C_n.

Let $V_l = \{\xi \in A_{l+1}\left[\frac{\partial}{\partial x_i}\right] : \xi(A_k) \to A_{k+l}\}$, where $A_{l+1}\left[\frac{\partial}{\partial x_i}\right]$ is the A_{l+1} module generated by $\frac{\partial}{\partial x_i}, i = 1, \ldots\ n$.

Definition 2.1 $[DG]$ *A set of local patching data for g consists of: for each face $\Delta \subset \Gamma(g)$, a set $\{\xi_{\Delta,i}\}$ of germs of polynomial vector fields in Θ_n such that*

(i) $\xi_{\Delta,i} = \sum g_{\Delta,i,j} x_j \frac{\partial}{\partial x_j}$

(ii) $\{in(\xi_{\Delta,i}(g)) | \overline{\Delta}\}$ generates an ideal of finite codimension in $k[[\overline{\Delta}]]$.

Proposition 2.2 [D.G pg 344] *If* $\left\{x_i \frac{\partial g}{\partial x_i}\right\}$ *generates an ideal of finite codimension in* C_n*, then there exists a set of local patching data for* g.

To measure the effect of a fixed local patching data applied to the filtration, Damon and Gaffney use the notion of jumps.

Definition 2.3 *For a local patching data* $\{\xi_{\Delta,i}\}$ *for* g*,let jump* $(\xi_{\Delta,i})$ *be defined by*

$$jump(\xi_{\Delta,i}) \;=\; \text{fil}\;(\xi_{\Delta,i}(g)) \;-\; min_j\Big\{\;\text{fil}\;\Big(g_{\Delta,i,j}\;x_j\;\frac{\partial}{\partial x_j}\Big)\Big\}$$

Also, for each face $\Delta \subset \Gamma(g)$ we define

$$jump(\Delta) \;=\; max_i\;\{jump(\xi_{\Delta,i})\}$$

For each vertex $v \in \Gamma(g)$, we define

$$jump(v) \;=\; max\{jump(\Delta)\;:\;\Delta \subset star(v)\}$$

(star(Δ) is defined as for any polyhedral decomposition $star(\Delta) = \bigcup\{\Delta' \in \Gamma(g) : \Delta \subset \overline{\Delta'}\}$

For any germ $\phi \in C_{n+r}$, we can define $fil_v(\phi)$. If ϕ_L denotes the Taylor expansion of ϕ in x up to filtration $\leq L$, then $fil_v(\phi_L)$ is independent of L for all sufficiently large L.

Then, $fil_v(\phi)$ is defined to be this common value.

Definition 2.4 *A germ* ϕ *satisfies a simple jump condition for* g *if*

$$fil_v(\phi) \;\geq\; m + jump(v)\;\;\text{for all vertices}\;\; v \in \Gamma(g)$$

We are now ready to describe Damon. Gaffney's result

Theorem 2.5 ([DG], Theorem 3) *Let* $G : k^{n+r}, 0 \;\rightarrow\; k, 0$ *be a deformation of* g*. Suppose that* g *has local patching data* $\{\xi_{\Delta,i}\}$ *so that:*

(i) $fil\,(\xi_{\Delta,i}(G)) \;=\; fil\,(\xi_{\Delta,i}(g))$ *for all* Δ, i.

(ii) The $\partial G/\partial t_i$ *satisfy a jump condition.*
Then, G *is a topologically trivial deformation.*

An important corollary of this result is:

Corollary 2.6 *[DG, pg 349] Suppose that $g : k^n, 0 \rightarrow k, 0$ is a non-degenerate germ with an isolated singularity. Any deformation of g of non-decreasing Newton filtration $fil(G) = fil(g)$ is a topologically trivial deformation.*

Let $G(x,t) = g(x) + tx^k$, $x^k = x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n}$
Suppose the following condition holds for G :

(2.7) there exists a set of local patching data $\mathcal{P}$ satisfying the hypothesis of Theorem 2.5.

Definition 2.8 *We define $\sum_{\mathcal{P}}^{+}(g)$ as the convex hull of the set*

$$\bigcup\{k + \mathbb{R}_+^n \mid \text{condition(2.7)holds}\}$$

Let $\sum^{+}(g) = \bigcup_{\mathcal{P}} \sum_{\mathcal{P}}^{+}(g)$
We call $\sum^{+}(g)$ the Damon-Gaffney polyhedron of g.
With this notation, we may reestate Damon - Gaffney's theorem as:

Theorem 2.9

(a) *If $\Theta \in \sum^{+}(g)$, then $G(x,t) = g(x) + t\Theta(x)$ is topologically trivial for sufficiently small t*

(b) *If $\Theta \in int \sum^{+}(g)$, then $G(x,t) = g(x) + t\Theta(x)$ is topologically trivial, $\forall t$.*

§3 - The gradient polyhedron of Yoshinaga.

The approach in [Y] to the question of topological triviality is also based on the method of controlled vector fields.
Let $|\, grad\ g|^2 = \sum_{i=1}^{n} |x_i \frac{\partial g}{\partial x_i}|^2$
For a fixed $k \in \mathbf{Z}_+^n$ consider the following condition:

(3.1) there exists a positive $\mathcal{E} = \mathcal{E}(k)$ such that $|\, grad\ g \,| \geq \mathcal{E}|x^k|$ in a neighbourhood of the origin in k^n.

Definition 3.2 [Y , pg 804] *The gradient polyhedron of* $g, \Lambda^+(g)$ *is the convex hull of the set*

$$\bigcup\{k + \mathbb{R}^n_+ \mid \text{condition (3.1) holds }\}$$

Theorem 3.3 [Y , pg. 805]

(a) *If* $\Gamma^+(\Theta) \subset int\Lambda^+(g)$, *then the family* $G(x,t) = g(x) + t\Theta(x)$ *is topologically trivial for all* $t \in [0,1]$

(b) *If* $\Gamma^+(\Theta) \subset \Lambda^+(g)$, *then* $G(x,t) = g(x) + t\Theta(x)$ *is topologically trivial for sufficiently small values of* t.

Remark 3.4: Yoshinaga also proves that $\Lambda^+(g) = \Gamma^+(g) \Leftrightarrow g$ is non-degenerate with respect to its Newton diagram ([Y, Theorem 1.7]).

§4: Equivalence among the approaches to topological triviality and equisingularity

In this section we use the notion of integral closure of ideals and the results of Teissier ([T_1]), ([T_2]) on the equisingularity of the pair $\{X_G, 0 \times k\}$ to show the equivalence between the approaches of Damon-Gaffney, and Yoshinaga.

Let $\mathcal{O}_{X,x_0}$ be the local ring of a complex analytic reduced space X, at x_0.

Definition 4.1 *Let* $I \subset \mathcal{O}_{X,x_0}$ *an ideal. An element* $h \in \mathcal{O}_{X,x_0}$ *is integral over* I, *if satisfies the relation of integral dependence*

$$h^k + a_1h^{k-1} + \ldots + a_k = 0, \quad \text{where } a_i \in I^i$$

The integral closure of I, denoted $\overline{I}$, is the ideal in $\mathcal{O}_{X,x_0}$ of the integral elements of I.

Teissier gives in [T_1] various equivalent notions to the above concept.

Proposition 4.2 ([T_1], [G]) *Let* I *be an ideal in* $\mathcal{O}_{X,x_0}, X$ *a reduced complex analytic space. The following statements are equivalent:*

(i) $h \in \overline{I}$

(ii) *(Hironaka) For each choice of generators* $\{g_i\}$ *of* I *there exists a neighbourhood* U *of* x *and a constant* $C > 0$ *such that*

$$\| h(x) \| \leq C \, sup_i \, \| g_i(x) \| \ \text{for all } x \in U$$

(iii) *(Valuative criterion) For each analytic curve* $\varphi : \mathbb{C}, 0 \rightarrow X, x$, $h \circ \varphi$ *lies in* $(\varphi^*(I))\mathcal{O}_1$

(iv) *There exists a faithful* $\mathcal{O}_{X,x_0}$ *module* L *of finite type such that* $h.L \subset I.L$.

Remark 4.3 When X is real analytic, the above definition of integral closure is not adequate.

We may take the valuative criterion (iii) above as the definition of $\overline{I}$. The equivalences $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ remain true.

Let $X = G^{-1}(0)$, where $G : k^n \times k.0 \rightarrow k, 0$ is a deformation of the germ $g : k^n, 0 \rightarrow k, 0$, which has isolated singularity.

We denote by $Jg = \left\langle x_i \frac{\partial g}{\partial x_i} \right\rangle_{i=1,\ldots,n}$

Definition 4.4 *We denote by* $C(\overline{Jg})$ *the convex hull of the set*

$$\bigcup \left\{ k + \mathbb{R}^n_+ \mid x^k = x_1^{k_1} \ldots x_n^{k_n} \in \overline{Jg} \right\}$$

We can state now our main result:

Proposition 4.5 $C(\overline{Jg}) = \Lambda^+(g) = \Sigma^+(g)$

Proof:

Let $x^k = x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n}$ be in $\Lambda^+(g)$. Then condition (3.1) holds, i. e, $|grad\ g| \geq c \cdot |x^k|$, for some constant $c > 0$. Now, this condition is clearly equivalent to (ii) in Proposition 4.1. Thus, $\Lambda^+(g) = C(\overline{Jg})$

To show the second equality, let us consider $\Theta \in \Sigma^+_{\mathcal{P}}(g)$, where $\mathcal{P}$ is some local patching data for $G(x,t) = g(x) + t\,\Theta(x)$.

Then, as in [D. G, pg. 343] it follows that there exists a level commode A_l in C_{n+1} such that

$$ver[A_{2l}]\ \Theta \subset A_{2l}\left\{x_j \frac{\partial G}{\partial x_j}\right\}$$

Hence, we may write the localized equation:

$$-\rho \cdot \Theta = \sum_i \xi_i(G)\ ,$$

where $\rho = \sum |x^\alpha|^2, x^\alpha$ are the vertices of A_l, and

$$\xi_i = \sum_j h_{i,j}\ x_j \frac{\partial}{\partial x_j}, \quad \text{with} \quad fil\,(h_{i,j}) \geq 2l.$$

Then: $\qquad \rho|\Theta| \leq n max_j\ |h_{i,j}|$

Now, we apply Lemma 3.5 in [D.G., pg 342], to obtain $|h_{i,j}| \leq C_1\rho$ for some constant C_1, in some neighbourhood U of 0 in $k^n, j = 1, \ldots, n$.

Hence,

$$\rho\ |\Theta| \leq c\rho\ \mathrm{max_j}|x_j \frac{\partial G}{\partial x_j}\ |$$

The following inequality also holds at $x = 0$

$$|\ \Theta\ | \leq c\ \mathrm{max_j}\ |x_j \frac{\partial G}{\partial x_j}\ |\ in\ U\ ,$$

which, by condition (ii) in Propostion 4.1 will imply that $\Theta \in \left\langle x_j \frac{\partial G}{\partial x_j}\right\rangle$, and consequently, $\Theta \in C(\overline{Jg})$.

Suppose $\Theta \in C(\overline{Jg})$.

We define the vector field in Θ_n by:

$$\xi = \overline{\sum_{i=1}^{n} \left[|grad\ g|^{-2}\ \overline{\Theta}|x_i|^2 \frac{\partial g}{\partial x_i}\right]} \frac{\partial}{\partial x_i}$$

Since Θ is *in* $\overline{J(g)}$, there exists a constant $c > 0$ such that $|\Theta| \leq c \cdot |gradg|$, that is $fil(\Theta) \geq fil(g) + [fil(grad^2 g) - fil[|x_i \frac{\partial g}{\partial x_i}|^2|]$. Then, Θ satisfies the jump conditions for ξ and $\Theta \in \sum^+(g)$.

In what follows, we define the polyhedron of equisingularity of the germ $g, \mathcal{E}(g)$, and show that $C(\overline{Jg})$ (equivalently, $\Lambda^+(g), \sum^+(g)$), characterizes a subset of it.

Definition 4.6 *The polyhedron of equisingularity of a germ g, denoted by $\mathcal{E}(g)$, is the convex hull of the set:*
$\bigcup\{k + \mathbb{R}^n_+ |$ the pair $\{X_G, 0 \times k\}$ is Whitney equisingular $\}$

For complex germs, Teissier gives the following algebraic characterization of $\mathcal{E}(g)$:

Theorem 4.7 $[T_2, pg604]$ *The pair $\{X_G, 0 \times k\}$ is equisingular if and only if $\frac{\partial G}{\partial t} \in \overline{\left\langle x_i \frac{\partial G}{\partial x_j} \right\rangle}_{i,j=1,\ldots,n}$*

Remark 4.8 The sufficient condition in Teissier's algebraic criterion for equisingularity of the pair $\{X_G, 0 \times k\}$ is also valid in the real analytic category.

The following proposition shows that the results of Damon-Gaffney, and of Yoshinaga can be stated in terms of the equisingularity of the pair $\{X_G, 0 \times k\}$

Proposition 4.9 $C(\overline{Jg}) \subset \mathcal{E}(g)$ *(or equivalently, $\Lambda^+(g) \subset \mathcal{E}(g)$, $\sum^+(g) \subset \mathcal{E}(g)$)*

The proof of this proposition follows from Teissier's result (Theorem 4.7) and the Lemma:

Lemma 4.10 *Let $G(x,t) = g(x) + t\Theta(x)$ be a deformation of g such that $\Gamma^+(\Theta) \subset C\ \overline{(J(g))}$. Then $\overline{\left\langle x_i \frac{\partial G}{\partial x_j} \right\rangle} \supset \overline{J(g)C_{n+1}}$*

Proof: The proof is the same as in [Lemma 2.1, Y]

Example 4.11 Let $g : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ be a complex germ with isolated singularity at $x = 0$, and nondegenerate with respect to $\Gamma^+(g)$. Then $\Gamma^+(g) = \Gamma^+(g) \bigcap \mathcal{E}(g) = C(\overline{Jg})$

§5. A special class of germs of plane curves for which $\Gamma^+(g) \bigcap \mathcal{E}(g) = C(\overline{Jg})$

(5.1) Let $g : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be a germ defining a plane curve, with (algebraically) isolated singularity at 0. Let us assume further that g is commode.

Preposition 5.2 *With the above hypothesis, we have:*

$$\Gamma^+(g) \bigcap \mathcal{E}(g) = C(\overline{Jg})$$

Lemma 5.3 *Let g be as in (5.1) and $\Theta(x_1, x_2) = x_1^{a_1} x_2^{a_2}$, with $a_i \neq 0, i = 1, 2$*

Then $\overline{\left\langle x_i \frac{\partial G}{\partial x_j} \right\rangle} = \overline{\left\langle x_i \frac{\partial G}{\partial x_i} \right\rangle}_{i=1,2}$, where $G(x_1, x_2) = g(x_1, x_2) + t\Theta(x_1, x_2)$.

Proof

Since $\overline{\left\langle x_i \frac{\partial G}{\partial x_j} \right\rangle} \supset \overline{\left\langle x_i \frac{\partial G}{\partial x_i} \right\rangle}_{i=1,\ldots,n}$, we need only show that if $\Theta \notin \overline{\left\langle x_i \frac{\partial G}{\partial x_i} \right\rangle}$, *then* $\Theta \notin \overline{\left\langle x_i \frac{\partial G}{\partial x_j} \right\rangle}$

Applying the avaliative criterion (Prop. 4.2, (*ii*)), we see that if $\Theta \notin \overline{\left\langle x_i \frac{\partial g}{\partial x_j} \right\rangle}$ then $\Theta \notin \overline{\left\langle x_i \frac{\partial G}{\partial x_j} \right\rangle}$.
Then, we may assume that

$$\Theta \notin \overline{\left\langle x_i \frac{\partial g}{\partial x_i} \right\rangle} \quad \text{and} \quad \Theta \in \overline{\left\langle x_i \frac{\partial g}{\partial x_j} \right\rangle}$$

Let $\phi : (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$ be such that fil $(\Theta \circ \phi) < fil(x_i \frac{\partial g}{\partial x_i} \circ \phi)$, $i = 1, 2$
Then, $\phi(\lambda) = (c_1 \lambda^{r_1} + \phi_1,\ c_2 \lambda^{r_2} + \phi_2)$, with degree $(\phi_i) > r_i, c_i \neq$ o, and $r_1 \leq r_2$.

Since $\Theta \in \overline{\left\langle x_i \frac{\partial g}{\partial x_j} \right\rangle}$ and $r_1 \leq r_2$, we have:

$$fil\left(x_1 \frac{\partial g}{\partial x_2} \circ \phi\right) \leq fil(\Theta \circ \phi) < fil\left(x_2 \frac{\partial g}{\partial x_2} \circ \phi\right) \tag{5.4}$$

Furthermore,, g is commode and $c_i \neq 0, i = 1, 2$ hence

$$fil\left(x_1 \frac{\partial g}{\partial x_2} \circ \phi\right) = fil\left(x_2 \frac{\partial g}{\partial x_2} \circ \phi\right) - (r_2 - r_1). \tag{5.5}$$

The conditions (5.4) and (5.5) imply that

$$fil\left(x_1 \frac{\partial \Theta}{\partial x_2} \circ \phi\right) < fil\left(x_1 \frac{\partial g}{\partial x_2} \circ \phi\right)$$

We can assume that
$x_1 \frac{\partial g}{\partial x_2} \circ \phi = \sum_{i \geq i_o} a_i \lambda^i$, $a_{i_o} \neq 0$, $x_1 \frac{\partial \Theta}{\partial x_2} \circ \phi = \sum_{j \geq j_o} b_j \lambda^j$, $b_{j_o} \neq 0$ and $j_o < i_o$.

Then we define $\psi : k, o \rightarrow k^2 \times k, o$ by $\psi(\lambda) = (\phi(\lambda), t(\lambda))$, where $t(\lambda) = \lambda^{(i_o - j_o)} S(\lambda).[\sum_i^q a_i \lambda^{i-i_o}]$ and $S(\lambda) = \left[\sum_j b_j \lambda^{j-j_o}\right]^{-1}$.

Since fil $\Theta \circ \phi = fil(\Theta \circ \psi)$ and $fil\left(x_1 \frac{\partial G}{\partial x_2} \circ \psi\right) = 0$, it follows that $\Theta \notin \overline{\left\langle x_i \frac{\partial G}{\partial x_j}\right\rangle}$

Remark 5.6 As a consequence of the above proof it follows that if $\Theta(x_1, x_2) = x_1^{a_1} x_2^{a_2}$ $a_1 \neq 0, a_2 \neq 0$ is in $\mathcal{E}(g)$ then it belongs to $\Gamma^+(g)$.

Proof of Proposition 5.2

Since g is commode, there are positive integers $p_i, i = 1, 2$ such that $g(x_1, x_2) = x_1^{p_1} + x_2^{p_2} + h(x_1, x_2)$ and ord $(h(x_i, 0)) > p_i$.

By a diffeomorphism that leaves $C(Jg)$ invariant, we can assume that g has the form:

(5.7) $$x_1^{p_1} + x_2^{p_2} + x_1 x_2 r(x_1, x_2),$$

For deformations of g by mixed terms $\theta(x_1, x_2) = x_1^{a_1} x_2^{a_2}, a_1 \neq 0, a_2 \neq 0$, the result follows from Lemma 5.3.

Let us assume that $\theta(x_1, x_2) = x_1^{p_1} \notin \overline{Jg}$. Then, it follows from the expression 5.7 for g that there exists a degenerate cell of the Newton diagram of g containing $x_1^{p_1}$. Hence, x_1^p does not belong to $\overline{\left\langle x_i \frac{\partial g}{\partial x_j}\right\rangle} \bigcap \Gamma^+(g)$ either.

The following example shows that even for complex plane curves, $\mathcal{E}(g)$ may contain $C(\overline{Jg})$ properly.

Example 5.8: $g(x_1, x_2) = x_1^a + x_1^4 x_2 + x_2^5$, $a > 5$.

In this example g is commode, $\Gamma^+(g)$ is non-degenerate but clearly, $x_1^5 \in \mathcal{E}(g)$ and $x_1^5 \notin C(\overline{Jg})$.

Remark 5.9: An algorithm to determine $C(\overline{Jg})$ for plane curves is described in [S].

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NOTAS DO ICMSC

Nº 99/91 - ACHCAR, J.A. & LOUZADA NETO, F. - A bayesian approach for accelerated life tests considering the Weibull distribution

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