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A BAYESIAN ANALYSIS OF THE ACCELERATED LIFE TESTS VIA THE ORTHOGONAL PARAMETERS

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SUMMARY

Accelerated life testing of an item under more severe than normal conditions is commonly used by industrial statisticians to reduce test time and costs. We consider, from the Bayesian point of view, the problem of accelerated life tests when the parameter of interest is estimated via the orthogonal parameters (Cox and Reid, 1987) and a type II censoring mechanism for exponential survival data. One important advantage of this orthogonality is to obtain a closed form for the posterior mode under the Jeffreys prior for the parameters involved in the model.

The method is illustrated by an example.

Key Words: accelerated life tests, power rule model, Bayesian analysis, orthogonality.

1. INTRODUCTION

Accelerated life testing of an item is often used to obtain information on its performance under normal use conditions. In engineering applications accelerated conditions are produced by testing items at higher than normal temperature, voltage, pressure, load, etc. Mann, Shafer and Singpurwalla (1974), derive least-squares and maximum likelihood estimators of model parameters when the underlying failure distribution is exponencial. Nelson (1972,c) describes a graphical solution and Achcar and Dias (1990) a Bayesian solution to this problem. The object of this paper is to consider, from the Bayesian point of view, the case when the parameter of interest is estimated via the orthogonal parameters, that is, the Fisher information matrix is diagonal (see for example, Cox and Reid, 1987). One important advantage of this orthogonality, as emphasized by Cox and Reid, 1987, is that the estimator of the nuisance parameter does not depend on the parameter of interest. This stability of the estimator of the nuisance parameter allows to obtain a closed form for the mode of the posterior distribution. An example to illustrate the performance of the procedure is considered.

2. FORMULATION OF THE MODEL

The problem considered in the sequel is as follows. Suppose that we decide to perform life test at k accelerated values of stress V_j, $j=1,...,k$ on a device, which under stress V_i, the life time T has an exponencial density

$$f(t;\ \theta_i) = \frac{1}{\theta_i} e^{-\frac{t}{\theta_i}}, \quad \theta_i > 0,\ t > 0. \tag{1}$$

The unknown parameter θ_i, $i=1,\dots,k$, is the mean time to failure under stress V_i.

A value V_i, $i=1,\dots,k$ is chosen, at random, from k values V_j, $j=1,\dots,k$, and n_i devices are put on a life test under stress V_i. The test terminates after a fixed number r_i of failures $t_{i1},\dots,t_{ir_i}$ have occurred, that is, we are using a type II censoring mechanism. This procedure is repeated for all k values V_j, each time choosing a V_i at random from the remaining value of V_j. Such a procedure ensures independence of the k life tests. Let θ_1 be the mean time of failure under the usual stress V_i. In this paper we adopt the power rule model considered by Mann at al. (1974),

$$\theta_i = \frac{\alpha}{V_i^{\beta}}, \quad \alpha > 0,\ -\infty < \beta < \infty \tag{2}$$

for all values of V_i. Our purpose is to get information of θ_1. To get this, first, we transform our original parameters (θ_1, β) to the orthogonal parameters (λ, β) (Cox and Reid, 1987) in order to estimate β independently of λ.

Let $\dot{V} = \prod_{i=1}^{k} \left(\frac{V_i}{V_1}\right)^{\frac{r_i}{r}}$ the wrighted geometric mean of the

$\frac{V_i}{V_1}$'s and $\hat{\theta}_i = \frac{A_i}{r_i}$, $i=1,\ldots,k$, where $A_i = \sum_{j=1}^{r_i} t_{ij} + (n_i - r_i)t_{ir_i}$, $r = \sum_{i=1}^{k} r_i$.

The k life tests yield, as data, the set $\{V_i, n_i, \hat{\theta}_i\}$, $i=1,\ldots,k$. Since randomization of the V_i ensures independence of the $\hat{\theta}_i$, $i=1,\ldots,k$, we have from (1) and (2) the log-likelihood function for θ_1 and β given by

$$\ell(\theta_1, \beta) = -r\ell n(\theta_1) + \beta r \ell n(\dot{V}) - \frac{1}{\theta_1} \Sigma A_i \left(\frac{V_i}{V_1}\right)^{\beta}. \qquad (3)$$

It is easy to verify that the Fisher information matrix for (θ_1, β) is given by,

$$I(\theta_1, \beta) = \begin{bmatrix} \frac{r}{\theta_1^2} & -\frac{r}{\theta_1} \ell n(\dot{V}) \\ -\frac{r}{\theta_1} \ell n(\dot{V}) & \sum_{i=1}^{k} r_i \ell n^2 \left(\frac{V_i}{V_1}\right) \end{bmatrix} \qquad (4)$$

To obtain an estimator of β which is stable with respect to θ_1, we solve the differencial equation

$$\frac{d\theta_1}{\theta_1} = \ell n(\dot{V}) d\beta, \qquad (5)$$

giving a one-to-one transformation

$$\theta_1 = \dot{V}^{\beta} \lambda, \qquad (6)$$

or,

$$\lambda = \frac{\alpha}{(V_1\dot{V})^{\beta}}. \tag{7}$$

Then, the new likelihood function in terms of (λ,β) is

$$\ell(\lambda,\beta) = -r\ell n(\lambda) - \frac{1}{\lambda}\sum_{i=1}^{k} A_i\left(\frac{V_i}{V_1\dot{V}}\right)^{\beta}. \tag{8}$$

It is easy to verify that the Fisher information matrix for (λ,β) is

$$I(\lambda,\beta) = \begin{bmatrix} \frac{r}{\lambda^2} & 0 \\ 0 & \sum_{i=1}^{k} r_i \ell n^2\left(\frac{V_i}{V_1\dot{V}}\right) \end{bmatrix}, \tag{9}$$

and $\hat{\beta}$ is the solution of

$$\sum_{i=1}^{k} A_i\left(\frac{V_i}{V_1\dot{V}}\right)^{\beta} \ell n\left(\frac{V_i}{V_1\dot{V}}\right) = 0. \tag{10}$$

As emphasized by Cox and Reid (1987), the estimator $\hat{\beta}$ is stable with respect to λ. The Jeffreys prior density for λ and β (see for example, Box and Tiao, 1973) is given by

$$\pi(\lambda,\beta) \propto \{\text{deter } I(\lambda,\beta)\}^{1/2} =$$

$$= \frac{1}{\lambda}\left[r\sum_{i=1}^{k} r_i \ell n^2\left(\frac{V_i}{V_1\dot{V}}\right)\right]^{1/2}. \tag{11}$$

That is,

$$\pi(\lambda,\beta) \propto \frac{1}{\lambda}.$$

Note that the Jeffreys prior is invariant under the transformation $(\theta_1,\beta) \to (\lambda,\beta)$.

For a Bayesian, the inferences are typically based on the marginal posterior

$$\pi(\lambda/\text{data}) \propto \frac{1}{\lambda} \int e^{\ell(\lambda,\beta)} d\beta. \tag{12}$$

For fixed λ, expanding $\ell(\lambda,\beta)$ in a second order Taylor series about the maximizing value $\hat{\beta}$ of $\ell(\lambda,\beta)$, we have the Laplace approximation

$$\pi(\lambda/\text{data}) \propto \frac{1}{\lambda} e^{\ell_p(\lambda)} \left|J(\lambda,\hat{\beta})\right|^{-\frac{1}{2}}, \tag{13}$$

where

$$\ell_p(\lambda) = \ell(\lambda,\hat{\beta}) \text{ and } J(\lambda,\beta) = -\frac{\partial^2 \ell(\lambda,\beta)}{\partial^2 \beta} =$$

$$= \frac{1}{\lambda} \sum_{i=1}^{k} A_i \ell n^2 \left(\frac{V_i}{V_1 \bar{V}}\right) \left(\frac{V_i}{V_1 \bar{V}}\right)^{\beta}.$$

For a general discussion of the Laplace method in Bayesian analysis and its connection with the conditional profile likelihood, see Tierney and Kadane, (1986) and Cox and Reid, (1987). So, the marginal posterior distribution for λ has an inverted gamma density given by

$$\pi(\lambda/\text{data}) \propto \lambda^{-(r+\frac{1}{2})} . e^{-\frac{1}{\lambda} \sum_{i=1}^{k} A_i (\frac{V_i}{V_1 \bar{V}})^{\hat{\beta}}} . \tag{14}$$

Taking $\beta = \hat{\beta}$, we have from (6) and (14) the inverted gamma posterior density,

$$\pi(\theta_1/\text{data}) \propto \theta_1^{-(r+\frac{1}{2})} e^{-\frac{1}{\theta_1} \sum_{i=1}^{k} A_i (\frac{V_i}{V_1})^{\hat{\beta}}} \tag{15}$$

and an intuitively and nice closed form for the posterior mode of (15) given by,

$$\hat{\hat{\theta}}_1 = \frac{\sum_{i=1}^{k} r_i \hat{\theta}_i (\frac{V_i}{V_1})^{\hat{\beta}}}{r + \frac{1}{2}} . \tag{16}$$

3. A SIMULATED EXAMPLE

Consider the data of Table 1 which was generated according to a power rule model for 5 stress with $\alpha = 500$ and $\beta = 0.8$ (see Achcar and Dias, 1990).

TABLE 1 - Generated data with $\alpha = 500$ and $\beta = 0.8$.

i	V_i	θ_i	n_i	r_i	UNCENSORED OBSERVATIONS
1	10	79.24	30	5	6, 8, 10, 12, 14
2	20	45.52	30	8	4, 5, 5, 6, 8, 8, 9,10
3	30	32,90	30	12	2, 3, 3, 5, 6, 7, 7, 8, 8, 9,10,17
4	40	26,14	30	18	3, 3, 4, 5, 6, 6, 8, 9,10, 10,12,12,13,14,14,14,15,24
5	50	21,87	30	22	2, 3, 4, 5, 5, 8, 8, 8, 9,10,12, 13,14,14,15,18,18,18,19,20,20,27

The maximum likelihood estimator of β is $\hat{\beta} = 0.79945$ and the posterior mode is $\hat{\hat{\theta}}_1 = 78.82657$. The plot of the marginal posterior $\pi(\theta_1/\text{data})$ versus θ_1 for the data of Table 1 is shown in Figure 1.

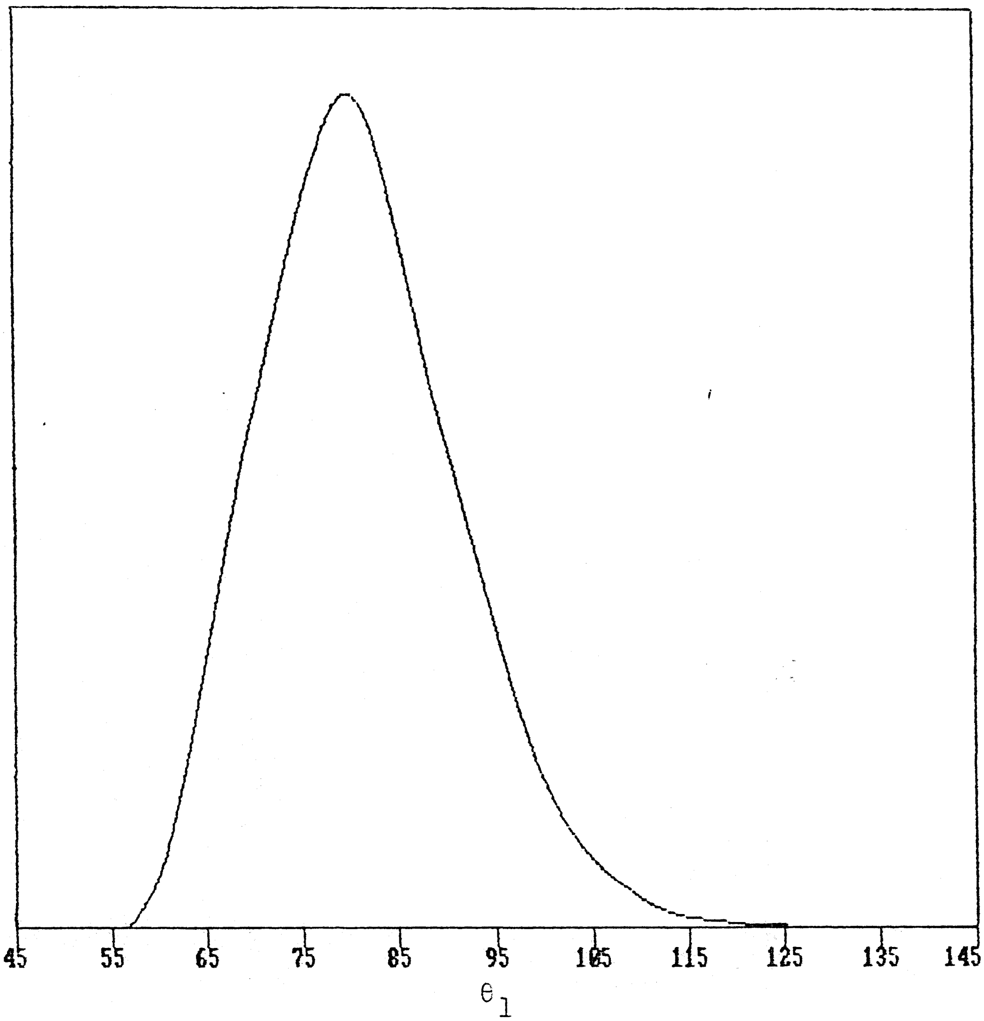

FIGURE 1 - Marginal Posterior Density for θ_1.

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