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A Bayesian approach for accelerated life tests considering the Weibull distribution

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A BAYESIAN APPROACH FOR ACCELERATED LIFE TESTS CONSIDERING THE WEIBULL DISTRIBUTION

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SUMMARY

"We develop a Bayesian analysis for accelerated life tests considering a Weibull distribution for the life times and a general stress-response model considering one stress variable which includes some usual models (for example, the Eyring model, the Power rule model and the Arrhenius model). We assume a noninformative Jeffreys prior density for the parameters and we use the Laplace's method to find the marginal posterior densities of interest. With this approach, we obtain simple expressions for the marginal posterior densities, a result that could be of great practical interest, specially for industrial applications. We illustrate the proposed method with a generated data set".

KEY WORDS: Accelerated tests, Weibull distribution, Bayesian approach, Laplace's method.

1. INTRODUCTION

Accelerated life tests are used in industrial applications when product life under normal operating conditions is very long, but we need to obtain measures of reliability of units under the usual stress level (see for example, Mann, Schaffer and Singpurwalla, 1974; Nelson, 1990). Consider T a random variable denoting the life time of an unit with a Weibull density,

$$f(t;\lambda,p) = \lambda p(\lambda t)^{p-1}\exp\{-(\lambda t)^{p}\} \tag{1}$$

where $t > 0$, $\lambda, p > 0$, and assume a stress variable V affecting the scale parameter λ, but with common shape parameter p for all stress levels.

With k levels of a stress variable V, assume the general stress-response model given (see Klein and Basu, 1981) by

$$\lambda_i = \exp\{-(Z_i + \beta_0 + \beta_1 X_i)\} \tag{2}$$

where $i = 1,2,\ldots,k$, β_0 and β_1 are unknown parameters. Some usual stress response models used in engineering applications are particular cases of (2). Thus,

i) If $X_i = -\log V_i$, $Z_i = 0$, $\beta_0 = \log\alpha$ and $\beta_1 = \beta$, we have the power rule model.

ii) If $X_i = 1/V_i$, $Z_i = 0$, $\beta_0 = -\alpha$ and $\beta_1 = \beta$, we have the Arrhenius model.

iii) If $X_i = 1/V_i$, $Z_i = -\log V_i$, $\beta_0 = -\alpha$ and $\beta_1 = \beta$, we have the Eyring model.

Also, assume a type II censoring mechanism, that is, the experiment terminates when we observe r_i failures for each stress level V_i. Thus, with n_i units at the beginning of each test with stress V_i, we have the ordered uncensored observations given by $t_{1_i}, t_{2_i}, \ldots, t_{r_i i}$ and $n_i - r_i$ censored observations equal to $t_{r_i i}$, $i = 1,2,\ldots,k$.

Considering the data of k stress levels $V_1, V_2, \ldots, V_k$ taken at random, the likelihood function for β_0, β_1 and p is given by

$$L(\beta_0,\beta_1,p) = p^r\left(\prod_{i=1}^{k}\prod_{j=1}^{r_i} t_{ij}^{p-1}\right) \times$$
$$\times \exp\{-pa_0-p\beta_0 r-p\beta_1 a_1-e^{-p\beta_0}\sum_{i=1}^{k} A_i(p)e^{-pZ_i-p\beta_1 X_i}\} \tag{3}$$

where $r = \sum_{i=1}^{k} r_i$, $a_0 = \sum_{i=1}^{k} r_i Z_i$, $a_1 = \sum_{i=1}^{k} r_i X_i$ and $A_i(p) = \sum_{j=1}^{r_i} t_{ij}^p + (n_i-r_i)t_{ir_i}^p$.

Standard classical methods for inferences on β_0, β_1 and p usually requires the use of iterative methods to obtain the maximum likelihood estimators and the use of asymptotic results. A Bayesian analysis could be of great practical interest.

In section 2, we develop a Bayesian analysis using the Jeffreys multiparameter rule (see for example, Box and Tiao, 1973) to find a noninformative prior density for β_0, β_1 and p, and we find the marginal posterior density for p using the Laplace's method for approximation of integrals (see for example, Tierney and Kadane, 1986; or Kass, Tierney and Kadane, 1990). Since T^p has an exponential distribution with parameter $\lambda^* = \lambda^p$, we can transform the data considering the value of p given by the mode of the marginal posterior density for p and consider a standard Bayesian analysis of the model (2) with a exponential distribution for the life times, as is given in section 3. In section 4, we have a numerical illustration of the proposed Bayesian method.

2. A BAYESIAN ANALYSIS ASSUMING β_0, β_1 AND p UNKNOWN

The prior density for β_0, β_1 and p can be written in the form

$$\pi(\beta_0,\beta_1,p) = \pi(\beta_0,\beta_1|p)\pi_0(p),$$

and using the Jeffreys multiparameter rule, we have,

$$\pi(\beta_0,\beta_1|p) \propto \{\det I_p(\beta_0,\beta_1)\}^{1/2} \tag{4}$$

where $I_p(\beta_0,\beta_1)$ is the Fisher information matrix given p.

The Fisher information matrix for β_0 and β_1 given p is given by,

$$I_p(\beta_0,\beta_1) = \begin{bmatrix} rp^2 & a_1p^2 \\ a_1p^2 & a_2p^2 \end{bmatrix} \tag{5}$$

where r and a_1 are given in (3) and $a_2 = \sum_{i=1}^{k} r_iX_i^2$.

Assuming a locally uniform prior density for $\log p$, that is, $\pi_0(p) \propto 1/p$, $p > 0$, a noninformative prior for β_0, β_1 and p based on the Jeffreys multiparameter rule is given (from (4)) by

$$\pi(\beta_0,\beta_1,p) \propto p \tag{6}$$

where $p > 0$ and $-\infty < \beta_0,\beta_1 < \infty$.

2.1 - MARGINAL POSTERIOR DENSITY FOR p

Considering the prior (6), the joint posterior density for β_0, β_1 and p is given by

$$\pi(\beta_0,\beta_1,p|\text{data}) \propto p^{r+1}\left(\prod_{i=1}^{k}\prod_{j=1}^{r_i} t_{ij}^{p-1}\right) \times \\ \times \exp\{-pa_0-p\beta_0 r-p\beta_1 a_1-e^{-p\beta_0}\sum_{i=1}^{k} A_i(p)e^{-pZ_i-p\beta_1X_i}\} \tag{7}$$

where $p > 0$ and $-\infty < \beta_0,\beta_1 < \infty$.

The marginal posterior density for p is given (from (7)) by

$$\pi(p|\text{data}) \propto p^{r+1}\left(\prod_{i=1}^{k}\prod_{j=1}^{r_i} t_{ij}^{p-1}\right)e^{-pa_0} \times$$

$$\times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-nh(\beta_0,\beta_1)}\, d\beta_0 d\beta_1$$

where

$$-nh(\beta_0,\beta_1) = -p\beta_0 r - p\beta_1 a_1 - e^{-p\beta_0}\sum_{i=1}^{k} A_i(p)e^{-pZ_i - p\beta_1 X_i}.$$

Using the Laplace's method for approximation of integrals (see for example, Tierney and Kadane, 1986), we find an approximate marginal posterior density for p given by

$$\pi(p|\text{data}) \propto \frac{p^{r-1}e^{-p(a_0+\hat{\beta}_1 a_1)}\left(\prod_{i=1}^{k}\prod_{j=1}^{r_i} t_{ij}^{p-1}\right)}{\left\{\sum_{i=1}^{k} A_i(p)e^{-p(Z_i+\hat{\beta}_1 X_i)}\right\}^{r}\left\{r^2\,\frac{\sum_{i=1}^{k} A_i(p)X_i^2 e^{-p(Z_i+\hat{\beta}_1 X_i)}}{\sum_{i=1}^{k} A_i(p)e^{-p(Z_i+\hat{\beta}X_i)}} - a_1^2\right\}^{1/2}} \tag{8}$$

where $p > 0$ and $\hat{\beta}_1$ maximizes

$$-r\log\left\{\frac{\sum_{i=1}^{k} A_i(p)e^{-p(Z_i+\beta_1 X_i)}}{r}\right\} - p\beta_1 a_1 - r,$$

for each value of p.

Observe that we also could use other priors for β_0, β_1 and p obtaining similar results working with Laplace's approximation for integrals.

2.2 - JOINT MARGINAL POSTERIOR DENSITY FOR THE MEAN LIFE TIME θ_1 UNDER THE NORMAL USE STRESS LEVEL V_1 AND p

Usually, industrial researchers have interest to obtain inferences about the mean life time under the normal use stress level V_1 given by $\theta_1 = \Gamma(1+1/p)/\lambda_1$, where $\lambda_1 = \exp\{-(Z_1+\beta_0+\beta_1 X_1)\}$. Considering the transformation of variables $\theta_1 = \Gamma(1+1/p)\exp\{Z_1+\beta_0+\beta_1 X_1\}$, $\beta_1 = \beta_1$ and $p = p$, the joint posterior density for θ_1, β_1 and p is given (from (7)) by

$$\pi(\theta_1,\beta_1,p|\text{data}) \propto p^{r+1}\theta_1^{-(pr+1)}\left(\prod_{i=1}^{k}\prod_{j=1}^{r_i} t_{ij}^{p-1}\right) \times$$

$$\times\, e^{-p(a_0-rZ_1)}[\Gamma(1+1/p)]^{pr} \times$$

$$\times \exp\left\{-p\beta_1(a_1-rX_1)-\theta^{-p}[\Gamma(1+1/p)]^{p}\sum_{i=1}^{k} A_i(p)e^{-p(Z_i-Z_1)+p\beta_1(X_1-X_i)}\right\} \qquad (9)$$

where θ_1, $p>0$ and $-\infty<\beta_1<\infty$.

The Laplace's approximate joint marginal posterior density for θ_1 and p is given by

$$\pi(\theta_1,p|\text{data}) \propto$$

$$\propto p^{r}\theta_1^{-p(r-1/2)-1}\left(\prod_{i=1}^{k}\prod_{j=1}^{r_i} t_{ij}^{p-1}\right)e^{-p(a_0-rZ_1)}(\Gamma(1+1/p)]^{p(r-1/2)} \times$$

$$\times \frac{\exp\left\{-p\hat{\beta}_1(a_1-rX_1)-\theta_1^{-p}[\Gamma(1+1/p)]^{p}\sum_{i=1}^{k} A_i(p)e^{-p(Z_i-Z_1)+p\hat{\beta}_1(X_1-X_i)}\right\}}{\left\{\sum_{i=1}^{k} A_i(p)(X_1-X_i)^2 e^{-p(Z_i-Z_1)+p\hat{\beta}_1(X_1-X_i)}\right\}^{1/2}} \qquad (10)$$

where θ_1, $p>0$ and $\hat{\beta}_1$ maximizes,

$$-p\beta_1(a_1-rX_1) - \theta_1^{-p}[\Gamma(1+1/p)]^p \sum_{i=1}^{k} A_i(p) e^{-p(Z_i-Z_1)+p\beta_1(X_1-X_i)}$$

for each value of (θ_1, p).

3. A BAYESIAN ANALYSIS ASSUMING AN EXPONENTIAL DISTRIBUTION FOR T

Assuming an exponential distribution for T with parameter λ ($p=1$ in (1)), and considering a Jeffreys prior density (from (6)) locally uniform, the joint posterior density for β_0 and β_1 is given by

$$\pi(\beta_0,\beta_1|\text{data}) \propto \exp\left\{-\beta_0 r - \beta_1 a_1 - e^{-\beta_0}\sum_{i=1}^{k} A_i e^{-Z_i-\beta_1 X_i}\right\} \tag{11}$$

where $-\infty < \beta_0, \beta_1 < \infty$ and A_i is given by $A_i(1)$ (see (3)).

The marginal posterior density for β_1 is given by

$$\pi(\beta_1|\text{data}) \propto \frac{e^{-\beta_1 a_1}}{\left\{\sum_{i=1}^{k} A_i e^{-Z_i-\beta_1 X_i}\right\}^r} \tag{12}$$

where $-\infty < \beta_1 < \infty$, and the marginal posterior density for β_0 approximated by the Laplace's method is given by

$$\pi(\beta_0|\text{data}) \propto \frac{\exp\left\{-\beta_0(r-1/2) - \hat{\beta}_1 a_1 - e^{-\beta_0}\sum_{i=1}^{k} A_i e^{-Z_i-\hat{\beta}_1 X_i}\right\}}{\left\{\sum_{i=1}^{k} A_i X_i^2 e^{-Z_i-\hat{\beta}_1 X_i}\right\}^{1/2}} \tag{13}$$

where $-\infty < \beta_0 < \infty$ and $\hat{\beta}_1$ maximizes $-\hat{\beta}_1 a_1 - e^{-\beta_0}\sum_{i=1}^{k} A_i e^{-Z_i-\beta_1 X_i}$, for each value of β_0.

Considering the transformation of variables $\theta_1 = \exp\{Z_1+\beta_0+\beta_1 X_1\}$ and $\beta_1 = \beta_1$, the joint posterior density for θ_1 and β_1 is given (from (11)) by

$$\pi(\theta_1,\beta_1|\text{data}) \propto \theta_1^{-(r+1)}\exp\Big\{-\beta_1(a_1-rX_1) - $$

$$- \theta_1^{-1}\sum_{i=1}^{k} A_i e^{-(Z_i-Z_1)+\beta_1(X_1-X_i)}\Big\} \tag{14}$$

where $\theta_1 > 0$ and $-\infty < \beta_1 < \infty$.

The approximate marginal posterior density for θ_1 is given by

$$\pi(\theta_1|\text{data}) \propto \frac{\theta_1^{-(r+1/2)}\exp\left\{-\hat{\beta}_1(a_1-rX_1)-\theta_1^{-1}\sum_{i=1}^{k} A_i e^{-(Z_i+Z_1)-\hat{\beta}_1(X_1-X_i)}\right\}}{\left\{\sum_{i=1}^{k} A_i(X_1-X_i)^2 e^{-(Z_i-Z_1)+\hat{\beta}_1(X_1-X_i)}\right\}^{1/2}} \tag{15}$$

where $\theta_1 > 0$ and $\hat{\beta}_1$ maximizes $-\beta_1(a_1-rX_1)-\theta_1^{-1}\sum_{i=1}^{k} A_i e^{-(Z_i-Z_1)+\beta_1(X_1-X_i)}$, for each value of θ_1.

4. AN EXAMPLE

In Table 1, we have a generated data set considering an Eyring model and a Weibull distribution (1) with $\alpha = -12.7$, $\beta = 6.2$ and $p = 2$ ($\beta_0 = 12.7$, $\beta_1 = 6.2$, $X_i = 1/V_i$ and $Z_i = -\log V_i$ in the model (2)).

TABLE 1 - Generated Data with $\alpha = -12.7$, $\beta = 6.2$ and $p = 2$.

i	V_i	n_i	r_i	NONCENSORED OBSERVATIONS
1	20	15	5	39, 45, 67, 88, 100
2	30	15	7	27, 34, 42, 61, 67, 79, 97
3	40	15	8	22, 32, 37, 48, 53, 70, 79, 88
4	50	15	10	18, 25, 32, 47, 56, 61, 70, 79, 82, 94
5	60	15	12	20, 22, 30, 34, 50, 58, 64, 73, 79, 88, 88, 100

In Figure 1, we have the graph of the approximate marginal posterior density for p (see (8)). The mode of this posterior density is given by $\tilde{p} = 1.9864$. A 95% HPD interval for p is given by (1.4750; 2.5750).

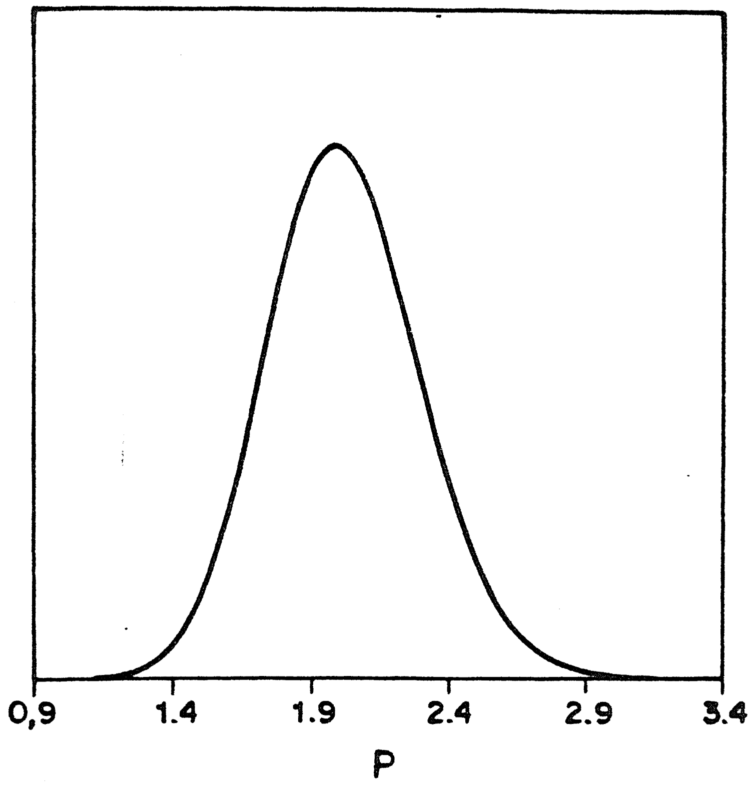

FIGURE 1 - Marginal Posterior Density for p.

Considering the logarithm scale ($\delta = \log\theta_1$ and $\gamma = \log p$), we have in Figure 2 the 95% contour plot for the joint posterior density for δ and γ. The mode of this joint posterior density is given by $\tilde{\delta} = 4.88053$ ($\tilde{\theta}_1 = 131.7005$) and $\tilde{\gamma} = 0.72506$ ($\tilde{p} = 2.06485$). A 95% approximate HPD interval for p is given (see Figure 2) by (1.5683; 2.6209). We observe good eliptical form for the contour plot of the joint posterior density for δ and γ (see Figure 2).

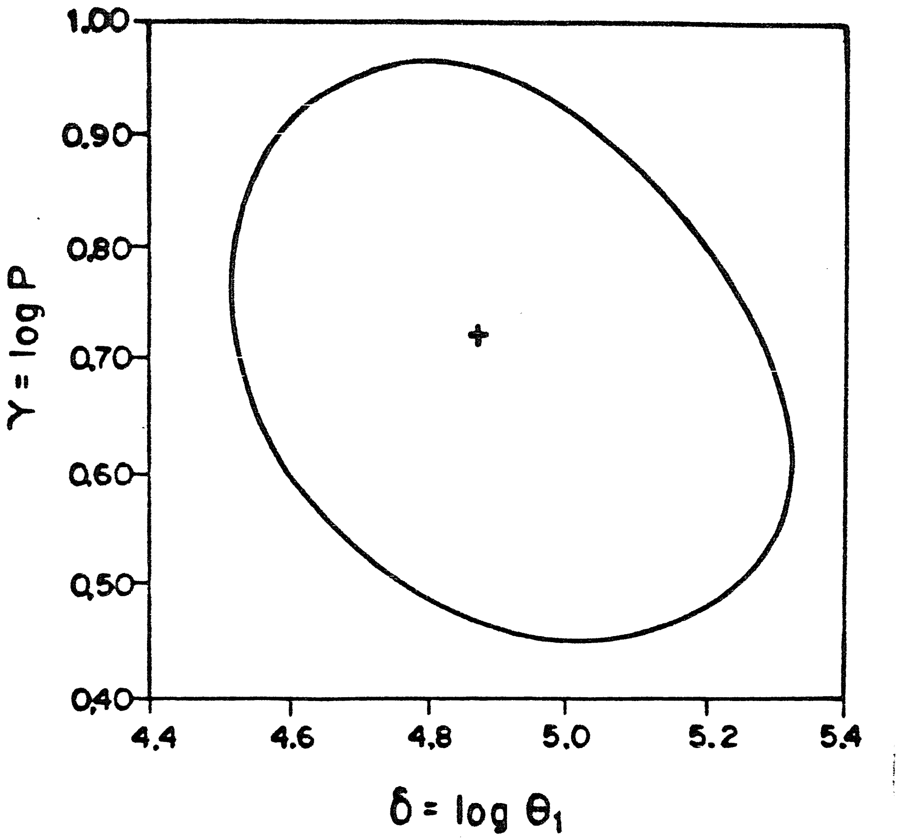

FIGURE 2 - 95% Contour for the Joint Posterior Density for δ and γ.

Since T^p has an exponential distribution with parameter $\lambda^* = \lambda^p$, we also could develop a Bayesian analysis considering p known (say, the mode $p = 1.9864$ of the approximate marginal posterior density for p). In Figure 3, we have the graph of the approximate marginal posterior for $\delta^* = \log(\theta_1^*)$ (from (15)), where θ_1^* is the mean life time under the normal use stress level $V_1 = 20$, considering the transformed data $T^{1.9864}$ with an exponential distribution with parameter $\lambda_1^* = 1/\theta_1^*$. The mode of this posterior density is given by $\tilde{\delta}^* = 10.0561$, that is, $\tilde{\theta}_1^* = 23297.9348$ ($\tilde{\lambda}_1^* = 0.000043$).

In the original scale, we have data with a Weibull distribution with scale parameter $\lambda_1 = \lambda_1^{*1/p}$. That is, $\tilde{\lambda}_1 = 0.00633$ and the mean life time under the normal use stress level $\theta_1 = \Gamma(1+1/p)/\lambda_1$ is estimated by $\tilde{\theta}_1 = 140.0410$.

In Table 2, we have approximate 95% credible intervals for θ_1

considering p known or p unknown.

TABLE 2 - Approximate 95% Credible Intervals for θ_1.

p	95% CREDIBLE INTERVALS FOR θ_1
Known	(99.1936; 214.6563)
Unknown	(92.1299; 205.2031)

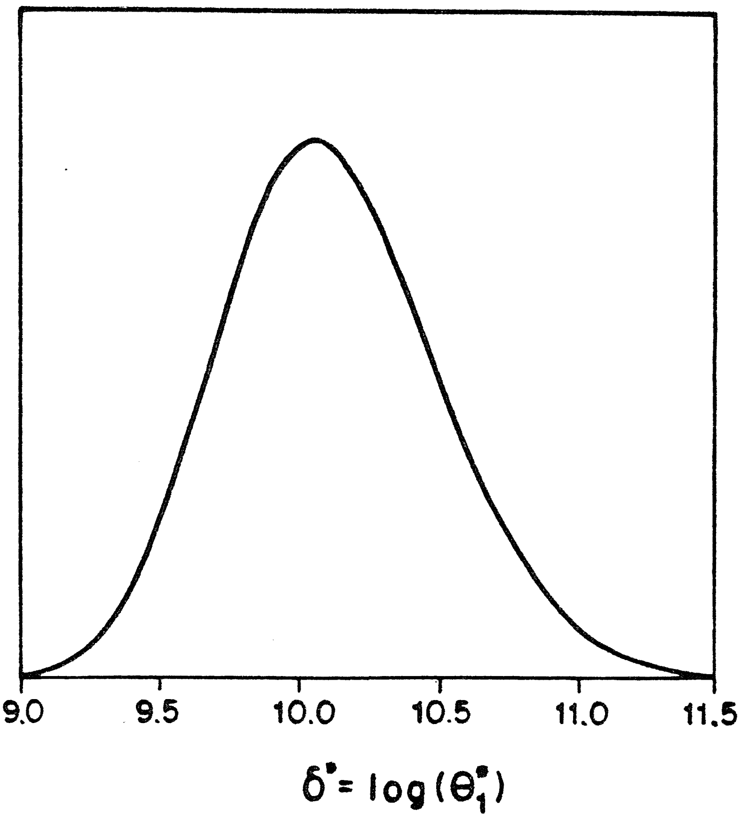

FIGURE 3 - Marginal Posterior Density for $\delta^* = \log(\theta_1^*)$.

5. CONCLUSIONS

We obtained simple formulas for the approximate marginal posterior densities of interest considering accelerated life tests with a Weibull distribution. Since the stress-response model (2) includes the most popular models (power rule, Eyring and Arrhenius model) used in accelerated life testing, the obtained results could be of great practical interest. It is important to point out that accuracy of the

Laplace's method used in the approximation of the marginal posterior densities usually requires carefull reparametrization dependent upon both the data and the choice of prior (see for example, Achcar and Smith, 1990).

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