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ACCELERATED LIFE TESTS WITH AN EXPONENTIAL DISTRIBUTION: A BAYESIAN APPROACH WITH THE GENERALIZED EYRING MODEL AND TYPE II CENSORED DATA

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SUMMARY

In this paper, we present a Bayesian Analysis of the generalized Eyring model considering two stress variables Z and V. We assume an accelerated life test with K combinations of the stress levels V_i and Z_i with an exponential distribution for the life data in each combination of stresses and type II censored data. We find the posterior densities of interest considering a noninformative Jeffreys prior for the parameters of the model. In special, we find the marginal posterior density for the mean life time of unities assuming the usual conditions stresses Z_1 and V_1. We also present an example and some comparisons with the usual existing methods of analysis of this model.

Key words: Generalized Eyring model. Accelerated life tests. Type II censored data. Bayesian analysis.

1. INTRODUCTION

Let T be a random variable denoting the life time of a component with an exponential distribution with density,

$$f(t;\lambda_i) = \lambda_i \exp\{-\lambda_i t\} \tag{1}$$

where $\lambda_i > 0$ and $t > 0$. The parameter λ_i denotes the constant rate of failures under the stress levels V_i and Z_i, $i = 1,2,\ldots,k$.

The mean life time under the stress levels V_i and Z_i is given by,

$$\theta_i = \frac{1}{\lambda_i} \tag{2}$$

We assume k combinations of the stress levels V_i and Z_i and the generalized Eyring model (see for example Mann, Schafer and Singpurwalla, 1974):

$$\lambda_i = \alpha Z_i \exp\{\gamma V_i + \frac{\delta V_i}{K_1 Z_i} - \frac{\beta}{K_1 Z_i}\} \tag{3}$$

where $\alpha > 0$, $-\infty < \beta, \gamma < \infty$, Z_i and V_i are two stress variables, α, β, γ and δ are unknown parameters and $K_1 = 1.38 \times 10^{-16}$ erg/degree Kelvin is the universal Boltzmann's constant.

With k combinations of stress levels Z_i and V_i, we assume a type II censoring mechanism, that is, the experiment terminates when we observe r_i failures for each combination of stress Z_i and V_i, $i=1,2,\ldots,k$. Thus, with n_i unities at the beginning of each test with the stress levels Z_i and V_i, we have the ordered uncensored observations given by

$t_{1_i} < t_{2_i} < \ldots < t_{r_i i}$ and $n_i - r_i$ censored observations equal to $t_{r_i i}$, $i=1,2,\ldots,k$.

2. THE LEAST-SQUARES ESTIMATORS OF THE PARAMETERS α, β, γ AND δ

Usually, with the generalized Eyring model, it is usual to obtain the least-squares estimators of the parameters of the linear model, since the maximum likelihood estimators require the solution of four simultaneous nonlinear equations (see for example, Mann, Schaffer and Singpurwalla, 1974).

Let $\hat{\lambda}_i = r_i/A_i$, where $A_i = \sum_{j=1}^{r_i} t_{j_i} + (n_i - r_i) t_{r_i i}$ is the maximum likelihood estimator of λ_i under the stress levels Z_i and V_i.

Considering a multiplicative error term and the model $\hat{\lambda}_i = \lambda_i \varepsilon_i$, we have:

$$\log\hat{\lambda}_i = \log\alpha + \log Z_i - \frac{\beta}{K_1 Z_i} + \gamma V_i + \frac{\delta V_i}{K_1 Z_i} + \log\varepsilon_i \tag{4}$$

Define,

$$\underset{\sim}{\alpha}' = (\log\alpha,\ \gamma,\ \delta,\ -\beta)$$

$$\underset{\sim}{\varepsilon}' = (\log\varepsilon_1,\ \log\varepsilon_2,\ \ldots,\ \log\varepsilon_k) \tag{5}$$

$$\underset{\sim}{Y}' = \left(\log\left(\frac{\hat{\lambda}_1}{Z_1}\right),\ \log\left(\frac{\hat{\lambda}_2}{Z_2}\right),\ \ldots,\ \log\left(\frac{\hat{\lambda}_k}{Z_k}\right)\right)$$

and $$X = \begin{bmatrix} 1 & V_1 & \frac{V_1}{K_1 Z_1} & (K_1 Z_1)^{-1} \\ 1 & V_2 & \frac{V_2}{K_1 Z_2} & (K_1 Z_2)^{-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & V_K & \frac{V_K}{K_1 Z_K} & (K_1 Z_K)^{-1} \end{bmatrix}$$

Then, the model in matrix form becomes,

$$\underset{\sim}{Y} = X\underset{\sim}{\alpha} + \underset{\sim}{\varepsilon} \tag{6}$$

where $E(\underset{\sim}{\varepsilon}) = \underset{\sim}{0}$, $E(\underset{\sim}{\varepsilon}\underset{\sim}{\varepsilon}') = \sigma^2 I$, σ^2 is a constant and I is the identity matrix.

Assuming $r_i = r$ for $i=1,2,\ldots,k$, and $\log\hat{\lambda}_i$ with a normal distribution, the least-squares estimators are given by $\hat{\underset{\sim}{\alpha}} = (X'X)^{-1}X'\underset{\sim}{Y}$, where $\hat{\underset{\sim}{\alpha}} = (\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\alpha}_4)$, $\alpha_1 = \log\alpha, \alpha_2 = \gamma$, $\alpha_3 = \delta$ and $\alpha_4 = -\beta$. Also, $\hat{\underset{\sim}{\alpha}} \sim N\{\underset{\sim}{\alpha}; \sigma^2(X'X)^{-1}\}$, and $\hat{\lambda}_i = Z_i \exp\{\hat{\alpha}_1 + \hat{\alpha}_2 V_i + \frac{\hat{\alpha}_3 V_i}{K_1 Z_i} + \frac{\hat{\alpha}_4}{K_1 Z_i}\}$ is an asymptotically umbiased estimator of λ_i under the use conditions stresses Z_i and V_i (see Singpurwalla and Goldschen, 1974).

A $100(1-\alpha^*)\%$ confidence interval for λ_i is given by,

$$P\{\hat{\lambda}_i \exp(t_{\alpha^*/2} J_i) \leq \lambda_i \leq \hat{\lambda}_i \exp(-t_{\alpha^*/2} J_i)\} = 1-\alpha^* \tag{7}$$

where $J_i = \{S^2[1 + \underset{\sim}{x}_i'(X'X)^{-1}\underset{\sim}{x}_i]\}^{1/2}$, $\underset{\sim}{x}_i' = (1,\ V_i,\ \frac{V_i}{K_1 Z_i},\ \frac{1}{K_1 Z_i})$, $S^2 = \{\underset{\sim}{Y}'[I - X(X'X)^{-1}X']\}/(K-4)$, and $t_{\alpha^*/2}$ is the $100(\alpha^*/2)$th percentage point of the Student's t distribution with K-4 degrees of freedom.

Observe that, with this approach we consider a linear model with $\log(\hat{\lambda}_i)$ in place of the data of each combination of stresses V_i and Z_i, $i=1,2,\ldots,k$ and a multiplicative error model, which can not be appropriate. This fact, motivate us to develop a Bayesian analysis of the generalized Eyring model.

3. A BAYESIAN ANALYSIS ASSUMING α, β, γ AND δ UNKNOWN

Assuming k randonized combinations of stress levels Z_i and V_i and a type II censoring mechanism, the likelihood function for α, β, δ and is given by,

$$L(\alpha,\beta,\gamma,\delta) = \prod_{i=1}^{k} \lambda_i^{r_i} \exp\{-\lambda_i A_i\} \tag{8}$$

where λ_i is given in (3) and $A_i = \sum_{j=1}^{r_i} t_{j_i} + (n_i - r_i)t_{r_i i}$. Observe that Z_i and V_i are fixed.

Thus,

$$L(\alpha,\beta,\gamma,\delta) = \alpha^r \left(\prod_{i=1}^{k} Z_i^{r_i}\right) \exp\{\gamma a_1 + \delta a_2 - \beta a_3\} \times \exp\{-\alpha \sum_{i=1}^{k} Z_i A_i e^{-b_i}\} \tag{9}$$

where $b_i = \frac{\beta}{K_1 Z_i} - \gamma V_i - \frac{\delta V_i}{K_1 Z_i}$, $a_1 = \sum_{i=1}^{k} r_i V_i$, $a_2 = \frac{1}{K_1} \sum_{i=1}^{k} \frac{V_i r_i}{Z_i}$,

$a_3 = \frac{1}{K_1} \sum_{i=1}^{k} \frac{r_i}{Z_i}$ and $r = \sum_{i=1}^{k} r_i$.

The second derivatives of the logarithm of the likelihood function for α, β, γ and δ are given by,

$$\frac{\partial^2 \log L}{\partial \alpha^2} = -\frac{r}{\alpha^2}; \quad \frac{\partial^2 \log L}{\partial \beta^2} = -\frac{\alpha}{k_1^2}\sum_{i=1}^{k}\frac{A_i}{Z_i}e^{-b_i};$$

$$\frac{\partial^2 \log L}{\partial \gamma^2} = -\alpha\sum_{i=1}^{k}V_i^2 Z_i A_i e^{-b_i}; \quad \frac{\partial^2 \log L}{\partial \delta^2} = -\frac{\alpha}{k_1^2}\sum_{i=1}^{k}\frac{V_i^2 A_i}{Z_i}e^{-b_i};$$

$$\frac{\partial^2 \log L}{\partial \alpha \partial \beta} = \frac{1}{k_1}\sum_{i=1}^{k}A_i e^{-b_i}; \quad \frac{\partial^2 \log L}{\partial \alpha \partial \gamma} = -\sum_{i=1}^{k}V_i A_i Z_i e^{-b_i};$$

$$\frac{\partial^2 \log L}{\partial \alpha \partial \delta} = -\frac{1}{k_1}\sum_{i=1}^{k}V_i A_i e^{-b_i}; \quad \frac{\partial^2 \log L}{\partial \beta \partial \gamma} = \frac{\alpha}{k_1}\sum_{i=1}^{k}V_i A_i e^{-b_i};$$

$$\frac{\partial^2 \log L}{\partial \beta \partial \delta} = \frac{\alpha}{k_1^2}\sum_{i=1}^{k}\frac{V_i A_i}{Z_i}e^{-b_i}; \quad \frac{\partial^2 \log L}{\partial \gamma \partial \delta} = -\frac{\alpha}{k_1}\sum_{i=1}^{k}V_i^2 A_i e^{-b_i}.$$

Since $E(A_i/r_i) = 1/\lambda_i$, where $\lambda_i = \alpha Z_i e^{-b_i}$, we have $E(A_i) = r_i e^{b_i}/(\alpha Z_i)$. Therefore, the Fisher information matrix is given by,

$$I(\alpha,\beta,\gamma,\delta) = \begin{bmatrix} \frac{r}{\alpha^2} & -\frac{1}{k_1\alpha}\sum_{i=1}^{k}\frac{r_i}{Z_i} & \frac{1}{\alpha}\sum_{i=1}^{k}r_i V_i & \frac{1}{k_1\alpha}\sum_{i=1}^{k}\frac{r_i V_i}{Z_i} \\ & \frac{1}{k_1^2}\sum_{i=1}^{k}\frac{r_i}{Z_1^2} & -\frac{1}{k_1}\sum_{i=1}^{k}\frac{r_i V_i}{Z_i} & -\frac{1}{k_1^2}\sum_{i=1}^{k}\frac{r_i V_i}{Z_i^2} \\ & & \sum_{i=1}^{k}\frac{r_i V_i}{Z_i} & \frac{1}{k_1}\sum_{i=1}^{k}\frac{r_i V_i^2}{Z_i} \\ \text{symmetric} & & & \frac{1}{k_1^2}\sum_{i=1}^{k}\frac{r_i V_i}{Z_i^2} \end{bmatrix}$$

The Jeffreys prior density for α, β, γ and δ is given by (see for example, Box and Tiao, 1973):

$$\pi(\alpha,\beta,\gamma,\delta) \propto \{\det I(\alpha,\beta,\gamma,\delta)\}^{1/2}$$

That is,

$$\pi(\alpha,\beta,\gamma,\delta) \propto \alpha^{-1} \tag{10}$$

where $\alpha > 0$, $-\infty < \beta,\gamma,\delta < \infty$.

Considering the Jeffreys prior (10), the joint posterior for α, β, γ and δ is given by,

$$\pi(\alpha,\beta,\gamma,\delta \mid \text{data}) \propto \alpha^{r-1}\exp\{\gamma a_1+\delta a_2-\beta a_3\} \times$$

$$\times \exp\{-\alpha \sum_{i=1}^{k} Z_i A_i \exp(\gamma V_i + \frac{\delta V_i}{K_1 Z_i} - \frac{\beta}{K_1 Z_i})\} \tag{11}$$

where $\alpha > 0$, $-\infty < \beta,\gamma,\delta < \infty$.

The marginal joint posterior density for β, γ and δ is given by,

$$\pi(\beta,\gamma,\delta \mid \text{data}) \propto \exp\{\gamma a_1+\delta a_2-\beta a_3\}\left(\sum_{i=1}^{k} Z_i A_i \exp(\gamma V_i + \frac{\delta V_i}{K_1 Z_i} - \frac{\beta}{K_1 Z_i})\right)^{-r} \tag{12}$$

where $-\infty < \beta,\gamma,\delta < \infty$.

The marginal joint posterior density for γ and δ is given by,

$$\pi(\gamma,\delta \mid \text{data}) \propto e^{\gamma a_1+\delta a_2} \int_{-\infty}^{\infty} e^{-nh_{\gamma,\delta}(\beta)}\, d\beta \tag{13}$$

where $-nh_{\gamma,\delta}(\beta) = -\beta a_3 - r\log\{\sum_{i=1}^{k} Z_i A_i e^{-b_i}\}$.

Thus, the marginal joint posterior density for γ and δ approximated by the Laplace's method (see for example, Tierney and Kadane, 1986) is given by,

$$\pi(\gamma,\delta\,|\,\text{data}) \propto \frac{\{h''_{\gamma,\delta}(\hat{\beta})\}^{-1/2}\exp\{\gamma a_1+\delta a_2-\hat{\beta}a_3\}}{\{\sum_{i=1}^{k} Z_i A_i \exp(\gamma V_i + \frac{\delta V_i}{K_1 Z_i} - \frac{\hat{\beta}}{K_1 Z_i})\}^{r}} \qquad (14)$$

where $\hat{\beta}$ maximizes $-nh_{\gamma,\delta}(\beta)$ and $h''_{\gamma,\delta}(\beta)$ is given by,

$$nh''_{\gamma,\delta}(\beta) = \frac{r}{k_1^2}\{[(\sum_{i=1}^{k}\frac{A_i}{Z_i}e^{-b_i})(\sum_{i=1}^{k}A_i Z_i e^{-b_i}) - (\sum_{i=1}^{k}A_i e^{-b_i})^2]/(\sum_{i=1}^{k}A_i Z_i e^{-b_i})^2\}$$

4. A BAYESIAN ANALYSIS ASSUMING γ AND δ KNOWN

Assuming γ and δ known, the likelihood function for α and β is given by,

$$L(\alpha,\beta) \propto \alpha^r \exp\{-a_3\beta - \alpha\sum_{i=1}^{k} Z_i A_i e^{-b_i}\} \qquad (15)$$

where $a_3 = \frac{1}{K_1}\sum_{i=1}^{k}\frac{r_i}{Z_i}$ and $b_i = \frac{\beta}{K_1 Z_i} - \gamma V_i - \frac{\delta V_i}{K_1 Z_i}$.

Considering a noninformative prior density for α and β given by $\pi(\alpha,\beta) \propto \alpha^{-1}$, $\alpha > 0$, $-\infty < \beta < \infty$, the joint posterior density for α and β is given by,

$$\pi(\alpha,\beta|\text{data}) \propto \alpha^{r-1}\exp\{-a_3\beta-\alpha\sum_{i=1}^{k} Z_i A_i e^{-b_i}\}. \tag{16}$$

The marginal posterior density for β is given by,

$$\pi(\beta|\text{data}) \propto e^{-a_3\beta}\{\sum_{i=1}^{k} Z_i A_i \exp(\gamma V_i + \frac{\delta V_i}{K_1 Z_i} - \frac{\beta}{K_1 Z_i})\}^{-r} \tag{17}$$

where $-\infty < \beta < \infty$.

Usually, the industrial researchers working with accelerated life testing have interest in inferences about the mean life time θ_1 assuming that Z_1 and V_1 are the use conditions stresses. That is, they have interest in the posterior density for θ_1.

Observe that (16) can be written in the form,

$$\pi(\alpha,\beta|\text{data}) \propto \alpha^{r-1}\exp\{-a_3\beta-\alpha\sum_{i=1}^{k} Z_i A_i \exp(c_i - \frac{\beta}{K_1 Z_i})\} \tag{18}$$

where $c_i = \gamma V_i + \frac{\delta V_i}{K_1 Z_i}$, $\alpha > 0$ and $-\infty < \beta < \infty$.

Considering the transformation $\lambda_1 = \exp\{\beta/K_1 Z_1\}$ and $\lambda_2 = \alpha Z_1 e^{c_1}$ (that is, $\beta = K_1 Z_1 \log \lambda_1$ and $\alpha = \lambda_2/(Z_1 e^{c_1})$), the joint posterior density for λ_1 and λ_2 is given by,

$$\pi(\lambda_1,\lambda_2|\text{data}) \propto \lambda_2^{r-1}\lambda_1^{-(a_3 K_1 Z_1+1)}\exp\{-\frac{\lambda_2}{Z_1 e^{c_1}}\sum_{i=1}^{k} Z_i A_i e^{c_i}\lambda_1^{-Z_1/Z_i}\} \tag{19}$$

where $\lambda_1 > 0$ and $\lambda_2 > 0$.

Observe that the mean life time θ_1 of the components under the stress levels Z_1 and V_1 is given by $\theta_1 = \lambda_1/\lambda_2$. Therefore, consider the transformation $\theta_1 = \lambda_1/\lambda_2$ and $\lambda_2 = \lambda_2$. Thus, we find the joint posterior density for θ_1 and λ_2 given by,

$$\pi(\theta_1,\lambda_2|\text{data}) \propto \frac{\exp\{-\frac{\lambda_2}{Z_1 e^{c_1}} \sum_{i=1}^{k} Z_i A_i e^{c_i} \lambda_2^{-Z_1/Z_i} \theta_1^{-Z_1/Z_i}\}}{\lambda_2^{a_3K_1Z_1-r+1}\, \theta_1^{a_3K_1Z_1+1}} \tag{20}$$

where $\lambda_1 > 0$ and $\lambda_2 > 0$.

The marginal posterior density for θ_1 is given by,

$$\pi(\theta_1|\text{data}) \propto \theta_1^{-(a_3K_1Z_1+1)} \int_0^\infty e^{-nh_{\theta_1}(\lambda_2)}\, d\lambda_2$$

where $-nh_{\theta_1}(\lambda_2) = -(a_3K_1Z_1-r+1)\log\lambda_2 - \frac{1}{Z_1e^{c_1}} \sum_{i=1}^{k} Z_iA_ie^{c_i}\lambda_2^{1-Z_1/Z_i}\theta_1^{-Z_1/Z_i}$.

Using the Laplace's approximation for integrals (see for example, Tierney and Kadane, 1986), we have:

$$\pi(\theta_1|\text{data}) \propto \frac{\exp\{-\frac{1}{Z_1e^{c_1}} \sum_{i=1}^{k} Z_iA_ie^{c_i}\hat{\lambda}_2^{1-Z_1/Z_i}\theta_1^{-Z_1/Z_i}\}}{\theta_1^{a_3K_1Z_1+1}\,\hat{\lambda}_2^{a_3K_1Z_1-r+1}\,\{h''_{\theta_1}(\hat{\lambda}_2)\}^{1/2}} \tag{21}$$

where $\theta_1 > 0$, $\hat{\lambda}_2$ maximizes $-nh_{\theta_1}(\lambda_2)$ for each value of θ_1 and

$$-nh''_{\theta_1}(\lambda_2) = \frac{(a_3K_1Z_1-r+1)}{\lambda_2^2} + e^{-c_1}\sum_{i=1}^{k} A_i e^{c_i}(1-\frac{Z_1}{Z_i})\theta_1^{-Z_1/Z_i}\lambda_2^{-Z_1/Z_i-1}$$

5. A BAYESIAN ANALYSIS ASSUMING β, γ AND δ KNOWN

Assuming β, γ and δ known, the likelihood function for α is given by,

$$L(\alpha) \propto \alpha^r \exp\{-\alpha \sum_{i=1}^{k} Z_iA_i \exp(\gamma V_i + \frac{\delta V_i}{K_1Z_i} - \frac{\beta}{K_1Z_i})\} \quad (22)$$

Considering a noninformative prior density for α proportional to α^{-1}, the posterior density for α is given by,

$$\pi(\alpha|\text{data}) \propto \alpha^{r-1}\exp\{-\alpha \sum_{i=1}^{k} Z_iA_i\exp(\gamma V_i+\frac{\delta V_i}{K_1Z_i} - \frac{\beta}{K_1Z_i})\} \quad (23)$$

where $\alpha > 0$.

The posterior density for $\theta_1 = e^{b_1}/(\alpha Z_1)$, where $b_1 = \frac{\beta}{K_1Z_1} - \gamma V_1 - \frac{\delta V_1}{K_1Z_1}$ is given by,

$$\pi(\theta_1|\text{data}) = \frac{B^r}{Z_1^r\Gamma(r)} \theta_1^{-(r+1)}\exp\{-\frac{B}{\theta_1Z_1}\} \quad (24)$$

where $\theta_1 > 0$, $B = e^{b_1}\sum_{i=1}^{k} Z_iA_ie^{-b_i}$ and $b_i = \frac{\beta}{K_1Z_i} - \gamma V_i - \frac{\delta V_i}{K_1Z_i}$.

The mode of the posterior density (24) is given by $\hat{\theta}_1^* = B/\{Z_1(r+1)\}$.

From (24), we observe that the posterior density for $2B/(Z_1\theta_1)$ is the density of a chi-square distribution with 2r degrees of freedom. Therefore, a $100(1-\alpha^*)\%$ HPD interval for the mean life time θ_1 of a component with the usual conditions stresses V_1 and Z_1 is given by,

$$\left(\frac{2B}{Z_1 X^2_{2r}(1-\alpha^*/2)}\,;\ \frac{2B}{Z_1 X^2_{2r}(\alpha^*/2)}\right) \tag{25}$$

where $X^2_{2r}(\alpha^*/2)$ is a quantile of the chi-square distribution given by $P\{X^2_{2r} \leq X^2_{2r}(\alpha^*/2)\} = \frac{\alpha^*}{2}$.

6. AN EXAMPLE

Certain capacitors are known to have an exponential distribution under all conceivalle environments. It is desired to predict the mean life of such a device at the usual temperature of $30^{o}C$ and voltage of 100. An accelerated life test is conducted by using voltage in conjunction with temperature as the accelerating mechanism. Ten capacitors are tested at each voltage-temperature combination, and each test is terminated after six failures are observed. In Table 1, we have 12 combinations of the stresses Z_i and V_i, $i = 1,2,\ldots,12$ where ten capacitors ($n_i = 10$) are tested at each voltage-temperature combination and each test is terminated after 6 failures are observed ($r_i = 6$). We also have in Table 1 the maximum likelihood estimators for $\theta_i = 1/\lambda_i$ for each combination of the stresses Z_i and V_i.

TABLE 1 - LIFE TIMES OF CAPACITORS (DATA WITH '+' ARE CENSORED).

i	Z_i	V_i	$\hat{\theta}_i$	DATA t_{ji}, $j=1,\ldots,10$; $i=1,\ldots,12$
1	30	100	1500	400,600,650,700,800,1170,1170+,1170+,1170+,1170+
2	30	200	1400	300,450,500,600,620,1186,1186+,1186+,1186+,1186+
3	30	300	1100	250,300,400,500,800,1320,1320+,1320+,1320+,1320+
4	30	400	1000	200,300,350,500,650, 800, 800+, 800+, 800+, 800+
5	40	100	1400	400,500,600,650,800,1090,1090+,1090+,1090+,1090+
6	40	200	1100	200,300,350,500,700, 910, 910+, 910+, 910+, 910+
7	40	300	1000	250,300,350,500,700, 780, 780+, 780+, 780+, 780+
8	40	400	900	150,250,380,400,500, 744, 744+, 744+, 744+, 744+
9	50	100	1100	180,260,350,480,750, 916, 916+, 916+, 916+, 916+
10	50	200	1000	200,280,320,460,780, 792, 792+, 792+, 792+, 792+
11	50	300	900	150,220,310,470,680, 714, 714+, 714+, 714+, 714+
12	50	400	800	120,180,280,360,620, 648, 648+, 648+, 648+, 648+

Assuming that the generalized Eyring model is appropriate for the data of Table 1, we can estimate the parameters α, β, γ and δ of (3) using the least-squares method. From (6), we have:

$$\log\left(\frac{\hat{\lambda}_i}{Z_i}\right) = \alpha_1 + \alpha_2 V_i + \alpha_3 \frac{V_i}{K_1 Z_i} - \alpha_4 \frac{1}{K_1 Z_i} + \varepsilon_i \qquad (26)$$

where $\hat{\lambda}_i$ are the maximum likelihood estimators of $\lambda_i = 1/\theta_i$, $i=1,2,\ldots,12$, $\alpha_1 = \log\alpha$, $\alpha_2 = \gamma$, $\alpha_3 = \delta$ and $\alpha_4 = \beta$. The least-squares estimators $\hat{\alpha}_1$, $\hat{\alpha}_2$, $\hat{\alpha}_3$ and $\hat{\alpha}_4$ (see Table 2) have an asymptotical normal distribution $N\{\underset{\sim}{\alpha};\ \sigma^2(X'X)^{-1}\}$, where,

$$
(X'X)^{-1} = \begin{bmatrix} 11.772 & -0.039 & 1.986\times10^{-16} & -5.958\times10^{-14} \\ & 0.0002 & -7.943\times10^{-19} & 1.986\times10^{-16} \\ & & 4.198\times10^{-33} & -1.050\times10^{-30} \\ \text{symmetric} & & & 3.149\times10^{-28} \end{bmatrix} \quad (27)
$$

The estimator of σ^2 based on the residual sum of squares (RSS) is given by $s^2 = RSS/(K-4) = 0.0128/8 = 0.0016$ (see for example, Draper and Smith, 1981). In Table 2, we have the least-squares estimators and 95% confidence intervals for the parameters α, β, γ and δ.

TABLE 2 - LEAST-SQUARES ESTIMATORS AND CONFIDENCE INTERVALS FOR α, β, γ AND δ.

i	LSE FOR α_i	LSE FOR $\alpha,\beta,\gamma,\delta$	95% CONFIDENCE INTERVALS FOR $\alpha,\beta,\gamma,\delta$
1	$\hat{\alpha}_1 = -11.282$	$\hat{\alpha} = e^{\hat{\alpha}_1} = 1.259\times10^{-5}$	(0.0000092; 0.0000173)
2	$\hat{\alpha}_2 = 0.000603$	$\hat{\gamma} = \hat{\alpha}_2$	(-0.0005546; 0.0017614)
3	$\hat{\alpha}_3 = 3.75\times10^{-18}$	$\hat{\delta} = \hat{\alpha}_3$	$(-2.2385\times10^{-18}; 9.7389\times10^{-18})$
4	$\hat{\alpha}_4 = -1.593\times10^{-15}$	$\hat{\beta} = \hat{\alpha}_4$	$(-3.2333\times10^{-15}; 0.0469\times10^{-15})$

With the least-squares estimators for α_1, α_2, α_3 and α_4, we find an estimator for $\lambda_1 = Z_1 \exp\{\alpha_1 + \alpha_2 V_1 + \frac{\alpha_3 V_1}{K_1 Z_1} + \alpha_4 \frac{1}{K_1 Z_1}\}$ where $Z_1 = 30$ and $V_1 = 100$, given by $\hat{\lambda}_1 = 0.000645$. Thus, an estimator of the mean life time of the components at the usual level stresses is given by $\hat{\theta}_1 = 1/\hat{\lambda}_1 = 1549.19$.

From (7), we find a 95% confidence interval for λ_1 given by (0.000574; 0.000726). Thus, an approximated 95% confidence interval for $\theta_1 = 1/\lambda_1$ is given by (1376.71; 1743.27).

Observe that this confidence interval for θ_1 is calculated considering the regression model (4) which considers $\log(\hat{\lambda}_i)$ as the response variable where $\hat{\lambda}_i$ is the maximum likelihood estimator for λ_i in each combination of stresses and a multiplicative error model $\hat{\lambda}_i = \lambda_i \varepsilon_i$. Therefore, a Bayesian analysis of the generalized Eyring model (3) can be of great practical interest.

In Figure 1, we have contour plots of the joint marginal posterior density (14) for γ and δ approximated by the Laplace's method. The mode of this posterior density is given by $\hat{\gamma}^* \cong 0.00000745$ and $\hat{\delta}^* \cong 7.332 \times 10^{-18}$.

Assuming $\gamma = 0.00000745$ and $\delta = 7.332 \times 10^{-18}$ known, we have in Figure 2 the contours for the posterior density for α and β given in (16).

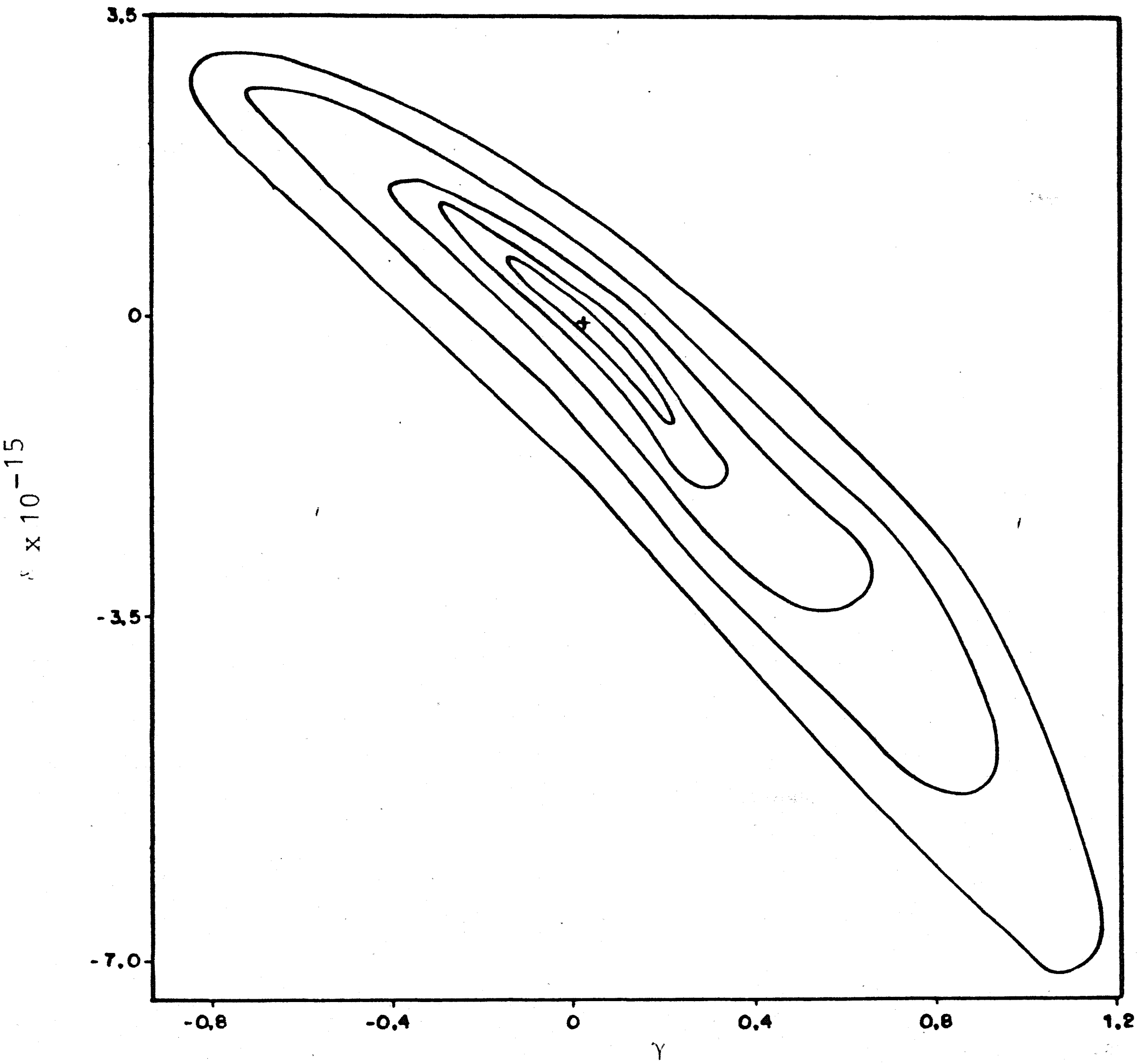

FIGURE 1 - Contours for the Posterior Density of γ and δ.

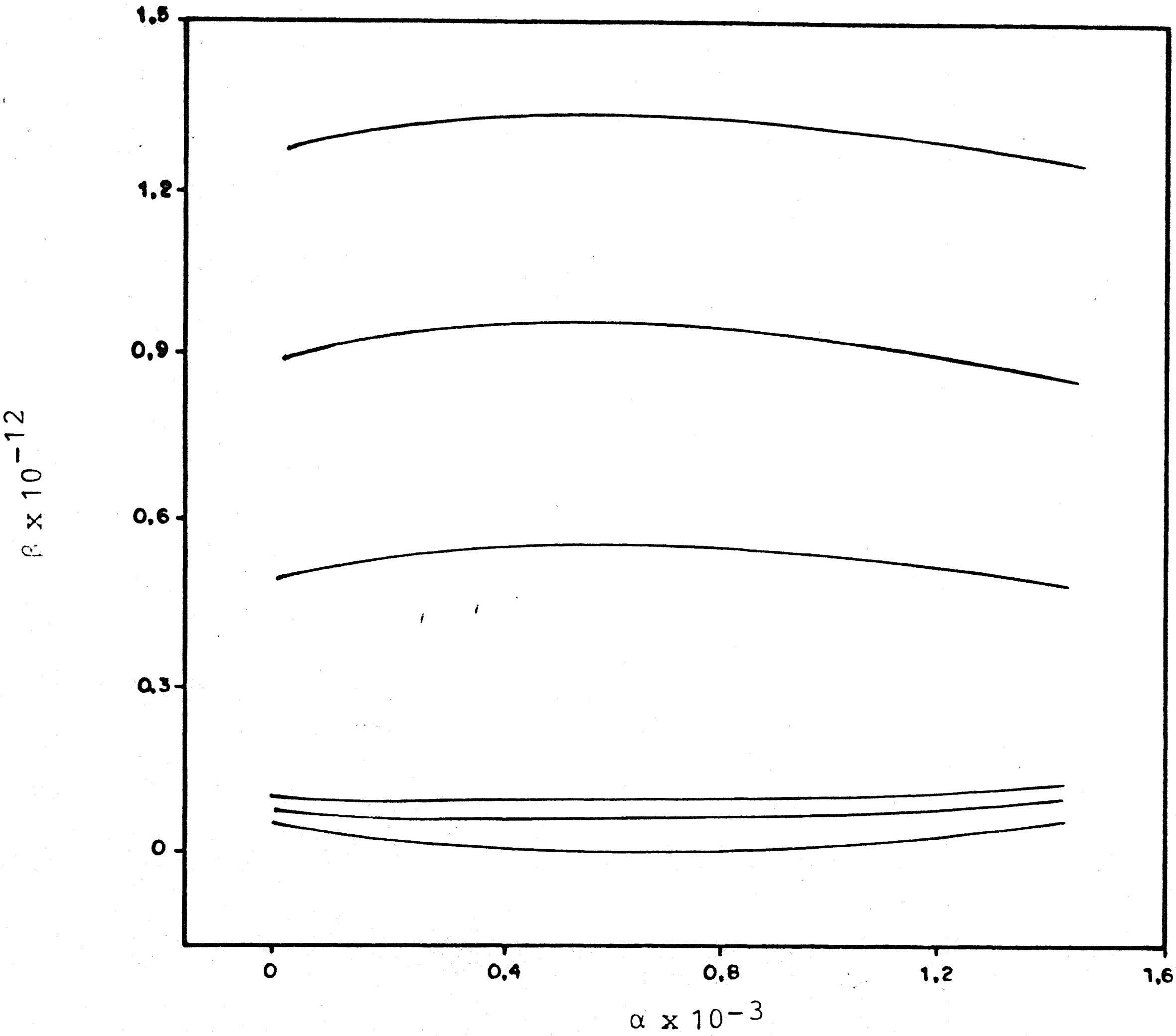

FIGURE 2 - Contours for the Posterior Density of α and β.

In Figure 3, we have the plot of the marginal posterior density for β given in (17) assuming γ and δ known. The mode of this posterior density is given by $\hat{\beta}^* = -0.683 \times 10^{-15}$.

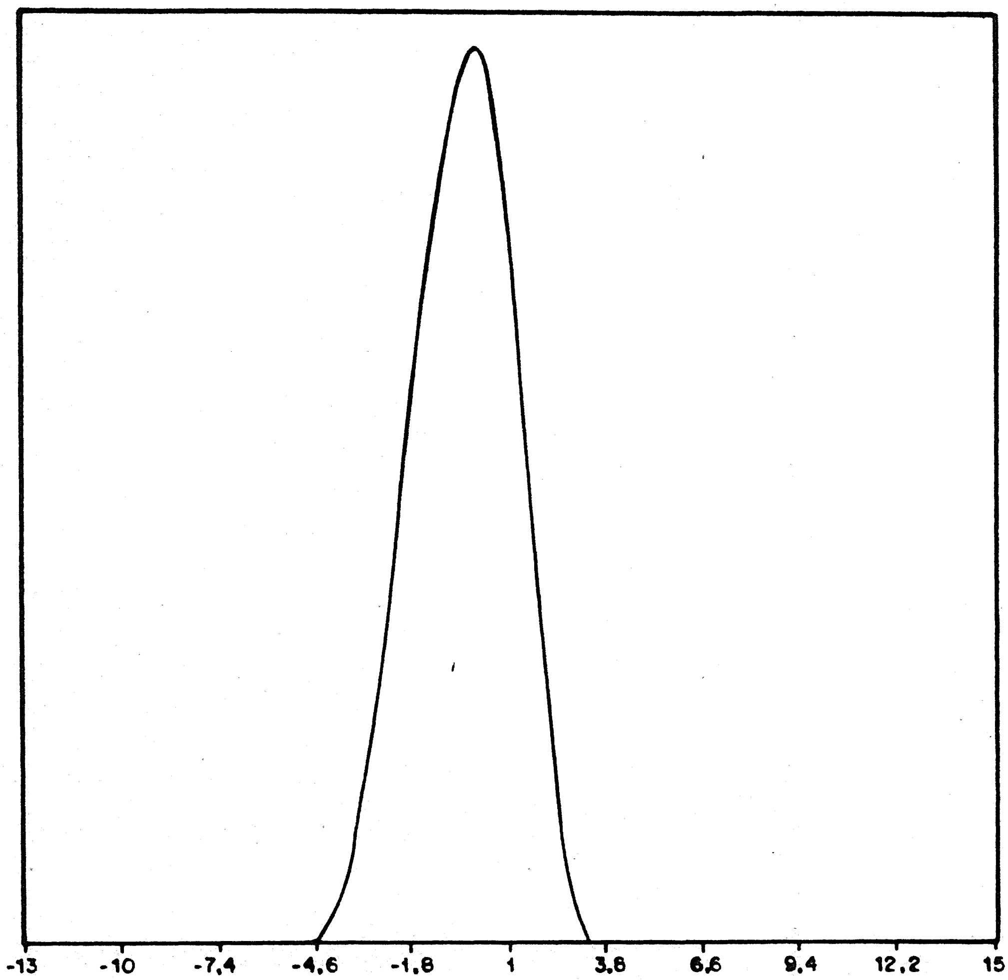

FIGURE 3 - Posterior Density for β Assuming γ and δ known.

In Figure 4, we have the plot of the posterior density for θ_1 given in (21) assuming $\gamma = 0.00000745$ and $\delta = 7.332 \times 10^{-18}$ known and approximated by the Laplace's method. The mode of this posterior density is given by $\hat{\theta}_1^* = 1553.33$. A 95% HPD interval for θ_1 is given by approximately (1200; 2200) (see Figure 4) which is different of the 95% confidence interval (1376.71;1743.27) considering (7).

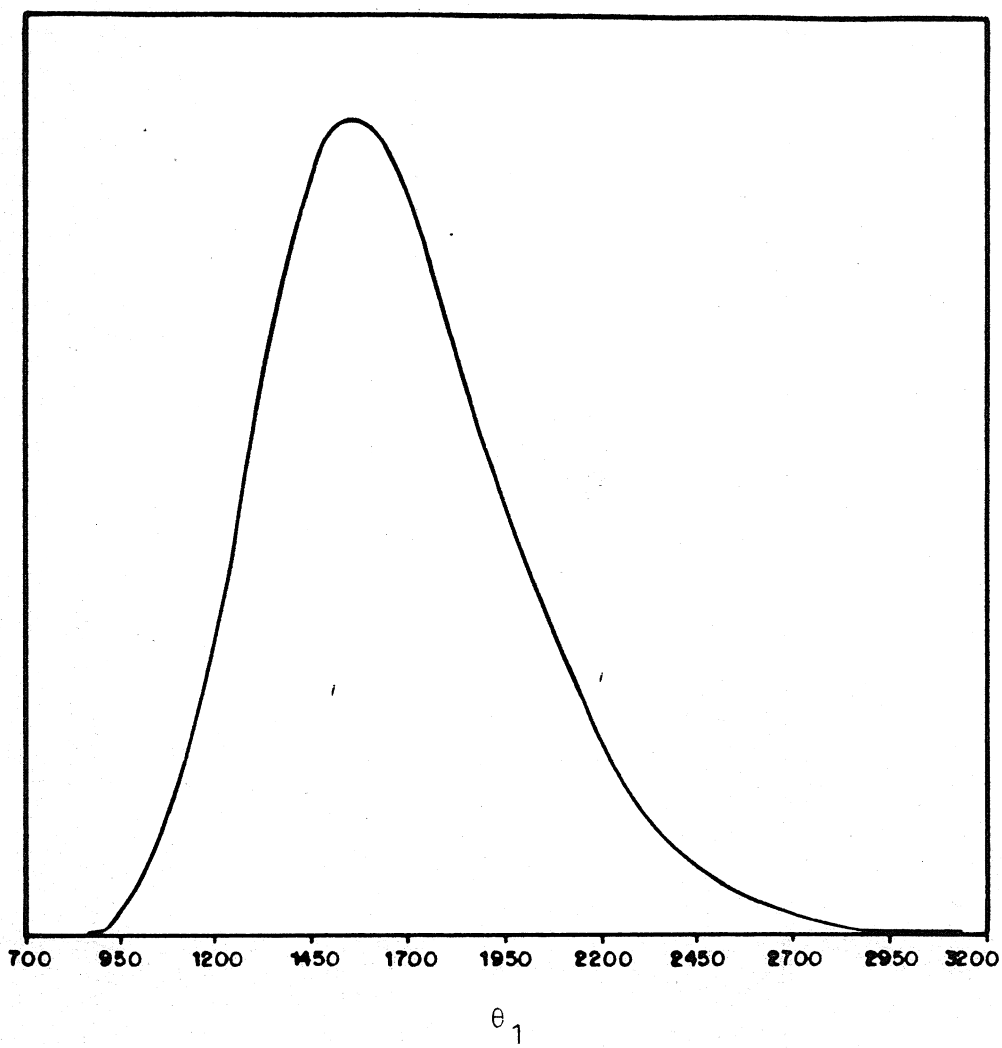

FIGURE 4 - Posterior Density for θ_1 Assuming γ and δ known.

Assuming $\gamma = 0.00000745$, $\delta = 7.332 \times 10^{-18}$ and $\beta = -0.683 \times 10^{-15}$ known, we have in Figure 5, the plot of the posterior density for α given in (23). The mode of this posterior density is given by $\hat{\alpha}^* = 1.444 \times 10^{-5}$.

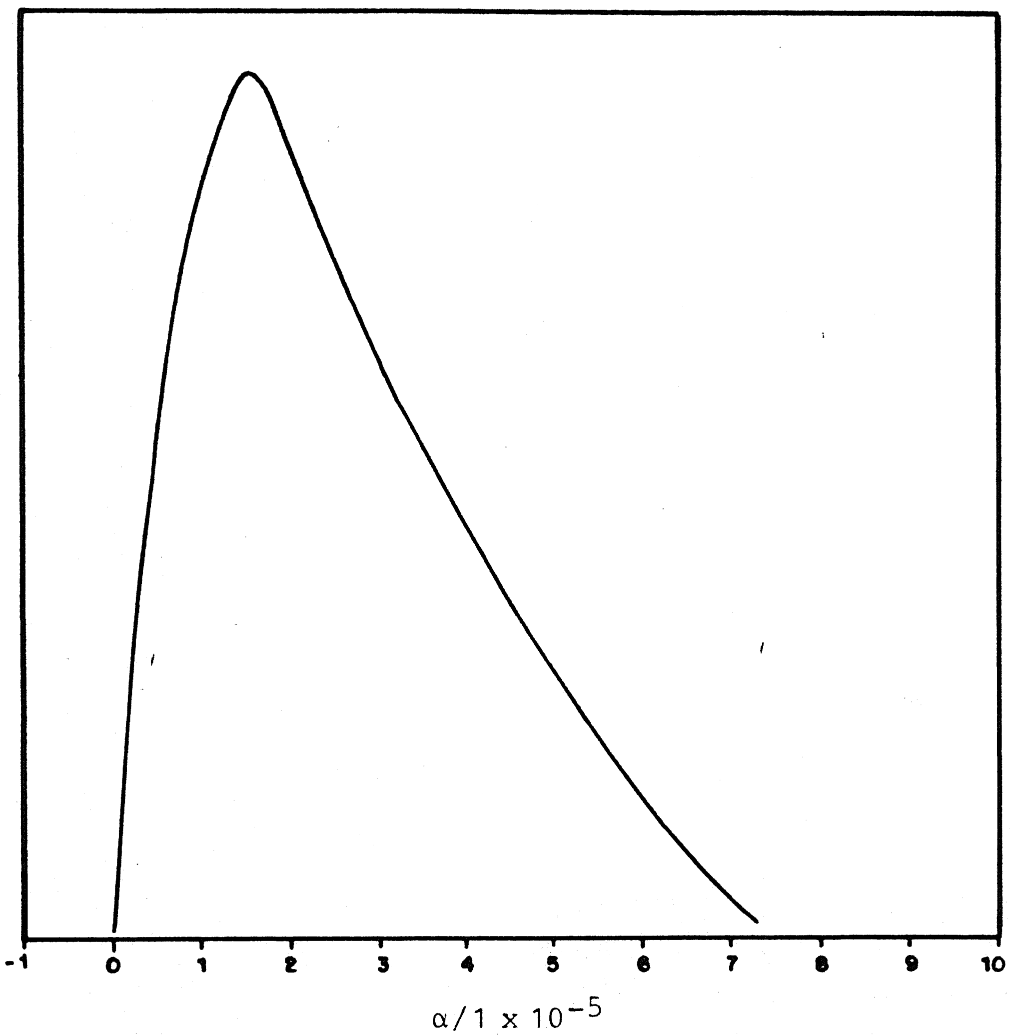

FIGURE 5 - Posterior Density for α Assuming γ, δ and β known.

In Figure 6, we have the plot of the posterior density for θ_1 assuming γ, δ and β known. The mode of this posterior density is given by $\hat{\theta}_1^* = 1593.125$ and a 95% HPD interval (see (25)) for θ_1 is given by (1302.15;2072.75).

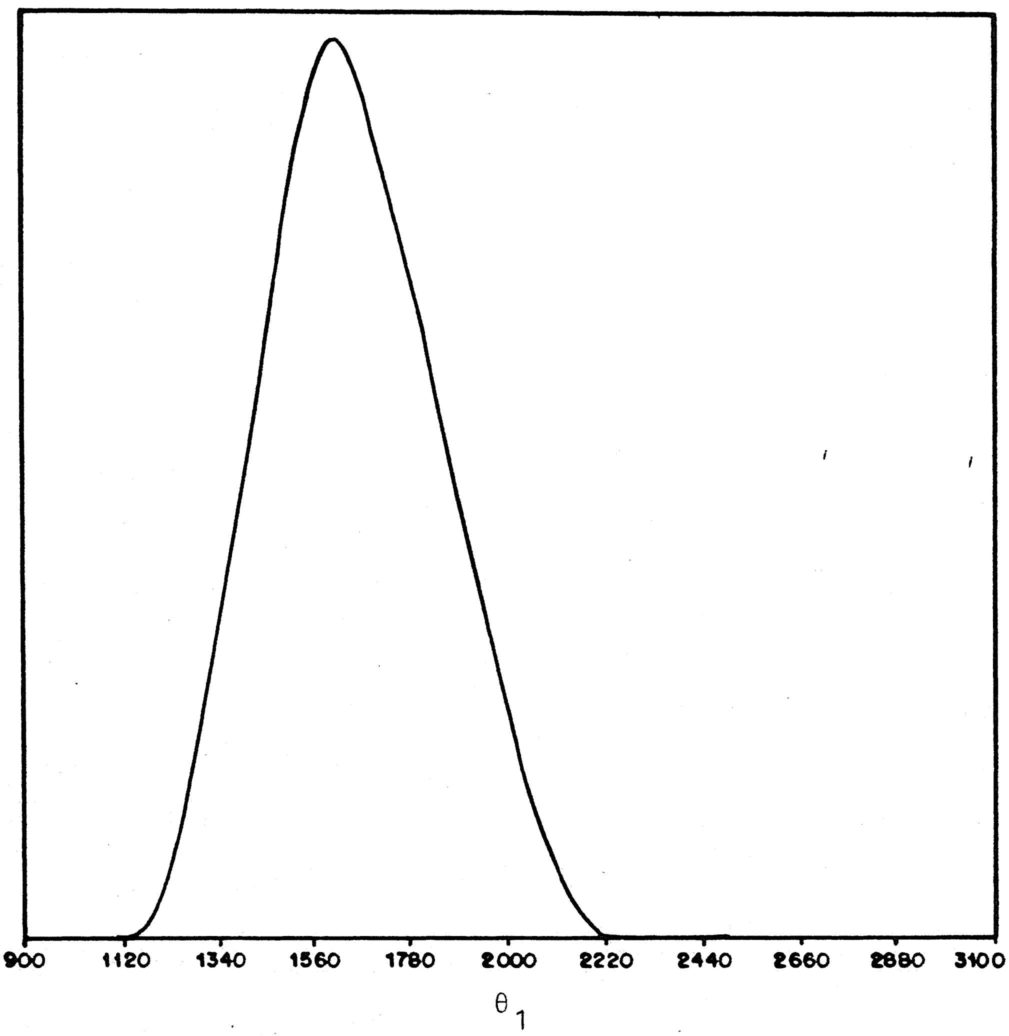

FIGURE 6 - Posterior Density for θ_1 Assuming γ, δ and β known.

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