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MARAR, W. L.

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The Euler Characteristic of the Disentanglement of the Image of a Corank 1 Map Germ

W. L. Marar*

Introduction

Let (X, x) be an ICIS of dimension n, i.e., (X, x) is a germ of a complex analytic space of dimension n, which is isomorphic to the fibre $(f^{-1}(0), 0)$ of an analytic map germ $f : (\mathbf{C}^{n+k}, 0) \to (\mathbf{C}^k, 0)$ and $x \in X$ is an isolated singular point of X. Let $F : (\mathbf{C}^{n+d}, 0) \to (\mathbf{C}^d, 0)$ be a miniversal deformation of f, let S be a neighbourhood of the origin in $\mathbf{C}^d$ and $F : \mathcal{X} = F^{-1}(S) \cap B_\varepsilon \to S$ a representative of F.

H. Hamm has proved that [Ha]:

If $\varepsilon > 0$ and S are sufficiently small and D_F is the discriminant of F, then the mapping $F \mid \mathcal{X} - F^{-1}(D_F) : \mathcal{X} - F^{-1}(D_F) \to S - D_F$ is the projection of a smooth fibre bundle (Milnor fibration). The typical fibre X_s, $s \in S - D_F$ of this bundle has the homotopy type of a wedge of spheres of dimension n.

Lê D.T. gave a formula for the number of spheres in that wedge (Milnor number) in terms of the mapping f [Lê] (see also [Lo] Chapter 2 and 5).

Let $f_0 : (\mathbf{C}^n, 0) \to (\mathbf{C}^p, 0)$, $2 \leq n < p$ be a finitely $\mathcal{A}$-determined map germ with (n, p) in the range of nice dimensions in the sense of Mather ([M1]).

D. Mond has proved that when $p = n + 1$ the image of a stabilization f_t of f_0 (see definition below) has the homotopy type of a wedge of n-spheres ([Mo]).

In this paper we express the number of these spheres in terms of the Milnor number of the multiple point schemes of f_0 ([M-M]) for corank 1 map germs f_0. Moreover, we express the Euler characteristic of the image of a stabilization f_t of a corank 1 map germ f_0 in terms of its multiple point schemes when $p > n \geq 2$.

*partially supported by CNPq and Fapesp.

1 Stabilization and disentanglement

Let $f_0 : (\mathbf{C}^n, 0) \to (\mathbf{C}^p, 0)$, $2 \le n < p$ be as above.

Let $F : (\mathbf{C}^n \times \mathbf{C}^d, 0) \to (\mathbf{C}^p \times \mathbf{C}^d, 0)$, $F(x,t) = (f_t(x), t)$ be an $\mathcal{A}_e$-versal unfolding of f_0, $\pi : \mathbf{C}^p \times \mathbf{C}^d \to \mathbf{C}^d$ be the natural projection and $B \subset \mathbf{C}^d$ the bifurcation set of F.

Theorem 1 *There exists an $\varepsilon > 0$, neighbourhoods U of the origin of $\mathbf{C}^n \times \mathbf{C}^d$ and T of the origin of $\mathbf{C}^d$ and a proper, finite-to-one representative of the unfolding F, $F : U \to B_\varepsilon \times T$ such that, if $X = F(\bar{U})$ then the stable type stratified mapping $F : \bar{U} \cap F^{-1}(\bar{B}_\varepsilon \times (T - B)) \to X \cap (\bar{B}_\varepsilon \times (T - B))$ is locally topologically trivial over $T - B$ with respect to the stratified submersion $\pi : \bar{B}_\varepsilon \times (T - B) \to T - B$.*

Proof. The proof uses standard techniques in stratification theory, in particular Thom's Second Isotopy Lemma (see [M2]). The stratification of X by stable types (see [Ga]) is locally analytically trivial, and hence certainly Whitney regular over $T - B$ (see [Ma]). □

Thus, we obtain:

(i) a 'fibration' of the mapping $F : \bar{U} \to X$ whose 'fibre' over a parameter $t \in T - B$ is the stable mapping $f_t : U_t \to X_t$, where $U_t = \{x \in \mathbf{C}^n : (x,t) \in \bar{U}\}$ is contractible and $X_t = X \cap (\bar{B}_\varepsilon \times \{t\})$.

ii) a topological fibration of the image X of F whose fibre over $t \in T - B$ is the image X_t of the stable mapping f_t.

Definition 1 *f_t as above is called a* stabilization *of f_0 and its image X_t the* disentanglement *of the image of f_0.*

2 Multiple point schemes

In [M-M] the multiple point shemes of a mapping $g : \mathbf{C}^s \to \mathbf{C}^t$, $2 \leq s < t$ are introduced.

If g is of corank at most 1, then the k-tuple point schemes embed in $\mathbf{C}^{s-1} \times \mathbf{C}^k$.

If $\gamma(k) = (r_1, \ldots, r_m)$ is an ordered partition of an integer k, i.e. $\Sigma r_i = k$ and $r_i \geq r_{i+1}$ then we denote by $\tilde{D}^k(g, \gamma(k))$ the k-tuple point sheme of g associated to the partition $\gamma(k)$.

If g is a corank 1 map then a generic point y of $\tilde{D}^k(g, \gamma(k)) \subseteq \mathbf{C}^{s-1} \times \mathbf{C}^k$ is of the form $y = (x, y_1, \ldots, y_1, \ldots, y_m, \ldots, y_m)$ with $x \in \mathbf{C}^{s-1}$, $y_i \in \mathbf{C}$, y_i repeated r_i times, $y_i \neq y_j$ for $i \neq j$, $g(x, y_1) = \cdots = g(x, y_m)$ and the local algebra of g at (x, y_i) isomorphic to $\mathbf{C}[z]/(z^{r_i})$.

Theorem 2 *'Let $f : (\mathbf{C}^n, 0) \to (\mathbf{C}^p, 0)$, $2 \leq n < p$ be a corank 1 map germ. Then, f is finitely $\mathcal{A}$-determined (resp. stable) if and only if $\tilde{D}^k(f, \gamma(k))$ is an ICIS (resp. smooth) of dimension $p - k(p - n + 1) + m$, if not empty.*

For the proof see [M-M]. □

Let $f_0 : (\mathbf{C}^n, 0) \to (\mathbf{C}^p, 0)$, $2 \leq n < p$ be a finitely $\mathcal{A}$-determined map germ of corank 1 at the origin and let $F : (\mathbf{C}^n \times \mathbf{C}^d, 0) \to (\mathbf{C}^p \times \mathbf{C}^d, 0)$, $F(x, t) = (f_t(x), t)$ be an $\mathcal{A}_e$-versal unfolding of f_0.

Thus, $(\tilde{D}^k(F, \gamma(k)), 0)$ is an ICIS. Let $\pi_{\gamma(k)}$ be the restriction to $\tilde{D}^k(F, \gamma(k))$ of the projection from $\mathbf{C}^{n-1} \times \mathbf{C}^k \times \mathbf{C}^d$ to $\mathbf{C}^d$. According to Theorem 2, the bifurcation set B of F contains the discriminant set of all mappings $\pi_{\gamma(k)} : \tilde{D}^k(F, \gamma(k)) \to T$. Therefore, the projection $\tilde{D}^k(F, \gamma(k)) \cap \pi^{-1}_{\gamma(k)}(T - B) \to T - B$ defines a Milnor fibration with typical fibre $\tilde{D}^k(f_t, \gamma(k))$ and critical fibre $\tilde{D}^k(f_0, \gamma(k))$ (see [Ma]).

So, over a parameter $t \in T - B \subseteq \mathbf{C}^d$ we have the following diagram:

$$
\begin{array}{ccccccccc}
 & & & & & & \vdots & & \\
 & & & & & & \downarrow & & \\
 & & & & & & \tilde{D}^r(f_t) & & \\
 & & & & & & \downarrow & & \\
 & & & & & & \tilde{D}^{r-1}(f_t) & & \\
 & & \vdots & & & & \downarrow & & \\
 & & \downarrow & & \vdots & & \vdots & & \\
 & & \tilde{D}^4(f_t,(2^2)) & & \downarrow & & \downarrow & & \\
 & \nearrow & & \searrow & & & & & \\
\tilde{D}^4(f_t,(4)) & & & & \tilde{D}^4(f_t,(2,1^2)) & \longrightarrow & \tilde{D}^4(f_t) & & \\
 & \searrow & & \nearrow & & & & & \\
 & & \tilde{D}^4(f_t(3,1)) & & \downarrow & & \downarrow & & \\
 & & \downarrow & & & & & & \\
 & & \tilde{D}^3(f_t,(3)) & \longrightarrow & \tilde{D}^3(f_t,(2,1)) & \longrightarrow & \tilde{D}^3(f_t) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \tilde{D}^2(f_t,(2)) & \longrightarrow & \tilde{D}^2(f_t) & & \\
 & & & & & & \downarrow & & \\
 & & & & & & U_t & \longrightarrow & X_t
\end{array}
$$

Notation: (1^k) stands for the partition $(1,\ldots,1)$ of k and $\tilde{D}^k(f_t) = \tilde{D}^k(f_t,(1^k))$.

Let $\rho_{\gamma(k)} : \tilde{D}^k(f_t,\gamma(k)) \to X_t$ be the mapping obtained as composition of the inclusion $\tilde{D}^k(f_t,\gamma(k)) \hookrightarrow \tilde{D}^k(f_t)$, the projection $\tilde{D}^k(f_t) \to U_t$ and f_t.

3 Statement of the theorem

The mappings $\rho_{\gamma(k)}$ and f_t are (by construction) proper and finite-to-one. Moreover, f_t is branched over the points of the image of $\rho_{(1^2)}$ (and hence over the points of the image of all $\rho_{\gamma(k)}$) in X_t, elsewhere it is one-to-one. $\rho_{\gamma(k)}$ is branched over the points of the image of all $\rho_{\gamma'(r)}$ with $r \geq k$ and such that $\gamma(k) < \gamma'(r)$. Here the symbol $\gamma(k) < \gamma'(r)$ means that $\tilde{D}^r(f_t, \gamma''(r)) \supseteq \tilde{D}^r(f_t, \gamma'(r))$, where $\gamma''(r) = (\gamma(k), 1, \ldots, 1)$ is a partition of r.

We recall that since $\tilde{D}^k(f_t, \gamma(k))$ is a typical Milnor fibre of $\tilde{D}^k(F, \gamma(k))$ and $\tilde{D}^k(f_0, \gamma(k))$ the critical fibre, $\chi(\tilde{D}^k(f_t, \gamma(k))) = 1 + (-1)^s \mu(\tilde{D}^k(f_0, \gamma(k)))$, where s is the complex dimension of the ICIS $\tilde{D}^k(f_0, \gamma(k))$ and $\mu(\tilde{D}^k(f_0, \gamma(k)))$ its Milnor number.

So, to express the Euler characteristic of the disentanglement X_t in terms of the Milnor number of the multiple point schemes $\tilde{D}^k(f_0, \gamma(k))$ is equivalent to finding numbers $\beta_{\gamma(k)}$ such that

$$\chi(X_t) = \beta_0 \chi(U_t) + \sum_{k \geq 2} \sum_{\gamma(k)} \beta_{\gamma(k)} \chi(\tilde{D}^k(f_t, \gamma(k))) \ , \qquad (1)$$

where $\gamma(k)$ runs through the set of all ordered partitions of k.

The properties of the mappings $\rho_{\gamma(k)}$ and f_t allow us to show, by combinatorial methods the following:

Theorem 3 *If $\gamma(k) = (r_1, \ldots, r_m)$, $r_1 \geq r_{i+1}$, $\Sigma r_i = k$ and $\alpha_i = \#\{j : r_j = i\}$ then the coefficients in (1) above are:*

$$\beta_0 = 1$$

$$\beta_{\gamma(k)} = \begin{cases} \frac{-(-1)^{\Sigma \alpha_i}}{\prod_{i \geq 1} i^{\alpha_i} \alpha_i!} \ , & \textit{if } \tilde{D}^k(f_0, \gamma(k)) \textit{ is non-empty} \\ 0 \ , & \textit{otherwise.} \end{cases}$$

The proof goes as follows: (details are given in Section 4 below).

Firstly we triangulate X_t in such way that the pull-back of this triangulation provides triangulations for U_t and all $\tilde{D}^k(f_t, \gamma(k))$. Those triangulations are such that the problem of proving the equality

$$\chi(X_t) = \beta_0 \chi(U_t) + \sum_{k \geq 2} \sum_{\gamma(k)} \beta_{\gamma(k)} \chi(\tilde{D}^k(f_t, \gamma(k)))$$

is reduced to proving that

$$C_0^{X_t} = \beta_0 C_0^{U_t} + \sum_{k\geq 2} \sum_{\gamma(k)} \beta_{\gamma(k)} C_0^{\gamma(k)} , \tag{2}$$

where $C_0^{X_t}$, $C_0^{U_t}$ and $C_0^{\gamma(k)}$ are respectively the number of zero-cells in the triangulation of X_t, U_t and $\tilde{D}^k(f_t, \gamma(k))$.

Secondly, in studying the degrees of the mappings $\rho_{\gamma(k)}$ and f_t, we shall determine the number of zero-cells in U_t and $\tilde{D}^k(f_t, \gamma(k))$ that come from the pull-back of each zero-cell of X_t. In this way, (2) is equivalent to the system of equations:

$$\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = M \begin{pmatrix} \beta_0 \\ \beta_{(1^2)} \\ \beta_{(2)} \\ \vdots \end{pmatrix} ,$$

where M is a square matrix whose entries are precisely the degrees of the mappings $\rho_{\gamma(k)}$ and f_t. M is constructed in the following way: its first column contains the degrees of f_t at a generic point of the image of f_t, $\rho_{(1^2)}$, $\rho_{(2)}$, $\rho_{(1^3)}$, $\rho_{(2,1)}$, $\rho_{(3)}$, $\rho_{(1^4)}$, $\ldots$ and so on for all partitions $\gamma(k)$, following the order $<$ on the partitions. The second column of M contains the degrees of the mappings $\rho_{(1^2)}$, the third column those of $\rho_{(2)}$, and so on, following the order $<$ on the partitions.

Thus M is a non-singular lower triangular matrix and hence the system above has unique solution.

Finally the solution will follow from the following:

Lemma 1 *Let $e^{(k)}$ be the k^{th} elementary symmetric function in the variables $x_1, \ldots, x_q$.*

$$e^{(k)} = \sum_{1\leq i_1 < \cdots < i_k \leq q} x_{i_1} \cdot \ldots \cdot x_{i_k} .$$

Let $\gamma(k) = (a_1, \ldots, a_h)$, $a_i \geq a_{i+1}$ be a partition of k and $\alpha_i = \#\{j : a_j = i\}$. Then $e^{(k)} = \sum_{\gamma(k)} \frac{(-1)^{k-\Sigma\alpha_i}}{\prod_{i\geq 1} i^{\alpha_i}\alpha_i!} \prod_{i\geq 1}(x_1^i + \cdots + x_q^i)^{\alpha_i}$.

Proof. (see [Mac] p.17). □

4 Proof. of the theorem

(1^{st} step) Triangulation of X_t, U_t and $\tilde{D}^k(f_t, \gamma(k))$.

Let $\mathcal{T}(X_t)$ denote the triangulation of X_t constructed in the following way. We first consider all the zero-dimensional multiple point schemes $\tilde{D}^r(f_t, \gamma(r))$ of f_t and start triangulating X_t by including the image of all mappings $\rho_{\gamma(r)} : \tilde{D}^r(f_t, \gamma(r)) \rightarrow X_t$ among the vertices of $\mathcal{T}(X_t)$. Next we build up the two-skeleton $X_t^{(2)}$ of $\mathcal{T}(X_t)$ so that the image in X_t of each multiple point scheme of complex dimension one is a subcomplex of $X_t^{(2)}$. Then we continue in this way until we obtain the $2(n-1)$-skeleton $X_t^{2(n-1)}$ of $\mathcal{T}(X_t)$, which should contain the image of $\rho_{(1^2)}$ as a subcomplex. Finally we complete $\mathcal{T}(X_t)$.

Since the mappings $\rho_{\gamma(k)}$ and f_t are proper and finite-to-one, then pulling back $\mathcal{T}(X_t)$ we obtain a triangulation for the source U_t and the pull-back of the $2s$-skeleton ${}'X_t^{(2s)}$ provides triangulations for all multiple point schemes of complex dimension s.

In fact, let $M_{\gamma(k)}$ denote the image of $\tilde{D}^k(f_t, \gamma(k))$ by the map $\rho_{\gamma(k)}$: $\tilde{D}^k(f_t, \gamma(k)) \rightarrow X_t$. By construction, $M_{\gamma(k)}$ is a subcomplex of $\mathcal{T}(X_t)$.

The triangulation $\mathcal{T}(\tilde{D}^k(f_t, \gamma(k)))$ of $\tilde{D}^k(f_t, \gamma(k))$ is obtained as follows:

- the vertices of $\mathcal{T}(\tilde{D}^k(f_t, \gamma(k)))$ will be all points over the vertices of $M_{\gamma(k)}$.
- for all one-simplices $\sigma : \Delta^1 \rightarrow M_{\gamma(k)}$, we denote the interior of Δ^1 by $\text{int}\Delta^1$. Then, by assumption, $\rho_{\gamma(k)}$ is a trivial covering over $\sigma(\text{int}\Delta^1)$, of degree say d_1. Since $\text{int}\Delta^1$ is simply connected, σ lifts to d_1 distinct maps $\sigma_0^{(i)} : \text{int}\Delta^1 \rightarrow \tilde{D}^k(f_t, \gamma(k))$, $i = 1, \ldots, d_1$, with disjoint images. Since $\rho_{\gamma(k)}$ is proper we can extend $\sigma_0^{(i)}$ to maps $\sigma^{(i)} : \Delta^1 \rightarrow \tilde{D}^k(f_t, \gamma(k))$ lifting σ.

$$\begin{array}{ccc} & & \tilde{D}^k(f_t, \gamma(k)) \\ & \sigma_0^{(i)} \nearrow & \downarrow \rho_{\gamma(k)} \\ \text{int}\Delta^1 & \longrightarrow & M_{\gamma(k)} \end{array}$$

- do the same for all two-simplexes of $M_{\gamma(k)}$ and so on. In this way we end up with a triangulation of $\tilde{D}^k(f_t, \gamma(k))$.

Now if we let $C_i^{X_t}$, $C_i^{U_t}$ and $C_i^{\gamma(k)}$ be respectively the number of i-cells in the triangulations of X_t, U_t and $\tilde{D}^k(f_t,\gamma(k))$ then,

$$(1) \qquad \chi(X_t) = \beta_0\chi(U_t) + \sum_{k\geq 2}\sum_{\gamma(k)} \beta_{\gamma(k)}\chi(\tilde{D}^k(f_t,\gamma(k)))$$

is equivalent to

$$\sum_i(-1)^iC_i^{X_t} = \beta_0\sum_i(-1)^iC_i^{U_t} + \sum_{k\geq 2}\sum_{\gamma(k)}\beta_{\gamma(k)}\sum_i(-1)^iC_i^{\gamma(k)}$$

Notice that if we find coefficients β_0 and $\beta_{\gamma(k)}$ (for all $\gamma(k)$) independent of i, such that

$$C_i^{X_t} = \beta_0C_i^{U_t} + \sum_{k\geq 2}\sum_{\gamma(k)}\beta_{\gamma(k)}C_i^{\gamma(k)}, \quad \text{for all } \ i, \ 0\leq i\leq 2n$$

then these coefficients will solve equation (1).

So let us concentrate on solving:

$$(2) \qquad C_0^{X_t} = \beta_0C_0^{U_t} + \sum_{k\geq 2}\sum_{\gamma(k)}\beta_{\gamma(k)}C_0^{\gamma(k)}$$

(where all coefficients $\beta_{\gamma(k)}$ appear).

(2^{nd} step) The degrees of the mappings $\rho_{\gamma(k)}$ and f_t.

The numbers $C_0^{U_t}$ and $C_0^{\gamma(k)}$ are obtained from $C_0^{X_t}$ according to the following:

Claim: If $\mathbf{x}$ is a generic point of U_t, (i.e., X is not on the image of the projection of $\tilde{D}^2(f_t)$ over U_t) then $\#f_t^{-1}(f_t(\mathbf{x})) = 1$.

If $\mathbf{y}$ is a generic point of $\tilde{D}^k(f_t,\gamma(k))$, where $\gamma(k) = (r_1,\ldots,r_m)$ then $\#f_t^{-1}(\rho_{\gamma(k)}(\mathbf{y})) = m$.

If $\mathbf{y}$ is a generic point of $\tilde{D}^r(f_t,\gamma'(r))$, where $\gamma'(r) = (b_1,\ldots,b_q)$ then $\#\rho_{\gamma(k)}^{-1}(\rho_{\gamma'(r)}(\mathbf{y}))$, with $\gamma(k) = (a_1,\ldots,a_h), r\geq k$ and $\gamma(k) < \gamma'(r)$, is the coefficient of the monomial $x_1^{b_1}\cdot\ldots\cdot x_q^{b_q}$ in the polynomial $\prod_{i=1}^h(x_1^{a_i}+\ldots+x_q^{a_i})$, if $k=r$ or $\#\rho_{\gamma(k)}^{-1}(\rho_{\gamma'(r)}(\mathbf{y}))$ is the sum of the coefficients of the monomials $x_1^{c_1}\cdot\ldots\cdot x_q^{c_q}$ in the polynomial $\prod_{i=1}^h(x_1^{a_i}+\ldots+x_q^{a_i})$, with $(c_i,\ldots,c_q)\in\mathbf{N}_0^q$, $c_i\leq b_i$ and $\Sigma c_i = k$, if $r>k$.

In fact, let $F_{\gamma(k)}^{\gamma'(r)}$ denote $\#\rho_{\gamma(k)}^{-1}(\rho_{\gamma'(r)}(\mathbf{y}))$ where $\mathbf{y} \in \tilde{D}^r(f_t, \gamma'(r))$ is a generic point.

Recall that a generic point $\mathbf{y} \in \tilde{D}^r(f_t, \gamma'(r) \subseteq \mathbf{C}^{n-1} \times \mathbf{C}^r$ is of the form $\mathbf{y} = (x, y_1, \ldots, y_1, \ldots, y_q, \ldots, y_q)$, with $x \in \mathbf{C}^{n-1}$, $y_i \in \mathbf{C}$, $y_i \neq y_j$ for $i \neq j$ and y_i repeated b_i times. On the other hand the points of the fibre $\rho_{\gamma(k)}^{-1}(\rho_{\gamma(r)}(\mathbf{y}))$ are the (generic or non-generic) points of $\tilde{D}^k(f_t, \gamma(k))$ whose coordinates are chosen out of the coordinates of the generic point $\mathbf{y}$ (the way to choose is such that the resulting point belongs to $\tilde{D}^k(f_t, \gamma(k))$. Thus $F_{\gamma(k)}^{\gamma'(r)}$ is the number of all such possible choices.

Finally, we consider the following one-to-one correspondence between the generic points $\mathbf{y} = (x, y_1, \ldots, y_1, \ldots, y_q, \ldots, y_q)$ of $\tilde{D}^r(f_t, \gamma'(r))$ and the monomials $x_1^{b_1}, \ldots, x_q^{b_q}$: namely, to the sequence of coordinates $y_i, \ldots, y_i$ (y_i repeated b_i times) we associate $x_i^{b_i}$. Now the claim follows by expanding the product $\prod_{i=1}^k (x_1^{a_i} + \ldots + x_q^{a_i})$. □

Now, if we consider the matrix M introduced in Section 3, whose entries are the degrees of the mappings f_t and $\rho_{\gamma(k)}$ then equality

$C_0^{X_t} = \beta_0 C_0^{U_t} + \sum_{k\geq 2} \sum_{\gamma(k)} C_0^{\gamma(k)}$ is equivalent to the system of equations:

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = M \begin{pmatrix} \beta_0 \\ \beta_{(1^2)} \\ \beta_{(2)} \\ \beta_{(1^3)} \\ \vdots \end{pmatrix} .$$

(3rd step) It follows from the construction of M that the equations of the system above are:

$$\begin{aligned} 1 &= \beta_0 \cdot 1 \\ 1 &= \beta_0 \cdot 2 + \beta_{(1^2)} \cdot 2 \\ 1 &= \beta_0 \cdot 1 + \beta_{(1^2)} \cdot 1 + \beta_{(2)} \cdot 1 \\ 1 &= \beta_0 \cdot 3 + \beta_{(1^2)} \cdot 6 + \beta_{(2)} \cdot 0 + \beta_{(1^3)} \cdot 6 \\ &\vdots \\ 1 &= \beta_0 \cdot q + \textstyle\sum_{k=2}^s \sum_{\gamma(k)} \beta_{\gamma(k)} \cdot F_{\gamma(k)}^{\gamma(s)} , \end{aligned}$$

where $\gamma(s) = (b_1, \ldots, b_q)$, with $b_i \geq b_{i+1}$, is a partition of $s \geq 2$.

Now, if $\gamma(k) = (a_1, \ldots, a_k)$, with $a_i \geq a_{i+1}$ and $\alpha_i = \#\{\, j : a_j = i \,\}$ then it follows from Lemma 1 that

$$\sum_{\gamma(k)} \frac{(-1)^{\Sigma\alpha_i}}{\prod_{i\geq 1} i^{\alpha_i}\alpha_i!} F_{\gamma(k)}^{\gamma(s)} = (-1)^{k-1} \binom{q}{k} .$$

Therefore, $\beta_{\gamma(k)}$ must be equal to $\frac{-(-1)^{\Sigma\alpha_i}}{\prod_{i\geq 1} i^{\alpha_i}\alpha_i!}$ (if $\tilde{D}^k(f_t, \gamma(k))$ is non-empty).

5 Some symmetries

First we recall that the k-tuple point scheme $\tilde{D}^k(f_t)$ is embedded in $\mathbf{C}^{n-1} \times \mathbf{C}^k$, where it lies invariant under the action of the symmetric group S_k, which permutes the coordinates in $\mathbf{C}^k$ ([M-M]).

Now, considering the finite mapping $\rho : \tilde{D}^k(f_t) \to \tilde{D}^k(f_t)/S_k$ we can repeat the procedure in Sections 3 e 4 above to obtain:

Proposition 1 *$\chi(\tilde{D}^k(f_t)/S_k) = \sum_{\gamma(k)} \nu_{\gamma(k)}\chi(\tilde{D}^k(f_t,\gamma(k)))$, where $\gamma(k) = (r_1,\ldots,r_m)$, with $r_i \geq r_{i+1}$, runs through the set of all ordered partitions of k, $\alpha_i = \#\{j : r_j = i\}$ and $\nu_{\gamma(k)} = 1/\prod_{i\geq 1} i^{\alpha_i}\alpha_i!$, or zero if the corresponding $\tilde{D}^k(f_t,\gamma(k))$ is empty.*

Proof. The proof is analogous to the proof in Section 4 and follows from the following:

Lemma 2 *Let h_r be the r^{th} complete symmetric function in the variables $x_1,\ldots,x_q$, i.e., h_r is the sum of all monomials of degree r in the variables $x_1,\ldots,x_q$. Then $h_r = \sum_{\gamma(r)} 1/\prod_{i\geq i} i^{\alpha_i}\alpha_i! \prod_{i\geq 1}(x_1^i + \cdots + x_q^i)^{\alpha_i}$ where $\gamma(r) = (a_1,\ldots,a_h)$; with $a_i \geq a_{i+1}$, runs through the set of all ordered partion of r and $\alpha_i = \#\{j : a_j = i\}$.*

Proof: ([Mac] p.16]). □

Now we can replace $\chi(\tilde{D}^k(f_t))$ by $\chi(\tilde{D}^k(f_t)/S_k)$ in equation 1 to obtain:

Theorem 4 *Let X_t be the disentanglement of the image of a corank 1 map germ $f_0 : \mathbf{C}^n,0 \to \mathbf{C}^p,0$, $2 \leq n < p$.*

Then

$$\begin{aligned}\chi(X_t) = 1 &+ \sum_{k\geq 2}(-1)^{k-1}\chi(\tilde{D}^k(f_t)/S_k + \\ &+ \sum_{k\geq 2}\sum_{\gamma(k)} \frac{(-1)^k - (-1)^{\Sigma\alpha_i}}{\prod_{i\geq 1} i^{\alpha_i}\alpha_i!}\chi(\tilde{D}^k(f_t,\gamma(k))) ,\end{aligned}$$

where $\gamma(k) = (a_1,\ldots,a_h) \neq (1^k)$ runs through the set of all ordered partitions of k, $\alpha_i = \#\{j : a_j = i\}$.

Examples: Suppose $f_0 : \mathbf{C}^n, 0 \to \mathbf{C}^p, 0$ has all multiple point schemes non-empty. So, if $n = 2$ and $p = 3$,

$$\chi(X_t) = 1 - \chi(\tilde{D}^2(f_t)/S_2) + \chi(\tilde{D}^3(f_t)/S_3) + \chi(\tilde{D}^2(f_t,(2)))$$

and if $n = 3$ and $p = 4$ then

$$\begin{aligned}\chi(X_t) = 1 &- \chi(\tilde{D}^2(f_t)/S_2) + \chi(\tilde{D}^2(f_t,(2)) + \chi(\tilde{D}^3(f_t)/S_3) - \\ &- \chi(\tilde{D}^3(f_t,(2,1)) - \chi(\tilde{D}^4(f_t)/S_4) .\end{aligned}$$

Now using the relations

$$\chi(\tilde{D}^k(f_t,\gamma(k))) = 1 + (-1)^s \mu(\tilde{D}^k(f_0,\gamma(k)))$$

we obtain:

if $n = 2$ and $p = 3$:

$$\chi(X_t) = 2 + \mu(\tilde{D}^2(f_0)/S_2) + \mu(\tilde{D}^3(f_0)/S_3) + \mu(\tilde{D}^2(f_t,(2)))$$

and

if $n = 3$ and $p = 4$:

$$\begin{aligned}\chi(X_t) = &- \mu(\tilde{D}^2(f_0)/S_2) - \mu(\tilde{D}^3(f_0)/S_3) - \mu(\tilde{D}^4(f_0)/S_4) - \\ &- \mu(\tilde{D}^2(f_0,(2))) - \mu(\tilde{D}^3(f_0,(2,1))) .\end{aligned}$$

In general, if $f_0 : (\mathbf{C}^n, 0) \to (\mathbf{C}^p, 0)$, $2 \le n < p$ is a corank 1 finitely $\mathcal{A}$-determined map germ then

$$\begin{aligned}\chi(X_t) = 1 &+ \sum_{k=2}^{p}((-1)^{k+1} + \sum_{\gamma(k)} \frac{(-1)^k - (-1)^{\Sigma\alpha_i}}{\prod_{i\geq 1} i^{\alpha_i}\alpha_i!}) + \\ &+ \sum_{k=2}^{p}(-1)^{p-k(p-n+1)+1}[\mu(\tilde{D}^k(f_0)/S_k) + \\ &\quad + \sum_{\gamma(k)} \frac{1+(-1)^{k+\Sigma\alpha_i+1}}{\prod_{i\geq 1} i^{\alpha_i}\alpha_i!}\mu(\tilde{D}^k(f_0,\gamma(k)))]\end{aligned}$$

where $\gamma(k) = (r_1, \dots, r_m) \neq (1^k)$ runs through the set of all ordered partitions of k, $2 \leq k \leq p$ and $\alpha_i = \#\{ j : r_j = 1 \}$.

Remark: If $p = n + 1$ then $\chi(X_t)$ is semicontinuous with respect to t since $(-1)^{p-k(p-n+1)+1}$ is independent of k.

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