UNIVERSIDADE DE SÃO PAULO

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Nº 71

NOTAS

Instituto de Ciências Matemáticas de São Carlos

ISSN 0103-2577

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N O T A S   D O   I C M S C

São Carlos (SP)
1990

ON THE DERIVATION OF THE STOCHASTIC PROCESSES ASSOCIATED TO LIE-ISOTOPIC GAUGE THEORY

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Abstract: We give the construction of a relativistic extension of Stochastic Mechanics starting from the conformal Riemann-Cartan structures on space-time of the Lie-isotopic lift of gauge theory. We obtain a geometrical-gauge basis for the theory of Dirichlet operators in potential theory and Quantum Mechanics and for Stochastic Mechanics.

INTRODUCTION

The notion of Lie isotopy lift of a Lie group was proposed by Santilli [1], and subsequently became the subject of fundamental research by Santilli, Myung and others(see these Proceedings). Gasperini [2] proposed its extension to gauge theories, identifying the underlying geometry as the Riemann-Cartan structures generated by a scale field which defines the Lie isotopic unit. It was expanded by Gasperini in numerous articles [3], and by the author and Tilli from the point of view of conformal structures [4], from which a principle of unification of the electromagnetic, weak and gravitational interactions was obtained. For an excellent partial review of this and more see [25].

Unrelated to this, Nelson has derived non-relativistic Quantum Mechanics from the theory of Markov processes, and the subsequent theory, Stochastic Mechanics, has been related to fundamental developments in Quantum Field Theory, in what is known today as Constructive Field Theory [5].

Yet, in Nelson's theory as well as in the fundamental studies of the theory of Markov processes in general manifolds, there is lacking a fundamental geometrical principle from which these theories should naturally stem from. Even though the necessity of the general theory of connections due to Cartan has been identified [6,7,24].

In this article we shall provide this unifying geometrical principle, as a theory of the gauge conformal Riemann-Cartan structures, being their associated wave operators, the infinitesimal generators of the Markov processes in space-time. This point of view is completely new to our present knowledge.

Therefore, this principle should provide for a fundamental geometrical and gauge perspective which has been lacking in Quantum Field Theory.

In this article, we shall present this principle, and we shall sketch the construction of the associated stochastic processes in space-time.

I. THE RIEMANN-CARTAN-WEYL GEOMETRY OF LIE-ISOTOPIC GAUGE THEORY

We shall assume from now on, otherwise stated, that all geometrical structures are infinitely differentiable, and that space-time M has dimension equal to 4.

The Riemann-Cartan geometry on space-time appears from a reduction of the bundle of Poincaré frames over space-time to that of the Lorentz frames.

The fundamental geometrical object is the Cartan soldering form θ, a Lorentz (or O(4)) invariant R^4-valued one-form on M, which allows for smooth identification of T_xM, the tangent space at $x \in M$, and the homogeneous space R^4 given by the quotient P/L, where P and L are the Poincaré and Lorentz (or O(4)) respectively. This soldering form is attributed the description of the gravitational field [2,3,4,20,21,22]. Thus, θ gives a tetrad field $\theta^a_\alpha \, dx^\alpha$, with *inverse* $e^\alpha_a \, \partial/\partial x^\alpha$ in a coordinate system (x^α) of M, where α and a= 1,...,4, with a representing the indices of an anholonomic basis in R^4. If $g = (g_{ab})$ denotes a metric on R^4, we can define a metric on M, by

$$g_{\alpha\beta} = g_{ab} \, \theta^a_\alpha \, \theta^b_\beta \qquad (I.1)$$

with the same signature of that of g. This is the *Lie-isotopic lift* of the metric g. If we have a Lorentz or orthogonal linear connection on R^4, $\Gamma = (\Gamma^{ab}_\mu)$, then Γ is skew-symmetric in a, b. From Γ we can define the space-time linear connection

$$\Gamma^\alpha_{\beta\mu} = e^\alpha_a \theta^b_\beta \Gamma^a_{b\mu} + e^\alpha_a \partial_\mu \theta^a_\beta \qquad (I.2)$$

which is metric-compatible, i.e. if ∇ denotes the exterior covariant differential with respect to the linear connection defined by (I.2), then

$$\nabla_\alpha \, g_{\beta\mu} = 0. \qquad (I.3)$$

This means that θ has reduced the bundle of linear frames to the orthogonal bundle (this is of great importance in assuring the *strong* Markov property for the stochastic processes we shall construct below).

What is essential to the connection on M defined by (I.3), is its non-symmetric character, i.e. it has a non-zero torsion tensor

$$T^{\alpha}_{\mu\nu} = 1/2\ (\Gamma^{\alpha}_{\mu\nu} - \Gamma^{\alpha}_{\nu\mu}) \quad . \qquad (I.4)$$

This geometry is called the Riemann-Cartan (RC) structure. It is well known to be the geometry subjacent to the Lie-isotopic lift of gauge theory [2,3]. In the latter, the identity I of the algebra of the local Lie-group is modified, to yield a scale-dependant local unit of the form ψI, where ψ is a function on M.

Let us introduce a conformal structure on the tangent space of M.

We define the Weyl transformation on the soldering form by

$$W(\theta^{a}_{\alpha}) = \psi\theta^{a}_{\alpha}, \qquad (I.5.i)$$

so that $W(e^{\alpha}_{a}) = (1/\psi)e^{\alpha}_{a}$, and a Weyl transformation on Γ (which by abuse of notation we denote by W as well as for the other derived transformations)

$$W(\Gamma^{a}_{b\mu}) = \Gamma^{a}_{b\mu} \qquad (I.5.ii).$$

If we assume that the original metric on $\mathbb{R}^4$ is scale-invariant, then we can *derive* the following transformation on the metric on M

$$W(g_{\alpha\beta}) = \psi^2 g_{\alpha\beta}. \qquad (I.6)$$

These are the well known conformal transformations of the *metric* on M. In the above definitions, ψ is a function defined on M with values on $\mathbb{R}^+$ smooth but for its singular domain (*node set*) $\{x: \psi(x) = 0\}$, which is closed in M.

The Riemann-Cartan structure, under the above transformations becomes

$$W(\Gamma^{\alpha}_{\beta\mu}) = \Gamma^{\alpha}_{\beta\mu} + \delta^{\alpha}_{\beta}\partial_{\mu}\ln\psi - g_{\beta\mu}g^{\gamma\alpha}\partial_{\alpha}\ln\psi \qquad (I.7)$$

with torsion tensor

$$W(T^{\alpha}_{\beta\mu}) = \delta^{\alpha}_{\beta}\partial_{\mu}\ln\psi - \delta^{\alpha}_{\mu}\partial_{\beta}\ln\psi \quad . \qquad (I.8)$$

This shows that only the trace of the torsion tensor: $Q = Q_{\mu}dx^{\mu}$ of the original connection is transformed as $W(Q) = Q - 3/2d\ln\psi$.

The fact that is to be remarked is that one could in principle start with a *torsionless and flat* connection, say $g = \mathrm{diag}(\pm 1,1,1,1)$, $(\theta^{a}_{\alpha}) = (\delta^{a}_{\alpha})$, and through a choice of conformal tetrads (I.5.i), we henceforth introduce a RC structure. It is important to notice that this also introduces a *metric compatible* connection. It is given by (we normalize the 3/2 factor above)

$$\Gamma^{\alpha}_{\beta\mu} = \{^{\alpha}_{\beta\mu}\} + 2/3\ (\delta^{\alpha}_{\beta}\ \partial_{\mu}\ \ln\psi - g_{\beta\mu}g^{\gamma\alpha}\partial_{\gamma}\ \ln\psi\ ) \qquad (I.9)$$

where $\{^{\alpha}_{\beta\mu}\}$ are the coefficients of the Levi-Civita connection associated to the metric defined by (I.1). Then, $Q = d\ln\psi$, the logarithmic differential of the scale field ψ, is a Weyl one-form of a RC metric compatible structure.

. This is in contradistinction with the usual Weyl-geometry, where the Weyl form precisely measures the non compatiblity with the metric of the Weyl geometry produced by the transformations *on the metric*.

Therefore, this geometry, which we shall call of Riemann-Cartan-Weyl (RCW) has *no historicity problem*, which moved Einstein to reject Weyl's attempt to construct the first gauge theory in which he associated the Weyl form to the electromagnetic field [8].

In our previous work [4], we have investigated this geometry as a natural principle for unification of the gravitational, electromagnetic and weak interactions.

We shall proceed to sketch in this article, the construction of Quantum Mecanics from this geometry, in the frame of Stochastic Mechanics. For this, we need to characterize the infinitesimal generators of the stochastic processes from which we intend to derive Quantum Mechanics.

We shall study then, the wave operator associated to the RC structures.

Henceforth, in this paragraph, the dimension of M will be arbitrary. Let ω be an arbitrary p-form on M. Locally

$$\omega = 1/p!\ \omega_{\alpha_1 \ldots \alpha_p} dx^{\alpha_1 \ldots \alpha_p}$$

Then,

$$\nabla\omega = 1/(p+1)!\ \left(\nabla_{\alpha}\omega_{\alpha_1\ldots\alpha_p} - \sum_{k=1}^{p} \nabla_{\alpha_k}\ \omega_{\alpha_1\ldots\hat{\alpha}_k\ldots\alpha_p} \right) dx^{\alpha}\wedge dx^{\alpha_1}\wedge\ldots\wedge dx^{\alpha_p}$$

(where $\hat{\alpha}_k$ denotes omission of α_k) is a (p+1)-form, which can be uniquely decomposed as $\nabla_g\omega + d_c\omega$, where ∇_g denotes the covariant derivative with respect to the Levi-Civita connection associated to g , and

$$d_c\omega = 2/(p+1)! \left[\sum_{m=1}^{p} \omega_{\alpha_1\ldots\alpha_{m-1}\beta\alpha_{m+1}\ldots\alpha_p}\ T^{\beta}_{\alpha\alpha_m} + \right.$$

$$+ \sum_{\alpha_t<\alpha_k}^{p} \omega_{\alpha_1\ldots\alpha_{t-1}\,\beta\alpha_{t+1}\ldots\alpha_{k-1}\gamma\alpha_{k+1}\ldots\alpha_p} T^{\beta}_{\alpha_t\alpha_k}\Big] dx^{\alpha}\wedge dx^{\alpha_1}\wedge\ldots\wedge dx^{\alpha_p} .$$

The covariant codifferential $\delta\omega$ of ω, is a (p-1)-form given by

$$\delta\omega = -1/(p+1)!\ g^{\beta\alpha}\nabla_{\beta}\omega_{\alpha\alpha_2\ldots\alpha_p} dx^{2}\wedge\ldots\wedge dx^{p}.$$

The covariant codifferential of a function is defined to be zero.

We define the wave operator $\mathcal{L}$ of the RC structure as

$$\mathcal{L} = \delta\nabla + \nabla\delta \qquad (I.10)$$

Hence, if ϕ denotes a function on M, $\mathcal{L}(\phi) = \delta\nabla\phi$, which can still be written as $\mathcal{L}(\phi) = \delta d\phi$.

δ can be defined from $*$, the (extension) of the duality Hodge operator

$$\delta\omega = (-1)^{p} *^{-1} \nabla *\omega \qquad (I.11)$$

($*^{-1} = (-1)^{p(n-p)}*$), where

$$*\omega = 1/(n-p)!\ e_{\alpha_1\alpha_2\ldots\alpha_{n-p}\beta_1\ldots\beta_p} Q^{\beta_1\ldots\beta_p} \qquad (I.12)$$

with $e_{\alpha_1\ldots\alpha_p}$ being the covariant components of the unit tensor field and

$$Q^{\beta_1\ldots\beta_p} = g^{\beta_1\gamma_1}g^{\beta_2\gamma_2}\ldots g^{\beta_p\gamma_p}\omega_{\gamma_1\ldots\gamma_p} \qquad (I.13)$$

are the components of the conjugate tensor field Q of ω. For example, if $\pi = \pi_{\alpha}dx^{\alpha}$ is an arbitrary one-form, its conjugate vector-field is $\pi^{\beta}\partial_{\beta}$, with $\pi^{\beta} = g^{\beta\gamma}\pi_{\gamma}$.

Returning to M four-dimensional, if we take now the 3-form $\omega = *\pi = 1/6\ e_{\alpha_1\ldots\alpha_4}\pi^{4}$, we obtain

$$\nabla\omega = -\left[\partial_{\alpha}|\det g|)^{1/2}\pi^{\alpha} + 2|\det g|\pi^{\alpha}T^{\beta}_{\alpha\beta}\right]dx^{1}\wedge\ldots\wedge dx^{4}.$$

Thus, the generalized divergence, $\mathrm{Div}(\pi)$ of π is a scalar-form given by

$$\mathrm{Div}\ \pi = |\det g|^{-1/2}\partial_{\alpha}(\ |\det g|^{-1/2}\pi^{\alpha}) + 2\ T^{\beta}_{\alpha\beta}\pi^{\alpha}.$$

Finally, the generalized Laplacian of a function ϕ, is given by

$$\mathcal{L}(\phi) = -\mathrm{Div}\ d\phi = -\left[|\det g|^{-1/2}\partial_{\alpha}(|\det g|^{-1/2}g^{\alpha\beta}\partial_{\beta}\phi) + 2g^{\alpha\gamma}T^{\beta}_{\alpha\beta}\partial_{\gamma}\phi\right] \qquad (I.14)$$

which, by definition, is invariant; we note, that only the torsion-trace one-form is involved in the second term, and that the first one is nothing else than the usual Laplace-Beltrami operator Δ_g associated to the space-time metric applied to ϕ. $\mathcal{L}(\phi)$ can be written in the simpler form

$$\mathcal{L}(\phi) = -(\Delta_g + 2Q^{\gamma}\partial_{\gamma})(\phi), \qquad (I.15)$$

which for the RCW structure defined by the choice of a conformal structure defined by ψ, is

$$\mathcal{L}_{\psi}(\phi) = -(\Delta_g + 2\partial^{\gamma}(\ln \psi)\partial_{\gamma})(\phi) \qquad (I.15)$$

where we haven written $\mathcal{L}$ as $\mathcal{L}_{\psi}$ to stress its dependance in the *Lie-isotopic element* ψ defining the RCW conformal structure. Therefore, the wave operator for the RCW structures has, additional to the usual propagation term, a coupling term which is a first-order operator given by the conjugate of twice the Cartan-Weyl form $d\ln\psi^2$. Yet, in Quantum Physics, to define correctly the ground state of the system under consideration, one is interested in Hamiltonean operators given by one-half the Laplacian, so we shall focuse our attention in the operator

$$H_{\psi} = -1/2\ \mathcal{L}_{\psi} = (1/2\ \Delta_g + \partial^{\alpha}(\ln \psi)\partial_{\alpha}) \quad . \qquad (I.16)$$

This natural renormalization will turn to be of fundamental importance in the sequel, for the construction of a relativistic extension of Stochastic Mechanics.

II. THE STOCHASTIC PROCESSES ASSOCIATED TO THE RIEMANN-CARTAN-WEYL STRUCTURES OF LIE-ISOTOPIC GAUGE THEORY

In Nelson's classical work of construction of Quantum Mechanics [9,10] from the theory of stochastic processes, specifically, the Markov processes, wave mechanics is constructed in terms of measures in the space of paths on configuration space, in principle R^3 or a three dimensional space manifold provided with a Levi-Civita connection. So, in principle, it is a non-relativistic theory, in which time is associated to the evolution parameter of the quantum system. The system itself is described by a Markov diffusion process $\xi = \xi(t)$, $t \in \mathbb{R}$, which is constructed from the assumptions of an initial probability density for ξ, and the infinitesimal generator of the process, a second order differential operator described by a Hamiltonean of the form $H = 1/2\Delta_g + b$, where b is a first-order differential operator with coefficients which depend both in the space and time coordinates. In the theory of stochastic processes, the Laplace-Beltrami operator is the generator of the inherently stochastic character of the process (if $b \equiv 0$, it is a process described by the wellknown Wiener measure) and b is the drift or velocity vector-field, thought as a classical contribution to the dynamics of the system, perturbed by the stochastic term. In Nelson's theory, b is *unrelated to any geometrical structure*, and it is assumed as a separate data for generation of the process. It is essential for his construction of the wave functions, that b be *assumed* to be a *gradient* vector field. All further developments of Stochastic Mechanics (SM), Guerra's et al., Albeverio and Høegh-Krohn's program of constructive Quantum Field Theory carry this imprint [11,12].

At this stage of our presentation, it is seems natural for us to pause to reflect to the fact that on our construction of the RCW structures we have a conceptually richer structure than the usual one contained in the known construction of SM.

Indeed, if we take for infinitesimal generators of the stochastic processes, the Hamiltonean operators H_ψ constructed from the wave operators of the RCW structures we have: firstly, Minkowski or Euclidean time is incorporated to the configuration space, and a priori should not be confused with the evolution parameter of the stochastic processes (which we shall denote from now on as τ, with $\tau \in [0,\infty)$; this proper-time parameter should be thought as a Kalutza-Klein fifth coordinate); secondly, *all* the information of the stochastic processes is contained uniquely in the Hamiltonean operators derived from the RCW structures, and *by construction* the drift is of gradient type (*this*, in fact, was the starting point of this research).

Certainly, b coincides with the conjugate vector field to $d\ln\psi$.

There is no accident to this remarkable perfect matching.

The key idea of construction of SM is that of parallel transport of a diffusion process, so the question of the linear connection to describe this parallel transport *is* the central issue. The choice of a Levi-Civita connection not only is of negative character (a torsionless connection), but also *chooses* a preferred system of coordinates, the so called *normal coordinates*, for which a well known theorem of Differential Geometry assures that a general linear connection reduces to a Levi-Civita one. In other words, this special type of parallel transport conflicts with the principle of relativity. So, whatever one has to define as a parallel transport, be this along a continuous but nowhere differentiable path as in Brownian motion, or along a smooth path as in Differential Geometry, it appears that the RCW structures will *precisely* be the natural ones defining the required notion of parallel transport. What one needs is to take in account the locally infinite variation of the paths of Brownian motion, i.e. a specific set of rules of calculation which will do the job of the usual differential calculus on manifolds. This is the so-called *Ito's stochastic calculus* [6,13], which we shall omit due to limitations of space.

The stochastic processes we shall build, constitute a class characterized by *both* the Markov property and the continuity of its trajectories in M; they are usually called *diffusion processes*. We shall give a formal (but incomplete) description of them. A more complete formal treatment of some of its aspects, can be extracted from [14], and a forthcoming paper by the author.

Let M be the one-point compactification of a topological space (one usually aggregates a terminal point at infinity for the case that the diffusion explodes, but for lack of space we shall not explicit this possibility). Let $W(M)$ be the set of all continuous functions $w : \mathbb{R} \longrightarrow M$. A *Borel cylinder set* in $W(M)$ is defined for a sequence of positive real numbers $\tau_1<\tau_2<\ldots<\tau_n$ and a Borel subset A in $M^n = M\times\ldots\times M$ n times as $\pi^{-1}_{\tau_1,\ldots,\tau_n}(A)$, where $\pi_{\tau_1,\ldots,\tau_n}(w) = (w(\tau_1),\ldots,w(\tau_n))$. We recall that a Borel subset in M is any set in the smallest σ-field containing all open sets. Let $\mathcal{B}(W(M))$ be the σ-field generated by all cylinder sets, and let $\mathcal{B}_\tau(W(M))$ be the σ-field generated by all cylinder sets up to time τ. A family of probabilities $\{P_x,\ x \in M\}$ on $(W(M),\mathcal{B}(W(M)))$ is called a *Markovian system* if it verifies the following conditions

i) $P_x\{w:\ w \in W(M),\ w(0) = x\} = 1\ ,\ \forall\ x \in M$

ii) $M \ni x \longrightarrow P_x(A)$ is Borel measurable, for each $A \in \mathcal{B}(W(M))$, and

iii) $\forall\ \tau\geq s$, $A \in \mathcal{B}_s(W(M))$, and Γ a Borel subset in M,

$$P_x(A \cap \{w:\ w(\tau) \in \Gamma\}) = \int_A P_{w'(s)}\{w:\ w(\tau-s) \in \Gamma\}\ P_x(dw'),\ \forall\ x \in M.$$

We set $P(\tau,x,\Gamma) = P_x\{w:\ w(\tau) \in \Gamma\}$. The family $\{P(\tau,x,\Gamma)\}$ is called the *transition probability of a Markovian system*. By succesive application of iii) we get

$$P_x[w(\tau_1) \in A_1,\dots,w(\tau_n) \in A_n] = \int_{A_1} P(\tau_1,x,dx_1)\times\int_{A_n} P(\tau_2-\tau_1,x_1,dx_2)\times \dots\times\int_{A_n} P(\tau_n-\tau_{n-1},x_{n-1},dx_n)$$

for $0<\tau_1<\dots<\tau_n$, and $A_1,\dots,A_n \in \mathcal{B}(M)$, so we can see that two Markovian systems defined on M with the same transition probability, coincide.

A stochastic process $\xi = (\xi(\tau))$ on M , is a $\mathcal{W}(M)$-valued random variable,(i.e. for fixed $\tau\geq 0$, the map $\mathcal{W}(M)\ni w \longrightarrow \xi(\tau)(w) = \xi(\tau,w)$ is a random variable with values in M), where $(\mathcal{W}(M),\mathcal{B}(\mathcal{W}(M),P)$ is a probability space provided with a probability measure P, is called a *diffusion process*, if there exists a Markovian system $\{P_x,\ x \in M\}$ on $(\mathcal{W}(M),\mathcal{B}(\mathcal{W}(M)))$ such that, for allmost all w, the trajectories $[\tau \longrightarrow \xi(\tau)] \in \mathcal{W}(M)$ and the probability law on $\mathcal{W}(M)$ (i.e., the image measure) of $[\tau \longrightarrow \xi(\tau)]$ coincides with $P_\mu(.) = \int_M P_x(.)\mu(dx)$, where μ is the Borel measure on M defined by $\mu(dx) = P\{w:\ \xi(0,w) \in dx\}$.

We shall say that the operator H is the *infinitesimal generator* of a diffusion process $\xi = (\xi(\tau))$ (or that $\{P_x, x \in M\}$ is *determined* by H), if the *stochastic derivative* Df of any twice differentiable bounded function with bounded derivatives f on M satisfies the condition

$$Df(\tau) = \lim_{h\xrightarrow{+}0} 1/h\ E_{\xi(\tau)}(\ f(\xi(\tau+h)) - f(\xi(\tau))) = (Hf)(\xi(\tau)), \qquad (II.1)$$

where $E_{\xi(\tau)}$ denotes the expected value with respect to $P_{\xi(\tau)}$.

We are interested in M being the one point compactification of space-time and the diffusion processes defined by the infinitesimal generators given by $H = H_\psi = -1/2\ \mathcal{L}_\psi$, which we shall locally write as (since $b^\alpha = \partial^\alpha \ln \psi$)

$$H = 1/2\ a^{\alpha\beta}\partial^2_{\alpha\beta} + \partial^\alpha \ln \psi\ \partial_\alpha. \qquad (II.2)$$

Let $\sigma = (\sigma^\beta_\alpha)$ be a square root of Δ_g, i.e. $x \longrightarrow \sigma(x)$ is continuous and the coefficients $a^{\alpha\beta}(x)$, $x \in M$, are given by $a^{\alpha\beta} = \sigma^\alpha_\gamma\sigma^\beta_\delta g^{\gamma\delta}$. This is a coordinate dependent construction and non-unique; this will not affect the unicity of the diffusion [15].

If ψ is constant, so that the drift is zero, then the diffusion ξ is standard Brownian motion with $P_x = W_x$, the Wiener measure on M starting at x and initial distribution $\mu(dx)$.

Thus, in the general case we might expect a departure of the Gaussian measures one usually encounters in quantum field theory, produced by ψ.

We now consider the following (proper-time τ homogeneous) stochastic differential equation for $\xi = (\xi(\tau))$

$$d\xi^{\alpha}(\tau) = \sigma^{\alpha}_{\beta}(\xi(\tau))dB^{\beta}(\tau) + b^{\alpha}(\xi(\tau))d\tau \quad , \alpha = 1,\dots,4 \qquad \text{(II.3)}$$

where $dB = (dB^{\alpha})$ denotes a Brownian motion on M, so that $E_{\xi(\tau)}(dB^{\alpha}(\tau)) = 0$, and the covariance satisfies $E_{\xi(\tau)}(dB^{\alpha}(\tau).dB^{\beta}(\tau)) = a^{\alpha\beta}d\tau$.

It is a theorem [6], in the case that Δ_g is an elliptic operator, so that g is a Riemannian metric (which we can think of as providing an Euclidean structure on TM), and σ and b are continuous and Lipschitz bounded on M, that for every $x \in M$, there exists a unique solution of (II.3) such that $\xi(0) = x$, and the probability law on $W(M)$ of the diffusion ξ is determined by H. The probability law of $\xi(0)$ coincides with μ, i.e. $P\{\xi(0) \in A\} = \mu(A)$, where μ is the given probability measure on M.

The condition on the metric can be relaxed to g being non-negative definite (still not including the hyperbolic case)[15]. Due to the zeros of ψ, b will be singular, yet the unicity of the diffusion can be constructed in assuming b to be locally bounded [15,16,19].

Thus, we have in principle, a one-to-one correspondance between the RCW structures and the diffusion processes generated by H_{ψ}.

Let $\{P_x: x \in M\}$ be the diffusion process determined by H. The *transition semigroup* T_{τ} of the diffusion generated by H, is defined by

$$(T_{\tau}f)(x) = E_x(f(\xi(\tau)) = \int_M f(\xi(\tau))P_x(dw) \qquad \text{(II.4)}$$

for f a bounded continuous function with continuous bounded derivatives). The function $u = u(\tau,x) = E_x(f(\xi(\tau))$ is the unique solution of the *heat equation*

$$\frac{\partial u}{\partial \tau} = Hu \qquad \text{(II.5)}$$

with initial condition given by $\lim_{\tau\to 0, y\to x} u(\tau,y) = f(x)$. It can be proved that T_{τ} and H commute: $T_{\tau}H = HT_{\tau}$, so that we can write $T_{\tau} = e^{\tau H}$, which, due to the Markov property of the diffusion, is a semigroup of operators: $T_{\tau+\tau'} = T_{\tau}T_{\tau'}$.

If $H = H_{\psi} = -1/2\mathcal{L}_{\psi}$, we obtain a family of *semigroups of Schrödinger operators* associated to the RCW structures defined by the choice of the conformal class defined by ψ.

In Quantum Physics, in the functional integral point of view, one is interested in the transition semigroups defined by infinitesimal generators of the form $1/2\Delta_g + V$, where V denotes a multiplication operator by a potential function, which one expresses through a Feynman-Kac formula.

In fact, the theory of diffusion processes assures that the potential perturbative factor can be assimilated to the diffusion process defined by $1/2\Delta_g$, as an exponencial factor in the path-integral representation of the Schrödinger operator, in terms of the Wiener measure [5,19].

There is a similar path-integral representation in the case of the general diffusion processes defined by the RCW structures. This is done through the so-called *Girsanov-Martin-Cameron transformation*; we shall present it below.

Returning to the problem of construction of the diffusion processes, we shall say that a Borel measure $\mu(dx)$ on M is an *invariant measure* by the diffusion process defined by H, if for $\tau\geq0$, and f bounded continuous on M

$$\int(T_\tau f)(x)\mu(dx) = \int f(x)\mu(dx) \qquad (II.6)$$

The diffusion $\{P_x,\ x \in M\}$ will be called *symmetrizable*, if there exists a Borel measure $\nu(dx)$ on M such that for any f,g bounded continuous on M

$$\int(T_\tau f)(x)g(x)\nu(dx) = \int f(x)(T_\tau g)(x)\nu(dx). \qquad (II.7)$$

It is easy to see that the measure for which the diffusion process is symmetrizable, is invariant.

It can be proved that $\mu(dx)$ is invariant for the process generated by H if and only if

$$\int(Hf)(x)\mu(dx) = 0 \qquad (II.8)$$

for every smooth function f on M.

We define an inner product on the smooth functions on M by

$$(f,g) = \int f(x)g(x)dx , \qquad (II.9)$$

where dx is the Riemannian volume element, so that $dx = |\det g|^{1/2}d^4x$. By integration by parts we prove that $(Hf,h) = (f,H^* h)$,where H^* is the adjoint operator of H, so that

$$H^* h = 1/2\ \Delta_g h - \operatorname{div}(hb). \qquad (II.10)$$

We can restate the condition for invariance of the measure μ in terms of H^*. Indeed, (II.10) is equivalent to μ to be a weak solution (in the sense of the theory of generalized functions) of the partial differential equation $H^*\mu = 0$.

Assuming that Δ_g is an elliptic operator, then H and in consequence H^* are elliptic too, and in consequence any solution of the equation $H^*\nu=0$ is of the form $\mu = \phi.dx$, for ϕ a smooth function vanishing on the zeros of ψ. In fact, ϕ is precisely an eigenfunction corresponding to the largest eigenvalue $\lambda = 0$ of the eigenvalue problem $(H^*-\lambda)\phi = 0$, it is simple and its associated eigenspace is of the form $\{c\phi^*_0(x),\ c \in \mathbb{R}^+\}$, and in consequence all invariant measures for H are of the form $\mu = (c\phi^*_0)dx$, for some constant c>0.

Let us determine precisely μ in the case of $H = H_\psi$. Choose a smooth function U(x) on M such that $\mu = U(x)dx$ is an invariant measure for H, i.e. $H^*(e^{-U}) = 0$. Since $H^*(e^{-U}) = -1/2\delta_g d(e^{-U}) + \delta_g(e^{-U}d\ln \psi)$, where δ_g denotes the codifferential of ∇_g, therefore $-1/2de^{-U}+ e^{-U}d\ln \psi = 0$, and $U = -\ln \psi^2$.

Therefore, we have proved that the *normalisation* of ψ^2dx gives an invariant probability density for the diffusion processes generated by $(-1/2)\mathcal{L}_\psi$. *The square of the conformal factor together with the Riemannian volume element determine an unique invariant measure for the RCW diffusion processes, away of the node set of* ψ, $\{ x \in M:\ \psi(x) = 0\}$. In fact, it can be proved that the diffusion is symmetrizable with respect to this invariant measure.

Due to the origin of ψ as a local $\mathbb{R}^+$-symmetry, this singularities *do exist*. It is a theorem due to Nelson [10], that the diffusion process *doesn't penetrate the node set, which can then be thought as a barrier for the diffusion process*, i.e. the probability of penetration is zero.

We have proved in our previous work [4], that these barriers have a natural interpretation. They are associated to a very general phenomenon of *charge quantisation, which can easily be described in terms of the principle of the argument in the theory of complex variables*. It is interesting also to remark that these nodes can be described in terms of Thom's theory of catastrophes. We shall give a description of these facts elsewhere.

Let us assume that ψ belongs to the Hilbert space $L^2(M)$ of square integrable functions on M with respect to dx. Let ρdx be the invariant probability density defined by $C\psi^2dx$, where $C^{-1}= \int\psi^2dx$. The semigroup $\exp(\tau H_\psi)$ can be defined on $L^2(\rho dx)$ by $(\exp(\tau.H_\psi)f)(x) = E_{\xi(0)}(f(\xi(\tau))) = \int f(x)\rho(x)dx$, for any

$\tau \geq 0$, where $\xi(\tau)$ is a solution of (II.3). In fact, the infinitesimal generator H_ψ of the semigroup, is defined on all smooth functions f which are zero on a neighborhood of the zeros of ψ, so that it is the Friedrichs selfadjoint extension of (I.16). The quadratic form on $L^2(\rho dx)$ with respect to which this extension is defined, is (minus) the so-called Dirichlet form [17]

$$\delta(f,g) = -1/2\int g(df,dg)\rho dx. \tag{II.11}$$

In the following, we shall still denote by H_ψ this extension. Let us know identify $L^2(\rho dx)$ with $L^2(dx)$ through the conformal isomorphism defined away of the nodes of ψ

$$C_\psi : L^2(dx) \longrightarrow L^2(\rho dx),\ \phi \longrightarrow \psi^{-1}\phi. \tag{II.12}$$

C_ψ is a unitary operator. We define the operator $D_\psi : L^2(dx) \longrightarrow L^2(dx)$, by $D_\psi = C_\psi^{-1} H_\psi C_\psi$. Then D_ψ is a positive self-adjoint Schrödinger operator unitarily equivalent to H_ψ. We compute

$$\begin{aligned} D_\psi &= 1/2(\Delta_g + V_\psi(x)) = 1/2(\Delta_g - \{|d\ln\psi|^2 + \Delta_g(\ln\psi)\} = \\ &= 1/2(\Delta_g - \Delta_g\psi/\psi) \end{aligned} \tag{II.13}$$

where we recognise in $V_\psi(x)$ the space-time expression for D. Bohm's quantum potential in the theory of hidden variables [18]. It is clear that ψ is the (generalized) *groundstate* of D_ψ.

To complete our construction of the diffusion process, we must give its transition density $p_\tau^\psi(x,y)$, so that $\exp(\tau H_\psi)$ can be represented , for any f in the domain of H_ψ, in terms of the *Riemannian density* determined by g

$$(\exp(\tau H_\psi)f)(x) = \int p_\tau^\psi(x,y)f(y)|\det g(y)|dy^1\wedge\ldots\wedge dy^4 \tag{II.14}$$

Due to the unitary equivalence of H_ψ and D_ψ, what we shall do is to determine p_τ^ψ in terms of a Wiener process generated by the Hamiltonean D_ψ (which, we stress, posseses the usual form of a Laplacean plus perturbative term), so we can apply to it, the usual Feynman-Kac formula but, as we shall see, on the coordinate functions of the general process ξ.

Let us do this for the case in which g is the Euclidean metric, so that Δ_g is the wave-operator on M, the one-point compactification of $\mathbb{R}^4$, so that in (II.13), σ is the identity matrix, and B_τ is the standard Wiener process

W_τ. By the Feynman-Kac formula [15,19,23] we have,

$$(\exp(\tau H_\psi)f)(x) = \psi(x)^{-1}E_{W_x}\left[\exp\left[-\int V_\psi(\xi_s)ds\right]\psi(\xi_s)f(\xi_s)ds\right] \qquad (II.15)$$

where W_x denotes Wiener measure on the coordinates $\xi_\tau(w) = w(\tau)$ of the solution of the stochastic differential equation

$$d\xi_\tau = dW_\tau + d\ln\psi(\xi_\tau)d\tau \qquad (II.16).$$

We define $P_x = Z_\tau.W_x|\mathcal{F}_\tau$, where $\mathcal{F}_\tau$ denotes the smallest σ-algebra for which the coordinate functions $w(\tau)$ are measurable, and Z_τ, for each $\tau>0$, is defined by

$$Z_\tau = \psi(x)^{-1}\psi(\xi_\tau)\exp\left[-\int V_\psi(W_s)ds/\ W_\tau = y\right] , \qquad (II.17)$$

If we assume that the components of $d\ln\psi$ are generalized functions in $L^2_{loc}(M)$ and that $\nabla_g\psi/\psi$ is a generalized function in $L^1_{loc}(M)$, then Z_τ is a random variable, positive W_x a.s. and $E_W(Z_\tau) = 1$. It follows that P_x is a probability measure on $(\mathcal{W}(M),\mathcal{F})$. Then, the diffusion process defined by

$$B_\tau = \xi_\tau - \int_0^\tau d\ln\psi(\xi_s)ds \qquad (II.18)$$

is proved to be standard Brownian motion starting at x with respect to the measure P_x, so that under this measure the coordinate maps of the process ξ_τ are the unique solution to the equation obtained in differentiating (II.17), which is (II.16), behave like a Brownian motion plus the drift. This is the Girsanov or *drift* transformation produced by the unique transformation of probability given by $W_x \longrightarrow P_x$.

Therefore, for all $\tau\geq 0$ and ξ starting at x, we have

$$E_{P_{(\tau)}}(f(\xi_\tau)) = E_{W_{\xi(\tau)}}(f(\xi_\tau)Z_\tau) \qquad (II.19)$$

so, that the transition density, with respect to the Riemannian volume element is

$$p^\psi_x(x,y) = \psi(x)^{-1}\psi(y)E_{W_x}\left[\exp\left[-\int V_\psi(W_s)ds|W_\tau=y\right]\right]p_\tau(x,y) \qquad (II.20)$$

where p_τ is the transition probability of the standard Brownian process, the well known

$$p_\tau(x,y) = (2\pi\tau)^{-2}\exp(-|x-y|^2/2\tau) \qquad (x,y \in M).$$

It can be proved that $p^{\psi}_{\tau}(x,y)$ converges to $\psi^2(y)$ when $\tau \longrightarrow \infty$, uniformly on x and y, and with exponential velocity on x away of the node set of ψ.

We have constructed a one to one correspondance between the RCW structures and diffusion processes in space-time.

We can extend the diffusion processes to $\mathbb{R}$, by defining $\xi^*(\tau) = \xi(-\tau)$. We shall obtain a relativistic extension of the *osmotic processes* described by Nelson [10].

Finally, returning to (II.11), we would like to comment that the *Dirichlet operator* $\mathcal{A}$, defined by

$$\delta(f,g) = (\mathcal{A}f,g)_\mu = \int(\mathcal{A}f)(x)g(x)\rho(x)dx \qquad (II.21)$$

coincides with H_ψ. Thus, in our formulation, the theory of Dirichlet forms receives a geometrical status which was previously unknown. We shall treat in detail this connection, and its relation to Quantum Mechanics in a forthcoming article. We shall only stress for the moment that we have a one to one correspondance between the RCW structures, its diffusion processes, and an *integrodifferential form* given in (II.11,21), which is conformal dependant.

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