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SUMMARY

In this paper, we explore the use of Laplace's method to develop a Bayesian analysis for accelerated life tests considering an inverse Gaussian distribution. We find simple expressions for the marginal posterior densities of interest, assuming a Jeffreys noniformative prior density for the parameters. We also find the predictive density for a future observation and we propose some criteria to be used in quality control problems. We illustrate the proposed method with a numerical example.

Key Words: Laplace's method, Jeffreys prior, inverse Gaussian distribution, accelerated life tests.

1 Introduction

Accelerated life tests have been used for a long time by industrial researchers when product life under normal operating conditions is very long, but estimates of length of life are needed in a relatively short time (see for example, Mann, Schaffer and Singpurwalla, 1974; or Nelson 1990). A parametric analysis of accelerated life tests data largely depends on the model chosen for the distribution of the life time and the relation of the parameter (s) to the stress variable. We consider a versatile model for the life time data given by the inverse Gaussian distribution $IG(\Theta, \lambda)$, whose probability density function (pdf) is given by

$$f(y; \Theta, \lambda) = \frac{\lambda^{1/2}}{(2\pi y^3)^{1/2}} exp\{ - \frac{\lambda}{2y} (\frac{y}{\Theta} - 1)^2\} \tag{1}$$

where $y > 0; \Theta, \lambda > 0$.

It is mean and variance are Θ and Θ^3/λ, respectively. The basic properties of this density are given in Tweedie (1957) and an extensive survey is given by Folks and Chhikara (1978). A Bayesian analysis of this model is given by Achcar, Bolfarine and Rodrigues (1991) considering a Jeffreys noninformative prior density for the parameters Θ and λ.

For accelerated life tests, we explore the relation of the parameter Θ to the stress variable given by the reciprocal-linear form,

$$\Theta^{-1} = \alpha + \beta X \tag{2}$$

where X is a stress variable associated to each failure time Y.

Bhattacharyya and Fries (1982), refered to the physical basis of the inverse Gaussian distribution in order to formulate the stochastic relation of the failure time to the intensity of the stress variable X given in (2).

In this paper, we use approximate Bayesian methods based on the Laplace's method for integrals (see for example, Tierney and Kadane, 1986; Kass, Tierney and Kadane, 1990) to find very simple expressions for marginal posterior densities and predictive densities when we cannot obtain explicitly solutions.

In section 2, we present the usual classical approach considering the asymptotical normal distribution for the maximum likelihood estimators of α and β. In section 3, we develop a Bayesian analysis assuming α, β and λ unknown, and we find very simple expressions for the marginal posterior densities of interest using the Laplace's method and a noninformative prior density for the parameters. We also find the marginal posterior density for the mean failure time Θ_1 under the usual stress level X_1. In section 4, we present a Bayesian analysis assuming λ known and in section 5, we find the predictive density for a future observation. In section 6, we present some criteria to be used in quality control using the predictive density for a future observation.

In section 7, we illustrate the proposed method considering a data set introduced by Nelson (1971).

2 The Classical Approach for the Problem

Let $Y_{i1}, \ldots, Yin_i$ be a random sample of size n_i of an inverse Gaussian distribution (1) with parameters Θ_i and λ in a stress level X_i with the reciprocal-linear model (2), $i = 1, 2, \ldots, K$. Considering the data of K stress levels taken at random, the likelihood function for α, β and λ is given by

$$L(\alpha,\beta,\lambda) \propto \lambda^{n/2} \, exp\{-\frac{\lambda}{2}\Sigma_{i=1}^{K} \Sigma_{j=1}^{n_i} \frac{1}{Yij}(\frac{Yij}{\Theta_i} - 1)^2\} \quad (3)$$

where $n = \Sigma_{i=1}^{K} n_i$ and $\Theta_i^{-1} = \alpha + \beta X_i$, $i = 1, 2, \ldots, K$.

The logarithm of the likelihood function (3) is given by

$$\begin{aligned} l(\alpha,\beta,\lambda) \propto & \frac{n}{2} ln\lambda - \frac{n\lambda\alpha^2 V_0}{2} - n\alpha\beta\lambda V_1 - \\ & - \frac{n\beta^2\lambda V_2}{2} + n\alpha\lambda + n\beta\lambda\overline{X} - \frac{n\lambda R}{2} \end{aligned} \quad (4)$$

where $V_l = n^{-1}\Sigma_{i=1}^{K} \Sigma_{j=1}^{n_i} YijX_i^l$, $l = 0, 1, 2$;
$R = n^{-1}\Sigma_{i=1}^{K} \Sigma_{j=1}^{n_i} Y_{ij}^{-1}$ and $\overline{X} = n^{-1}\Sigma_{i=1}^{K} \Sigma_{j=1}^{n_i} X_i$.

The likelihood equations yield the unique root $(\hat{\alpha}_L, \hat{\beta}_L, \hat{\lambda}_L)$, given by

$$\begin{aligned} \hat{\alpha}_L &= (V_2 - \overline{X}V_1) \,/\, (V_0V_2 - V_1^2 \\ \hat{\beta}_L, &= (\overline{X}V_0 - V_1) \,/\, (V_0V_2 - V_1^2) \\ \hat{\lambda}_L^{-1} &= R - \hat{\alpha}_L - \hat{\beta}_L \overline{X} \end{aligned} \quad (5)$$

where $V_0V_2 > V_1^2 > 0$ with probability 1 (see Bhattacharyya and Fries, 1982).

In order to obtain the maximum likelihood estimators (mle), one needs to examine whether or not the root lies in the parameter space $\Omega = \{(\alpha,\beta,\lambda) \, ; \, \alpha \geq 0 \, , \, \beta \geq 0 \, , \, \alpha + \beta > 0 \, , \, \lambda > 0\}$. Bhattacharyya and Fries (1982), state that the maximum likelihood estimators for α, β and λ are given by

$$(\hat{\alpha}, \hat{\beta}) = \begin{cases} (\hat{\alpha}_L, \hat{\beta}_L) & if \;\; V_1 < min(\overline{X}V_0, \overline{X}^{-1}V_2) \\ (0, \overline{X}V_2^{-1}) & if \;\; \overline{X}^{-1}V_2 \leq V_1 \leq \overline{X}V_0 \\ (V_0^{-1}, 0) & if \;\; \overline{X}V_0 \leq V_1 \leq \overline{X}^{-1}V_2 \end{cases} \quad (6)$$

and $\hat{\lambda}^{-1} = R - \hat{\alpha} - \hat{\beta}\,\overline{X}$.

The inferences on α, β and λ can be based on the limiting normal distributions of the mle's considering the observed information matrix I_0 or the Fisher information matrix $I(\alpha, \beta, \lambda)$ (see for example, Lawless, 1982).

The observed information matrix I_o is given by

$$I_0 = \begin{pmatrix} n\,\hat{\lambda}_L\,V_0 & n\,\hat{\lambda}_L\,V_1 & 0 \\ n\,\hat{\lambda}_L\,V_1 & n\,\hat{\lambda}_L\,V_2 & 0 \\ 0 & 0 & \frac{n}{2\hat{\lambda}_L^2} \end{pmatrix} \tag{7}$$

In view of the asymptotic equivalence of $(\hat{\alpha}, \hat{\beta}, \hat{\lambda})$ and $(\hat{\alpha}_L, \hat{\beta}_L, \hat{\lambda}_L)$ (see Bhattacharyya and Fries, 1982), it suffices to consider the limiting distribution of $(\hat{\alpha}_L, \hat{\beta}_L, \hat{\lambda}_L)$. Since $(\hat{\alpha}_L, \hat{\beta}_L, \hat{\lambda}_L) \overset{a}{\sim} N\ \{(\alpha, \beta, \lambda)\ ;\ I_0^{-1}\}$, we find the approximate variances of $\hat{\alpha}, \hat{\beta}$ and $\hat{\lambda}$ given by $(n\,\hat{\lambda}\,D)^{-1}V_2$, $(n\,\hat{\lambda}\,D)^{-1}V_0$ and $2n^{-1}\,\hat{\lambda}^2$, respectively, where $D = V_0V_2 - V_1^2$. The reciprocal mean failure time $\Theta_i^{-1} = \alpha + \beta X_i^*$, at a specified stress level X_i^*, is estimated by $\hat{\alpha} + \hat{\beta}\,X_i^*$, whose approximate variance obtained by the use of the "delta" method (see for example, Miller, 1981) and considering the observed information matrix (7), is given by $(nD\,\hat{\lambda})^{-1}(V_2 - 2X_i^*V_1 + X_i^{*2}V_0)$.

The Fisher information matrix for α, β and λ is given by,

$$I(\alpha, \beta, \lambda) = \begin{pmatrix} n\lambda W_0 & n\lambda W_1 & n\alpha W_0 + n\beta W_1 - n \\ & n\lambda W_2 & n\alpha W_1 + n\beta W_2 - n\overline{X} \\ Symmetric & & \frac{n}{2\lambda^2} \end{pmatrix} \tag{8}$$

where $W_l = E(V_l) = n^{-1}\Sigma_{i=1}^{K}\ \Sigma_{j=1}^{n_i}\ X_i^l\ (\alpha + \beta X_i)^{-1}, l = 0, 1, 2$.

3 A Bayesian Analysis Assuming α, β and λ Unknown

A noninformative prior density for α, β and λ can be obtained by using the Jeffreys rule (see for example, Box and Tiao, 1973) in the following way:

$$\begin{aligned} \Pi(\alpha, \beta, \lambda) &= \Pi(\alpha, \beta|\lambda)\Pi_0(\lambda) \\ &\propto \{detI(\alpha, \beta|\lambda)\}^{1/2}\Pi_0(\lambda) \end{aligned} \tag{9}$$

where $I(\alpha, \beta|\lambda)$ is the Fisher information matrix given λ,

$$I(\alpha, \beta|\lambda) = \begin{pmatrix} n\lambda W_0 & n\lambda W_1 \\ n\lambda W_1 & n\lambda W_2 \end{pmatrix}$$

where $W_l, l = 0,1,2$ is defined in (8).

Assuming $\Pi_o(\lambda) \propto \lambda^{-1}, \lambda > 0$, we find (see (9)) a prior density for α, β and λ given by

$$\Pi(\alpha,\beta,\lambda) \propto \{W_0(\alpha,\beta)W_2(\alpha,\beta) - W_1^2(\alpha,\beta)\}^{1/2} \tag{10}$$

where $\alpha, \beta, \lambda > 0$.

3.1 The Joint Posterior Density for α, β and λ

Considering the noninformative prior (10), the joint posterior density for α, β and λ is given by,

$$\Pi(\alpha,\beta,\lambda/DATA) \propto \lambda^{n/2} H(\alpha,\beta) exp\{-\frac{\lambda}{2}A(\alpha,\beta)\} \tag{11}$$

where $\alpha, \beta, \lambda > 0, H(\alpha,\beta) = \{W_0(\alpha,\beta)W_2(\alpha,\beta) - W_1^2(\alpha,\beta)\}^{1/2}$ and $A(\alpha,\beta) = nV_0\alpha^2 + 2nV_1\alpha\beta + nV_2\beta^2 - 2n\alpha - 2n\overline{X}\beta + nR$.

Integrating out λ in (11), we obtain the joint marginal posterior density for α and β given by

$$\Pi(\alpha,\beta|DATA) \propto \frac{H(\alpha,\beta)}{\{A(\alpha,\beta)\}^{n/2+1}} \tag{12}$$

where $\alpha, \beta > 0$, $H(\alpha,\beta)$ and $A(\alpha,\beta)$ are given in (11).

3.2 The Marginal Posterior Densities for α, β and λ

The marginal posterior densities for α, β or λ can be obtained using the Laplace's method for approximation of integrals (see for example, Tierney, Kass and Kadane, 1989; Tierney and Kadane, 1986), or using numerical methods.

The marginal posterior density for α is (from (12)) given by,

$$\Pi(\alpha|DATA) \propto \int_0^\infty f_\alpha(\beta) e^{-nh_\alpha(\beta)} d\beta \tag{13}$$

where $f_\alpha(\beta) = H(\alpha,\beta)$ and $-nh_\alpha(\beta) = -(\frac{n}{2}+1)\, ln\{A(\alpha,\beta)\}$.

Using the Laplace's method to approximate the integral in (13), we find from (A.2) in the appendix, the approximate marginal posterior density for α given by

$$\Pi(\alpha|DATA) \propto \frac{\{K_0(\alpha)K_2(\alpha) - K_1^2(\alpha)\}^{1/2}}{\{D\alpha^2 - 2(V_2 - \overline{X}V_1)\alpha + RV_2 - \overline{X}^2\}^{(n+1)/2}} \tag{14}$$

where $\alpha > 0$, and

$$K_l(\alpha) = \frac{1}{n}\Sigma_{i=1}^{K}\ \Sigma_{j=1}^{n_i}\frac{X_i^l V_2}{[\alpha V_2 + (\overline{X} - \alpha V_1)X_i]}\ , l = 0, 1, 2.$$

In the same way, we find an approximate marginal posterior density for β given by,

$$\Pi(\beta|DATA)\ \propto\ \frac{\{L_0(\beta)L_2(\beta) - L_1^2(\beta)\}^{1/2}}{\{D\beta^2 + 2(V_1 - V_0\overline{X})\beta + RV_0 - 1\}^{(n+1)/2}} \tag{15}$$

where $\beta > 0$, and

$$L_l(\beta) = \frac{1}{n}\ \Sigma_{i=1}^{K}\ \Sigma_{j=1}^{n_i}\ =\ \frac{X_i^l V_0}{[1 - \beta V_1 + \beta V_0 X_i]}, l = 0, 1, 2.$$

The marginal posterior density for λ is obtained by integrating out α and β in the joint posterior density (11). Using the Laplace's method (see(A.3) in the appendix), we find,

$$\Pi(\lambda|DATA)\ \propto\ \lambda^{\frac{n}{2}-1}exp\{-\frac{n\lambda}{2}\ [R - \frac{(V_2 - 2\overline{X}V_1 + \overline{X}^2 V_0)}{D}]\} \tag{16}$$

where $\lambda > 0$.

3.3 Marginal Posterior Density for the Mean Failure Time Θ_1 Under the Usual Stress Level X_1

Usually, industrial researchers have interest in inferences about $\Theta_1 = (\alpha + \beta X_1)^{-1}$, where X_1 is the usual stress level. Considering the transformation of variables $\Theta_1 = (\alpha + \beta X_1)^{-1}$, and $\alpha = \alpha$, the joint posterior density for α and Θ_1 (from (12)) is given by

$$\Pi(\alpha, \Theta_1|DATA)\ \propto\ \frac{\{W_0(\alpha, \Theta_1)W_2(\alpha, \Theta_1)\ -\ W_1^2(\alpha, \Theta_1)\}^{1/2}}{\Theta_1^2\{a_1\alpha^2 + a_2(\Theta_1)\alpha + a_3(\Theta_1)\}^{\frac{n}{2}+1}} \tag{17}$$

where $\alpha > 0, \Theta_1 > 0$,

$$\begin{aligned}
a_1 &= n(V_0 - \frac{2V_1}{X_1} + \frac{V_2}{X_1^2})\ , \\
a_2(\Theta_1) &= 2n(\frac{V_1}{X_1\Theta_1} - \frac{V_2}{X_1^2\Theta_1} + \frac{\overline{X}}{X_1}\ - 1)\ , \\
a_3(\Theta_1) &= n(\frac{V_2}{X_1^2\Theta_1^2} - \frac{2\overline{X}}{X_1\Theta_1} + R)\ ,
\end{aligned}$$

and $nW_l(\alpha, \Theta_1) = n_1 X_1^l\Theta_1 + \Sigma_{j=2}^{K} n_j X_j^l[\alpha(1 - \frac{X_j}{X_1}) + \frac{X_j}{X_1\Theta_1}]$ for $l = 0, 1, 2$.

The Laplace's approximate marginal posterior density for Θ_1 is given by

$$\Pi(\Theta_1|DATA) \propto \frac{\{J_0(\Theta_1)J_2(\Theta_1) - J_1^2(\Theta_1)\}^{1/2}}{\Theta_1^2\{4a_1a_3(\Theta_1) - a_2^2(\Theta_1)\}^{(n+1)/2}} \tag{18}$$

where $\Theta_1 > 0$, and

$$nJ_l(\Theta_1) = n_1X_1^l\Theta_1 + \Sigma_{j=2}^K n_jX_j^l[\frac{X_j}{X_1\Theta_1} - \frac{a_2(\Theta_1)}{2a_1}(1 - \frac{X_j}{X_1})] \text{ for } l = 0,1,2.$$

4 A Bayesian Analysis Assuming λ Known

Assuming λ Known, the Jeffreys prior density for α and β is given by

$$\Pi(\alpha,\beta) \propto \{W_0(\alpha,\beta)W_2(\alpha,\beta) - W_1^2(\alpha,\beta)\}^{1/2} \tag{19}$$

where $\alpha,\beta > 0$ and $W_l(\alpha,\beta)$ is given in (8), for $l = 0,1,2$.

The joint posterior density for α and β is given by

$$\Pi(\alpha,\beta|DATA) \propto H(\alpha,\beta)\, exp\{-\frac{\lambda}{2}A(\alpha,\beta)\} \tag{20}$$

where $\alpha,\beta > 0, H(\alpha,\beta)$ and $A(\alpha,\beta)$ are given in (11).

Using the Laplace's method for approximation of integrals, we obtain the marginal posterior densities for α and β.

The approximate marginal posterior density for α is given by

$$\Pi(\alpha|DATA) \propto \frac{\{K_0(\alpha)K_2(\alpha) - k_1^2(\alpha)\}^{1/2}}{exp\{\frac{n\lambda}{2V_2}[D\alpha^2 - 2(V_2 - \overline{X}V_1)\alpha + RV_2 - \overline{X}_2]\}} \tag{21}$$

where $\alpha > 0$, and $K_l(\alpha)$ is given in (14), $l = 0,1,2$.

The approximate marginal posterior density for β is given by

$$\Pi(\beta|DATA) \propto \frac{\{L_0(\beta)L_2(\beta) - L_1^2(\beta)\}^{1/2}}{exp\{\frac{n\lambda}{2V_0}[D\beta^2 + 2(V_1 - V_0\overline{X})\beta + RV_0 - 1]\}} \tag{22}$$

where $\beta > 0$, and $L_l(\beta)$ is given in (15), $l = 0,1,2$.

4.1 Marginal Posterior Density for Θ_1 Assuming λ Known

From (20), we find the joint posterior density for $\Theta_1 = (\alpha + \beta X_1)^{-1}$ and α given by

$$\Pi(\alpha, \Theta_1 | DATA) \ \propto \ \frac{\{W_0(\alpha,\Theta_1)W_2(\alpha,\Theta_1) - W_1^2(\alpha,\Theta_1)\}^{1/2}}{\Theta_1^2 exp\{\frac{\lambda}{2}[a_1\alpha^2 + a_2(\Theta_1)\alpha + a_3(\Theta_1)]\}} \tag{23}$$

where $\alpha, \Theta_1 > 0, a_1, a_2(\Theta_1), a_3(\Theta_1)$ and $W_l(\alpha, \Theta_1)$ are given in (17).

The approximate marginal posterior density for Θ_1 is given by

$$\Pi(\Theta_1 | DATA) \propto \frac{\{J_0(\Theta_1)J_2(\Theta_1) - J_1^2(\Theta_1)\}^{1/2}}{\Theta_1^2 exp\{\frac{\lambda}{8a_1}[4_1a_3(\Theta_1) - a_2^2(\Theta_1)]\}} \tag{24}$$

where $\Theta_1 > 0$ and $J_l(\Theta_1), l = 0,1,2,$ is given in (18).

5 Predictive Density for a Future Observation

Assuming α and β Known, and a noninformative prior density for λ proportional to $\lambda^{-1}, \lambda > 0$, the posterior density for λ is given by

$$\Pi(\lambda | DATA) = \frac{A^{n/2}(\alpha,\beta)}{2^{n/2}\Gamma(\frac{n}{2})}\lambda^{\frac{n}{2}-1} exp\{-\frac{\lambda}{2}A(\alpha,\beta)\} \tag{25}$$

$\lambda > 0$.

The predictive density for a future observation $Y_{(n+1)i}$, where $n = \Sigma_{i=1}^{K} n_i$ is the number of observations in test considering a stress level X_i, is given by (see for example, Press, 1989):

$$f^i(Y_{(n+1)i} | DATA) = \int_0^\infty f^i(Y_{(n+1)i}|\lambda)\Pi(\lambda|DATA)d\lambda \tag{26}$$

where $f^i Y_{(n+1)i}|\lambda) =$

$$= \frac{\lambda^{1/2}}{(2\Pi Y_{(n+1)i}^3)^{1/2}} exp\{-\frac{\lambda}{2Y_{(n+1)i}}[(\alpha + \beta X_i)Y_{(n+1)i} - 1]^2\} \quad ,$$

and $\Pi(\lambda | DATA)$ is the posterior density for λ given in (25).

That is, $f^i(Y_{(n+1)}i | DATA) =$

$$= \frac{A^{\frac{n}{2}}(\alpha,\beta)\Gamma(\frac{n+1}{2})}{\sqrt{\Pi}\Gamma(\frac{n}{2})Y_{(n+1)i}^{3/2}\{A(\alpha,\beta) - 2(\alpha+\beta X_i) + (\alpha + \beta X_i)^2 Y_{(n+1)i} - Y_{(n+1)i}^{-1}\}^{\frac{(n+1)}{2}}} \tag{27}$$

where $Y_{(n+1)i} > 0$.

If we consider α, β and λ unknown, we also can use the Laplace's method to find an approximate predictive density for a future obsevation $Y_{(n+1)i}$

6 Use of the Predictive Density for a Future Observation in Quality Control

We can use the predictive density (27) to formulate a quality control procedure in life testing. Usually, quality engineers select random samples of each batch of manufactured components to verify if the process line is under control. To minimize the cost and time of test, they consider units in life tests with a high stress level X_i and a fixed period of time L_i.

Using the predictive density (27) assuming α and β Known, and considering a fixed probability $1-\gamma$, we can find the required values of X_i and L_i to have, $\mathbb{P}\{\mathrm{Y_{(n+1)i} > L_i|DATA}\} = 1-\gamma$, that is,

$$\int_{L_i}^{\infty} \frac{dY_{(n+1)i}}{Y^{3/2}_{(n+1)i}\{c_i + d_i Y_{(n+1)i} + Y^{-1}_{(n+1)i}\}^{\frac{n+1}{2}}} = \frac{(1-\gamma)\sqrt{\Pi}\Gamma(\frac{n}{2})}{A^{n/2}(\alpha,\beta)\Gamma(\frac{n+1}{2})}) \qquad (28)$$

where $c_i = A(\alpha,\beta) - 2(\alpha+\beta X_i)$ and $d_i = (\alpha+\beta X_i)^2$.

From (28), we can find the required values of X_i, and L_i to be used in quality control tests and considering the following procedure: put m new units in test with the stress level X_i and during the period of time L_i. Let N be the number of failures, and assume that N has a binomial distribution $b(m,p^i)$, where $p^i = \mathbb{P}\{\mathrm{Y_{(n+1)i} \leq L_i|DATA}\}$. The production line is under control if we do not reject the hypothesis $H_0 : p^i \leq \gamma$.

7 An Example

In table 1, we have the data on the failures times of an insulation material in a motorette test performed at three elevated temperature settings ranging from 190° C to 240 ° C (data given in Nelson, 1971).

Assuming the reciprocal-linear regression model (2), $\Theta^{-1} = \alpha + \beta X$ where $X = 10^{-8}[T^3 - 180^3]$ and an inverse Gaussian distribution for Y = FAILURE HOURS/1000, the maximum likelihood estimators for $\alpha, \beta, \lambda, \lambda^{-1}$ and Θ_1, and their approximate standard errors (SE) obtained by the limiting distribution of the MLE'S are given in table 2.

The approximate marginal posterior density for α (see (14)) is given by

$$\Pi(\alpha|DATA) \propto \frac{\{K_0(\alpha)K_2(\alpha) - K_1^2(\alpha)\}^{1/2}}{\{0.01165\alpha^2 - 0.00087\alpha + 0.00007\}^{15.5}} \qquad (29)$$

i	TEMPERATURE (T)	FAILURE TIMES (IN HOURS)
1	$T_1 = 190\ °C$	7228,7228,7228,8448,9167 9167,9167,9167,10511,10511.
2	$T_2 = 220\ °C$	1764,2436,2436,2436,2436 2436,3108,3108,3108,3108.
3	$T_3 = 240\ °C$	1175,1175,1521,1569,1617, 1665,1665,1713,1761,1953

Table 1: Motorette Failure Times

	α	β	λ	λ^{-1}	Θ_1^{-1}
MLE	0.0371	7.3248	100.6721	0.0099	0.1124
SE USING THE OBSERVED INFORMATION	0.0127	0.3510	25.9934	0.0026	0.0104
SE USING THE FISHER INFORMATION	0.0127	0.3495	25.9934	0.0026	0.0103

Table 2: Maximum Likelihood Estimators

where $\alpha > 0, K_l(\alpha) = \frac{1}{30}\Sigma_{i=1}^{3}\Sigma_{j=1}^{10}\frac{0.00571X_i^l}{\{0.0057\alpha+(0.04612-0.11453\alpha)X_i\}}$ $l = 0,1,2$ and $X_1 = 0.0103, X_2 = 0.0482$ and $X_3 = 0.0799$. The mode of the posterior density (29) is given by $\hat{\alpha}^* \cong 0.037$ (see figure 1).

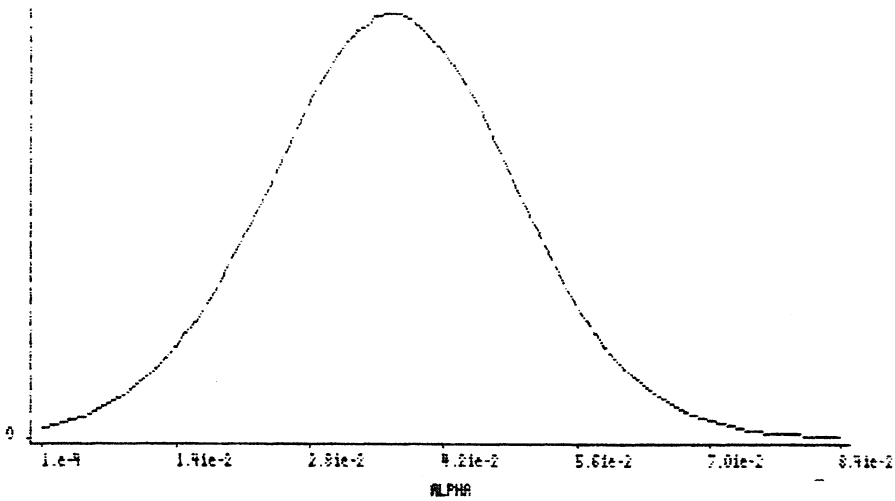

Figure 1. Marginal Posterior Density for α

The approximate marginal posterior density for β (see (15)) is given by,

$$\Pi(\beta|DATA) \propto \frac{\{L_0(\beta)L_2(\beta) - L_1^2(\beta)\}^{1/2}}{\{0.01165\beta^2 - 0.17065\beta + 0.66801\}^{15.5}} \tag{30}$$

where $\beta > 0, L_l(\beta) = \frac{1}{30}\Sigma_{i=1}^{30}\ \Sigma_{j=1}^{10}\ \frac{4.33373X_i^l}{\{1-0.11453\beta+4.33373\beta X_i\}}$,
$l = 0,1,2$, and $X_1 = 0.0103, X_2 = 0.0482$ and $X_3 = 0.0799$. The mode of (30) is given by

$\hat{\beta}^* \cong 7.35$ (see figure 2).

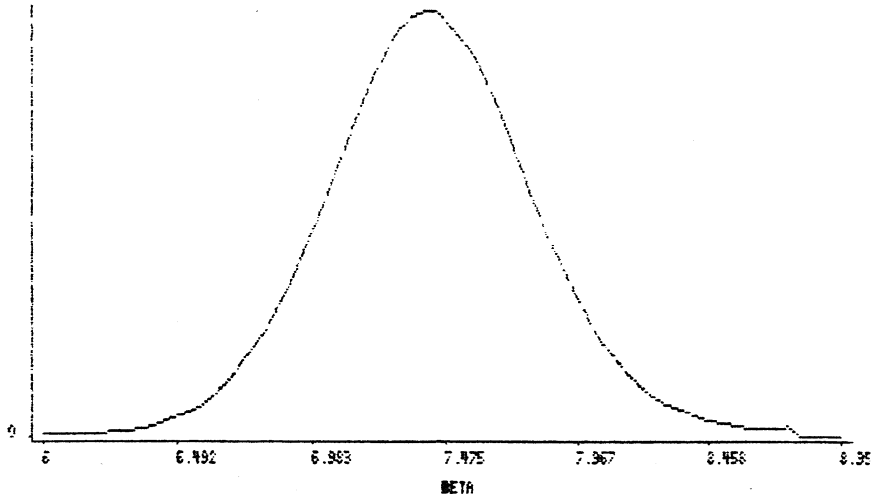

Figure 2. Marginal Posterior Density for β.

The approximate posterior density for λ (see (16)) is given by

$$\Pi(\lambda|DATA) \;\propto\; \lambda^{14} exp\{-0.14854\lambda\} \tag{31}$$

where $\lambda > 0$. The mode of (31) is $\hat{\lambda}^{*} \cong 94$ (see figure 3).

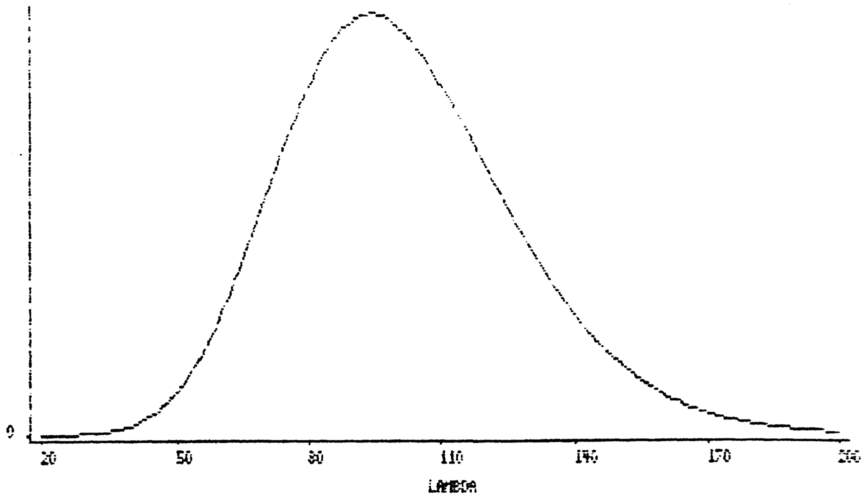

Figure 3. Marginal Posterior Density for λ.

The approximate marginal posterior density for the mean failure time Θ_1 under the usual stress level X_1 is (from (18)) given by,

$$\Pi(\Theta_1|DATA) \;\propto\; \frac{\{J_0(\Theta_1)J_2(\Theta_1) - J_1^2(\Theta_1)\}^{1/2}}{\Theta_1^2\{4315.59a_3(\Theta_1) - a_2^2(\Theta_1)\}^{15.5}} \tag{32}$$

where $\Theta_1 > 0$, $J_l(\Theta_1)$ is given in (18) for $l = 0, 1, 2, a_2(\Theta_1) = 208.64 - 2564.97/\Theta_1$, $a_3(\Theta_1) = 11.55 - 268.64/\Theta_1 + 1616.08/\Theta_1^2$. The mode of the posterior density (32) is given by $\hat{\Theta}_1^{*} \cong 8.75$

(see figure (4)).

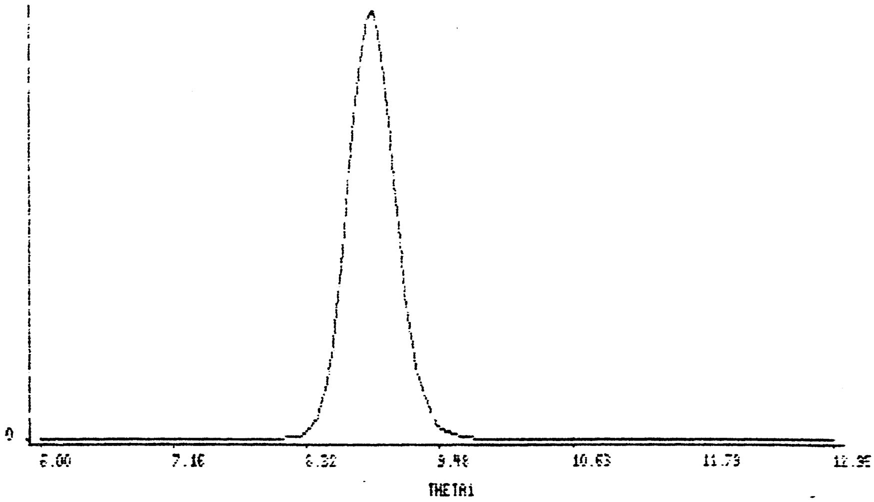

Figure 4. Marginal Posterior Density for Θ_1.

Assuming $\lambda = 94$ known, we obtain (from (21) e (22)) similar marginal posterior densities for α and β (see figures 5 and 6).

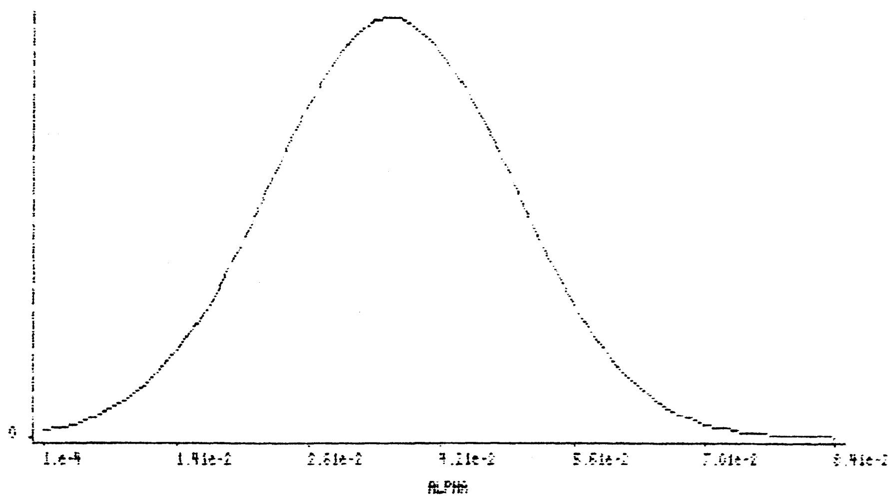

Figure 5. Marginal Posterior Density for α Assuming $\lambda = 94$ Known.

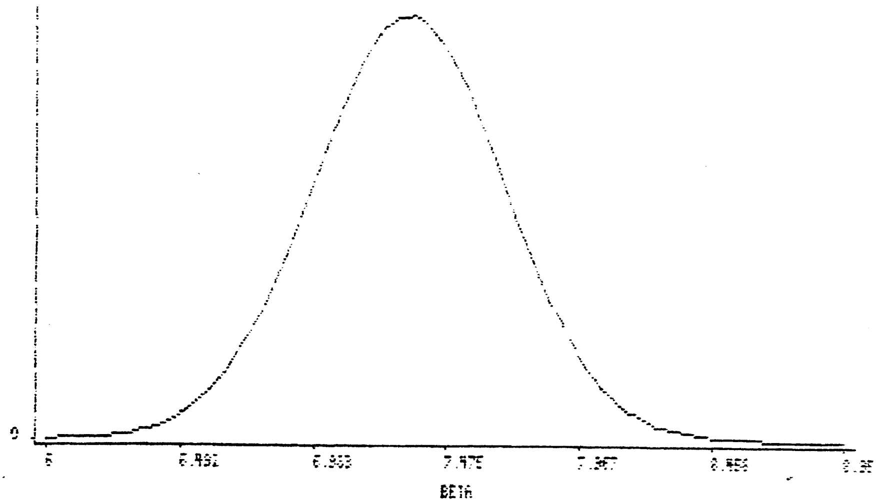

Figure 6. Marginal Posterior Density for β assuming $\lambda = 94$ Known.

In table 3, we have 95% approximate confidence intervals and credible intervals for the parameters α, β, λ and Θ_1. In figure 7, we have the graphs of the predictives densities for the future observation $Y_{(31)i} = FAILURE\ HOURS/1000$ considering each stress level $X_i, i = 1, 2, 3$.

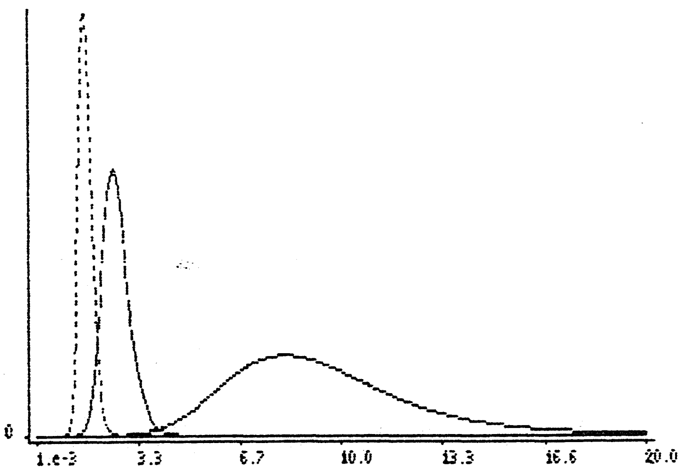

Figure 7. Predictive Densities for $Y_{(31)i}$.

	α	β	λ	Θ_1
Approximate 95% Confidence Intervals Using the Observed Information	(0.0122 ; 0.0621)	(6.6368 ; 8.0127)	(49.7260 ; 151.6183)	(7.5290 ; 10.8720)
Approximate 95% Confidence Intervals Using the Fisher Information	(0.0123;0.0620)	(6.6397;8.0098)	(49.7259;151.6183)	(7.5358;10.8578)
95% Credible Intervals	(0.0125;0.0617)	(6.6729;8.0271)	(44.7607 ; 143.2393)	(7.2234 ; 10.2766)

Table 3: Confidence and Credible Intervals for the Parameters

In table 3, we observe similar results for the asymptotic classical confidence intervals and the approximate Bayesian method based on a noinformative prior density for the parameters. We also could use informative priors for the parameters and the Laplace's method to obtain approximate marginal posteriors of interest.

8 Overall Conclusions

We obtained very simple expressions for the marginal posterior densities of interest considering Jeffreys noinformative priors and the Laplace's method for approximation of integrals. We also obtained predictive densities to be used in quality control problems. These results could be of great practical interest in the industrial life testing applications considering the inverse Gaussian distribution for the life times.

APPENDIX
THE LAPLACE'S METHOD

Assuming h is a smooth function of an m-dimensional parameter Θ with $-h$ having a maximum at $\hat{\Theta}$, the Laplace's method approximates an integral of the form

$$I = \int f(\Theta) exp\{-nh(\Theta)\} d\Theta \qquad (A.1)$$

by expanding h and f in a Taylor series about $\hat{\Theta}$ (see for example, Kass, Tierney and Kadane, 1990).

Considering first the case in which Θ is one-dimensional, the Laplace's method gives the approximation,

$$\hat{I} \cong \left(\frac{2\Pi}{n}\right)^{1/2} \sigma f(\hat{\Theta}) exp\{-nh(\hat{\Theta})\} \qquad (A.2)$$

where $\sigma = \{h''(\hat{\Theta})\}^{-1/2}$.

In the multiparameter case, with $\Theta \in R^m$, we have,

$$\hat{I} \cong (2\Pi)^{m/2}\{det(nD^2h(\hat{\Theta}))\}^{-1/2}f(\hat{\Theta})exp\{-nh(\hat{\Theta})\} \qquad (A.3)$$

where $\hat{\Theta}$ maximizes $-h(\Theta)$ and $D^2h(\Theta)$ is the Hessian matrix of h evaluated at $\hat{\Theta}$.

The accuracy of these approximations are studied by Kass, Tierney and Kadane (1990). A special case of the Laplace's approximation is given for integrals of the form $\int e^{-nh(\Theta)}d\Theta$ (see Tierney and Kadane, 1986; Tierney, Kass and Kadane, 1989).

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