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USE OF PREDICTIVE DENSITIES IN QUALITY CONTROL WITH ACCELERATED LIFE TESTS ASSUMING A POWER RULE MODEL AND AN EXPONENTIAL DISTRIBUTION

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SUMMARY

"In this paper, we consider accelerated life tests with an exponencial distribution and a power rule model. Assuming type II censored data and Jeffreys priors for the parameters of the model, we find the predictive density for a future observation in a specified stress level. Using the predictive density for a future observation, we present some criteria to be used by quality engineers to choose the required time on test or the stress level needed in a quality control test".

Key words: accelerated life tests, power rule model,
exponential distribution, predictive densities.

1. INTRODUCTION

Usually, industries of components such as semiconductors use accelerated life tests to obtain measures of the reliability of the devices and to determine quality control tests. The quality engineers consider random samples of components where all unities are submitted to a test with a high stress level V during a fixed period of time L. They verify if the production line is under control if the proportion of failed unities in the sample is less than a specified quantity. Often, the quality engineers need to find which stress level and which fixed period of time should be considered for the accelarated life test.

In this paper, we assume an exponential distribution for the life times of the components and a power rule model $\theta_i = \alpha/V_i^{\beta}$ where α and β are unknown parameters, $i = 1,2,..,K$ (number of stress levels) and θ_i is the mean life time of a component under the stress level V_i (see for example, Mann, Schaffer and Singpurwalla, 1974). We also assume an experiment with censored data under a type II censoring mechanism (see for example, Lawless, 1982) and noninformative Jeffreys priors for the parameters (see for example, Box and Tiao, 1973). We find the predictive distribution of a future observation in a stress level V_i and with this density, we construct designs of accelerated life tests to be used in quality control.

2. THE STATISTICAL MODEL

Let $T \geq 0$ be a random variable representing the life time of a component with exponential density,

$$f(t, \lambda_i) = \lambda_i \exp\{-\lambda_i t\} \qquad (1)$$

where $t \geq 0$ and $\lambda_i > 0$. The parameter λ_i denotes the constant hazard rate under a stress level V_i, $i = 1,2,\ldots,k$. The mean time to failure under the stress V_i is given by $\theta_i = 1/\lambda_i$.

We also assume a power rule model given by,

$$\theta_i = \frac{\alpha}{V_i^{\beta}} \qquad (2)$$

where $\alpha > 0$, and $-\infty < \beta < \infty$ are unknown parameters (see for example, Mann, Schaffer and Singpurwalla, 1974).

Let us consider a type II censoring mechanism, that is, the experiment terminates when we observe r_i failures for each stress level i, $i = 1,2,\ldots,k$. Thus, with n_i unities at the beginning of each test with stress V_i, we have the ordered noncensored observations given by $t_{1i}, t_{2i},\ldots,t_{r_i i}$ and $n_i - r_i$ censored observations equal to $t_{r_i i}$, $i = 1,2,\ldots,k$.

The likelihood function for α and β considering the data under the stress level V_i is given by,

$$L(\alpha,\beta) = \prod_{j=1}^{r_i} f(t_{ji};\ \lambda_i) S^{n_i - r_i}(t_{r_i i};\lambda_i) \qquad (3)$$

where $S(t_{r_i i};\lambda_i) = \mathbb{P}\{T \geq t_{r_i i}\} = \exp\{-\lambda_i t_{r_i i}\}$ is the reliability function.

That is,

$$L(\alpha,\beta) = \frac{V_i^{\beta r_i}}{\alpha^{r_i}} \exp\{-\frac{A_i V_i^{\beta}}{\alpha}\} \tag{4}$$

where $A_i = \sum_{j=1}^{r_i} t_{ji} + (n_i - r_i) t_{r_i i}$.

Considering the data of k stress levels $V_1, V_2, \ldots, V_k$ taken at random, the logarithm of the likelihood for α and β is given by,

$$\ell(\alpha,\beta) = \beta \sum_{j=1}^{k} r_j \ln V_j - r \ln \alpha - \frac{1}{\alpha} \sum_{j=1}^{k} A_j V_j^{\beta} \tag{5}$$

where $r = \sum_{j=1}^{k} r_j$ (total number of observed failures).

The maximum likelihood estimators for α and β are given by,

$$\hat{\alpha} = \frac{1}{r} \sum_{j=1}^{k} A_j V_j^{\hat{\beta}}$$

$$\sum_{j=1}^{k} r_j \ln V_j = \frac{r \sum_{j=1}^{k} A_j V_j^{\hat{\beta}} \ln V_j}{\sum_{j=1}^{k} A_j V_j^{\hat{\beta}}} \tag{6}$$

For inferences about α and β, or even functions of the parameters, researchers usually use the asymptotical normality of the maximum likelihood estimators, given by,

$$(\hat{\alpha}, \hat{\beta}) \overset{a}{\sim} N\{(\alpha,\beta);\ I^{-1}(\hat{\alpha}, \hat{\beta})\} \qquad (7)$$

where $I(\alpha,\beta)$ is the Fisher information matrix for α and β (see for example, Lawless, 1982).

3. PREDICTIVE DENSITY FOR A FUTURE OBSERVATION IN A STRESS LEVEL V_i ASSUMING β KNOWN

Assuming β known, the likelihood function for α is given by,

$$L(\alpha) \propto \prod_{j=1}^{k} \frac{V_j^{\beta r_j}}{\alpha^{r_j}} \exp\{-\frac{A_j V_j^{\beta}}{\alpha}\} . \qquad (8)$$

Considering a Jeffreys prior density $\pi(\alpha) \propto \alpha^{-1}$, $\alpha > 0$ (see for example, Box and Tiao, 1973), the posterior density for α is given by,

$$\pi(\alpha|\text{data}) = \frac{\{\sum_{j=1}^{k} A_j V_j^{\beta}\}}{\Gamma(r)} \alpha^{-(r+1)} .$$

$$\exp\{-\frac{1}{\alpha} \sum_{j=1}^{k} A_j V_j^{\beta}\} \qquad (9)$$

where $\alpha > 0$ and $r = \sum_{j=1}^{k} r_j$ is the fixed number of failures.

The predictive density for a future observation $T_{(n+1)i}$, where $n = \sum_{j=1}^{k} n_j$ is the number of observations in test, considering a stress level V_i is given by (see for example Aitchison and Dunsmore, 1975)

$$f^i(t_{(n+1)i}|\text{data}) = \int_0^\infty f^i(t_{(n+1)i}|\alpha)\pi(\alpha|\text{data})d\alpha$$

$$= \underset{\alpha|\text{data}}{E}\{f^i(t_{(n+1)i}|\alpha)\} \tag{10}$$

where $f^i(t_{(n+1)i}|\alpha) = \frac{V_i^\beta}{\alpha} \exp\{-\frac{V_i^\beta}{\alpha} t_{(n+1)i}\}$ and $\pi(\alpha|\text{data})$ is the posterior density for α given in (9).

That is,

$$f^i(t_{(n+1)i}|\text{data}) = \frac{\Gamma(r+1)\ V_i^\beta\{\sum_{j=1}^{k} A_j V_j^\beta\}^r}{\Gamma(r)\{V_i^\beta\ t_{(n+1)i} + \sum_{j=1}^{k} A_j V_j^\beta\}^{r+1}}$$

Since $\Gamma(r+1) = r\Gamma(r)$, we have,

$$f^i(t_{(n+1)i}|\text{data}) = \frac{r\ V_i^\beta\{\sum_{j=1}^{k} A_j V_j^\beta\}^r}{\{V_i^\beta t_{(n+1)i} + \sum_{j=1}^{k} A_j V_j^\beta\}^{r+1}}$$

where $t_{(n+1)i} > 0$ (a Pareto density).

The mean and the variance of the predictive density (11) are given by,

$$E^{i}(T_{(n+1)i}|\text{data}) = \frac{\sum_{j=1}^{k} A_j V_j^{\beta}}{(r-1)\ V_i^{\beta}}$$

(12)

$$\text{var}^{i}(T_{(n+1)i}|\text{data}) = \frac{r(\sum_{j=1}^{k} A_j V_j^{\beta})^2}{(r-1)^2 (r-2) V_i^{2\beta}}$$

where $i = 1,2,\ldots,k$ (see for example Johnson and Kotz, 1970).

4. PREDICTIVE DENSITY FOR A FUTURE OBSERVATION IN A STRESS LEVEL V_i ASSUMING α AND β UNKNOWN

The Jeffreys prior density for α and β (see for example, Box and Tiao, 1973) is given by,

$$\pi(\alpha,\beta) \propto \{\det I(\alpha,\beta)\}^{1/2}$$

$$\propto \frac{1}{\alpha} \{r \sum_{i=1}^{k} A_i (\ell n V_i)^2 - (\sum_{i=1}^{k} r_i \ell n\ V_i)^2\}^{1/2} \quad (13)$$

That is,

$$\pi(\alpha,\beta) \propto \frac{1}{\alpha} \quad (14)$$

where $\alpha > 0$ and $-\infty < \beta < \infty$.

The joint posterior density for α and β is given by,

$$\pi(\alpha,\beta|\text{data}) \propto \alpha^{-(r+1)} \left(\prod_{j=1}^{k} V_j^{\beta r_j}\right) \exp\left\{-\alpha^{-1} \sum_{j=1}^{k} A_j V_j^{\beta}\right\} \tag{15}$$

The predictive density for $T_{(n+1)i}$ in a stress level V_i is given by,

$$f^i(t_{(n+1)i}|\text{ data}) \propto \int_0^{\infty}\int_{-\infty}^{\infty} V_i^{\beta}\left(\prod_{j=1}^{k} V_j^{\beta r_j}\right) \alpha^{-(r+2)} \,.$$

$$.\ \exp\left\{-\frac{1}{\alpha}\left[V_i^{\beta} t_{(n+1)i} + \sum_{j=1}^{k} A_j V_j^{\beta}\right]\right\} d\alpha d\beta$$

That is,

$$f^i(t_{(n+1)i}|\text{data}) \propto \int_{-\infty}^{\infty} e^{-nh(\beta)} d\beta \tag{16}$$

where $-nh(\beta) = -\beta \ln V_i - \beta \sum_{j=1}^{k} r_j \ln V_j +$

$$+ (r+1) \ln\{V_i^{\beta} t_{(n+1)i} + \sum_{j=1}^{k} A_j V_j^{\beta}\} \,.$$

Using the Laplace method for approximation of integrals (see for example Tierney and Kadane, 1986), we have:

$$f^i(t_{(n+1)i}|\text{data}) \propto \frac{\{nh''(\hat{\beta})\}^{-1/2}\, V_i^{\hat{\beta}} \left(\prod_{j=1}^{k} V_j^{\hat{\beta} r_j}\right)}{\{V_i^{\hat{\beta}}\, t_{(n+1)i} + \sum_{j=1}^{k} A_j\, V_j^{\beta}\}^{r+1}} \tag{17}$$

where $t_{(n+1)i} > 0$, $\hat{\beta}$ maximizes $-nh(\beta)$ and

$$nh''(\beta) = (r+1)\left\{\frac{V_i^{\beta}(\ell n V_i)^2 t_{(n+1)i} + \sum_{j=1}^{k} A_j V_j^{\beta}(\ell n V_j)^2}{(V_i^{\beta} t_{(n+1)i} + \sum_{j=1}^{k} A_j V_j^{\beta})} - \frac{[V_i^{\beta}(\ell n V_i) t_{(n+1)i} + \sum_{j=1}^{k} A_j V_j^{\beta} \ell n\, V_j]^2}{(V_i^{\beta} t_{(n+1)i} + \sum_{j=1}^{k} A_j V_j^{\beta})^2}\right\}$$

5. USE OF THE PREDICTIVE DENSITY $f^i(t_{(n+1)i}|\text{data})$ IN QUALITY CONTROL

We can use the predictive density $f^i(t_{(n+1)i}|\text{data})$ to formulate a quality control procedure in life testing. Usually, the quality engineers select random samples of each batch of manufactured components to verify if the process line is under control. To minimize the cost and time of test, they consider unities in life tests with a high stress level V_i and a fixed period of time L_i.

In this quality control approach, usually the engineers have two kinds of questions:

i) Considering a fixed stress level V_i, which value of L_i should be taken?

That is, using the predictive density $f^i(t_{(n+1)i}|\text{data})$ and considering a fixed probability $1-\gamma$, which value of

L_i should be taken to have $\mathbb{P}^i(T_{(n+1)i} > L_i|\text{data}) = 1-\gamma$?

ii) Considering L_i fixed, which value of V_i should be taken to have $\mathbb{P}^i(T_{(n+1)i} > L_i|\text{data}) = 1-\gamma$?

Assuming β known, we have from (11),

$$\mathbb{P}^i(T_{(n+1)i} > L_i|\text{data}) = \int_{L_i}^{\infty} f^i(t_{(n+1)i}|\text{data})\, dt_{(n+1)i}$$

$$= r\left(\frac{C}{V_i^{\beta}}\right) \int_{L_i}^{\infty} \frac{dt_{(n+1)i}}{\left(t_{(n+1)i} + \frac{C}{V_i^{\beta}}\right)^{r+1}}$$

where $C = \sum_{j=1}^{k} A_j V_j^{\beta}$.

Considering $\mathbb{P}^i(T_{(n+1)i} > L_i|\text{data}) = 1-\gamma$, we have,

$$\frac{(C/V_i^{\beta})^r}{\left(L_i + \frac{C}{V_i^{\beta}}\right)^r} = 1 - \gamma \tag{18}$$

Thus, with β known, we have for both questions above, the following solutions:

i) Given the stress level V_i, we find L_i from (18), given by,

$$L_i = \frac{C[1 - (1-\gamma)^{1/r}]}{(1-\gamma)^{1/r}\, V_i^{\beta}} \tag{19}$$

ii) Given L_i, we find V_i from (18), given by,

$$V_i = \left\{\frac{C\left[1-(1-\gamma)^{1/r}\right]}{(1-\gamma)^{1/r}\, L_i}\right\}^{1/\beta} \tag{20}$$

Once we have the values of V_i and L_i such that $\mathbb{P}^i(T_{(n+1)i} > L_i \mid \text{data}) = 1-\gamma$, we can consider the following procedure for quality control:

a) Put m unities in test with a stress level V_i and during a period of time L_i. Define $p^i = \mathbb{P}^i(T_{(n+1)i} \leqslant L_i \mid \text{data})$ and consider the hypothesis test H_o: $p^i \leqslant \gamma$ (the production line is under control) against $H_1: p^i > \gamma$ (the production line is out of control).

b) Let X be the number of failed unities in the sample of size m, that is, the unities such that $T \leqslant L_i$. We assume $X \sim b(m; p^i)$. With m large (say $m \geqslant 30$), we have $X \overset{a}{\sim} N\{mp^i; mp^i(1-p^i)\}$.

c) A criterion for quality control can be based on the usual hypothesis test (with m large): the production line is out of control if $Z \geqslant z\alpha$, where α is a significance level,

$Z = (X - m\gamma) / \sqrt{m\gamma(1-\gamma)} \overset{a}{\sim} N\{0;1\}$ and

$z\alpha$ is such that $\mathbb{P}\{Z \geqslant z\alpha\}=\alpha$.

If β is unkown, we can use numerical methods to find V_i and L_i such that $\mathbb{P}^i\{T_{(n+1)i} > L_i | \text{data}\} = 1 - \gamma$ and using the predictive density (17) approximated by the Laplace's method.

6. AN EXAMPLE

Consider the data of table 1 generated by a power rule model (2) for 5 stress levels with $\alpha = 500$ and $\beta = 0.8$. The maximum likelihood estimators for α and β given in (6) are $\hat{\alpha} = 502.0524$ and $\hat{\beta} = 0.8003$.

i	V_i	θ_i	n_i	r_i	NONCENSORED OBERVATIONS
1	10	72.24	30	5	6, 8, 10, 12, 14
2	20	45.52	30	8	4, 5, 5, 6, 8, 8, 9, 14
3	30	32.90	30	12	2, 3, 3, 5, 6, 7, 7, 8, 8, 9, 10, 17
4	40	26.14	30	18	3, 3, 4, 5, 6, 6, 8, 9, 10, 10, 12, 12, 13, 14, 14, 14, 15, 24
5	50	21.87	30	22	2, 3, 4, 5, 5, 8, 8, 8, 9, 10, 12, 13, 14, 14, 15, 18, 18, 18, 19, 20, 20, 27

Table 1. Generated Data with $\alpha = 500$ and $\beta = 0.8$

(Life Times in Hours)

Assuming $\beta = 0.8$ known, we have from table 1, $r = 65$ and $C = \sum_{j=1}^{5} A_j V_j^{\beta} = 32598.92$.

Considering $1 - \gamma = 0.80$ and $V_i = 30$, we find L_i (from (19)) given by $L_i = 7.3777$ hours. In a quality control test, we put m unities in the stress level $V_i = 30$ during the period of time $L_i = 7.38$ and we observe the number X of failed unities. The production line is out of control if we reject H_o: $p^i \leq 0.20$ where $p^i = P^i\{T_{(n+1)i} \leq L_i | \text{data}\}$ in a specified significance level α. That is, the production line is out of control if $X \geq 0.20m + 0.40 z_{\alpha} \sqrt{m}$.

Another possibility is to fix a period of time $L_i = 20$ hours, $1 - \gamma = 0.80$ and find a stress level V_i (from (20)) given by $V_i = 8.6246$.

Observe that we have a criterion for quality control very flexible in terms of good choices of V_i and L_i to minimize time and cost of the tests. In table 2, we have some special cases in the determination of V_i and L_i considering $1 - \gamma = 0.80$, $r = 65$, $\beta = 0.8$ and $C = 32598.92$.

V_i FIXED		L_i FIXED	
V_i	REQUIRED L_i (19)	L_i	REQUIRED V_i (20)
5	30.93455	2	153.369
10	17.76723	4	64.4837
15	12.84538	6	38.8450
20	10.20459	8	27.1120
25	8.53626	10	20.5128
30	7.37773	12	16.3323
35	6.52177	14	13.4700
40	5.86100	16	11.3992
45	5.33396	18	9.8386
50	4.90279	20	8.6246

Table 2. Required Numbers of V_i or L_i Assuming $1-\gamma = 0.80$

7. OVERALL CONCLUSIONS

The results of this paper could be of great practical interest. In some industrial situations, we can have a high cost associated to each failed unity. As a special case, consider an industry of computers that needs a quality control test with a small number of failed unities during a short period of time. With the proposed method of this paper, the quality engineers could decide for the best strategy in each practical case, since they could choose $1-\gamma$, V_i and L_i more appropriate for their problems.

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