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Bayes predective likelihood function for the accelerated life tests via the orthogonal parameters

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1 Summary

Accelerated life testing of an item under more severe than normal conditions is commonly used by industrial statisticians to reduce test time and costs. In this paper, the Bayes predictive function via the orthogonal parameters for a future random time under the usual stress level is considered. It is assumed a power rule model and an exponential distribution with the type ll censoring mechanism.

Key words: accelerated life tests, power rule model, orthogonal parameters, predictive densities.

2 Introduction

In engineering applications accelerated conditions are produced by testing items at higher than normal temperature, voltage, pressure, load, etc (see Mann, Shafer and Singpurwalla (1974)). In this paper, we discuss from the Bayes point of view the construction of the predictive density function for a future random time under the exponential censored data. This predictive density is obtained via the orthogonal parameters, that is, the Fisher information matrix is diagonal (see Cox and Reid, 1987)). The main advantage of this orthogonality is to obtain a known predictive density and a closed form for the predictor of a future random time. We assume an exponential model with a power rule model and a type ll censored data.

3 Formulation of the model

We will consider the following model: Suppose that we perform life tests on a set of n_i devices under the stress V_i and the life time T has an exponential density

$$
\begin{aligned}
f(t;\theta_i) &= \frac{1}{\theta_i} exp\{-\frac{t}{\theta_i}\}, \quad \theta_i > 0, \quad t > 0, \\
i &= 1, \ldots, k.
\end{aligned} \tag{1}
$$

The unknown parameter $\theta_i, i = 1, \ldots, k$, is the mean time to failure under the stress V_i. The k tests yield, as data, the set $\{V_i, n_i, \hat{\theta}_i\}, i = 1, \ldots, k$, where $\hat{\theta}_i = \frac{A_i}{n_i}$, $A_i = \sum_{j=1}^{n_i} t_{ij} + (n_i - r_i)t_{ir_i}$ and $t_{ij}, j = 1, \ldots, r_i$ are observable failures under the stress V_i for n_i items on test ($r_i < n_i$). Given the data set and the usual stress level V_1, our purpose is to get information of the mean time θ_1. To make inference on θ_1, we adopt the power rule model

$$
\begin{aligned}
\theta_i &= \frac{\alpha}{V_i^{\beta}}, \quad \alpha > 0, \quad -\infty < \beta < \infty, \\
i &= 1, \ldots, k.
\end{aligned} \tag{2}
$$

The log-likelihood function for θ_1 and β is given by

$$
\begin{aligned}
l(\theta_1, \beta) &= -rln(\theta_1) + \beta rln(\dot{V}) - \frac{1}{\theta_1}\sum_{i=1}^{k} A_i(\frac{V_i}{V_1})^{\beta}, \quad \text{where} \\
\dot{V} &= \prod_{1}^{k}(\frac{V_i}{V_1})^{\frac{r_i}{r}} \quad \text{and} \quad r = \sum_{i=1}^{k} r_i.
\end{aligned} \tag{3}
$$

First, we estimate β in the presence of the nuisance parameter θ_1 . The procedure can be simplified by transforming the origin parameters (θ_1, β) to the orthogonal parameters (λ, β). The orthogonal parameter λ is obtained by solving the differencial equation (Cox and Reid, 1987)

$$
\frac{\partial \theta_1}{\theta_1} = ln(\dot{V})\partial\beta, \quad \text{giving} \tag{4}
$$

$$
\lambda = \frac{\theta_1}{\dot{V}^{\beta}}. \tag{5}
$$

Then, the new likelihood function in terms of (λ, β) is

$$l(\lambda, \beta) = -rln(\lambda) - \frac{1}{\lambda}\sum_{i=1}^{k} A_i(\frac{V_i}{\overset{\bullet}{V} V_1})^{\beta}. \tag{6}$$

It is easy to verify that $\hat{\beta}$, the estimator of β given λ, is the solution of

$$\frac{\partial l(\lambda, \beta)}{\partial \beta} = \sum_{i=1}^{k}(\frac{V_i}{\overset{\bullet}{V} V_1})^{\beta} ln(\frac{V_i}{\overset{\bullet}{V} V_1})^{\beta} = 0. \tag{7}$$

Note that $\hat{\beta}$ is obtained independently of λ, that is, $\hat{\beta}$ is stable with respect to λ. The Jeffrey's prior for λ and β (Box and Tiao, 1973) is given by

$$\pi(\lambda, \beta) \propto \frac{1}{\lambda}. \tag{8}$$

Using Laplace approximation (Tierney and Kadane, 1986), the marginal posterior for λ is given by

$$\pi(\lambda \mid \text{data}) \propto \lambda^{-(r+\frac{1}{2})} exp\{-\frac{1}{\lambda}\sum_{i=1}^{k} A_i(\frac{V_i}{\overset{\bullet}{V} V_1})^{\hat{\beta}}\} \tag{9}$$

Taking $\beta = \hat{\beta}$, we have from (5) and (9) the marginal posterior of θ_1 given by

$$\pi(\theta_1 \mid \text{data}) \propto \theta_1^{-(r+\frac{1}{2})} exp\{\frac{1}{\theta_1}\sum_{i=1}^{k} A_i(\frac{V_i}{V_1})^{\hat{\beta}}\} \tag{10}$$

4 Predictive density function for a future random time under the usual stress level V_1

The Bayes predictive density for a future random time $T_{n+1,1}$ w.r.t. (10),under the usual stress V_1 and $n = \sum_{i=1}^{k} n_i$, is giving by (Zellner, 1975)

$$p(t_{n+1,1} \mid \text{data}) = \frac{(r-\frac{1}{2})V_1^{\hat{\beta}}\{\sum_{i=1}^{k} A_i V_i^{\hat{\beta}}\}^{r-\frac{1}{2}}}{\{\sum_{i=1}^{k} A_i V_i^{\hat{\beta}} + V_1^{\hat{\beta}} t_{n+1,1}\}^{r+\frac{1}{2}}} \quad \text{(Pareto density)} \tag{11}$$
$$t_{n+1,1} > 0 \ .$$

The mean and the variance of the predictive density of $T_{n+1,1}$ are

$$E[T_{n+1,1} \mid \text{data}] = \frac{\sum_{i=1}^{k} r_i \hat{\theta}_i \{\frac{V_i}{V_1}\}^{\hat{\beta}}}{r - \frac{3}{2}} \tag{12}$$

$$V[T_{n+1,1} \mid \text{data}] = \frac{r - \frac{1}{2}}{r - \frac{5}{2}} \{ \frac{\sum_{i=1}^{k} r_i \hat{\theta}_i (\frac{V_i}{V_1})^{\hat{\beta}}}{r - \frac{3}{2}} \}^2. \tag{13}$$

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