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THE TOMOGRAPHY FROM THE VIEWPOINT OF THE TOPOLOGIST

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Abstract

The aim of tomography is the reconstruction of the inner structure of an object $\mathcal{O}$ through the study of the traces it determines on a family of cuts. From the point of view of the topologist, the "inner structure" is given by a differentiable function $f : \mathcal{O} \to \mathbf{R}$. We want to determine the class of f (with respect to the classic equivalence relation given by the action of the group $\mathcal{A}$) from a convenient equivalence relation defined on the family of cuts $(f|_{p^{-1}(c)})_c$, where $p : \mathcal{O} \to \mathbf{R}$ is the "microtome". We use technics from differential topology to study this problem.

1 Introduction

All the maps and manifolds we consider along this work are smooth. The topologies on the function spaces are the Whitney C^∞–topologies.

The denomination "tomography" is given to any process of reconstruction of an object, by analysing the traces it determines on a family of slices. This is, for instance, the problem of the paleontologist who studies a fossil by making cuts on the piece of rock where its is encrusted.

We shall be considering the case where the *structure* to be studied is given by the level surfaces of a given function f, which is, for instance, a

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density function. The *cuts* in the object $\mathcal{O}$ will be given by the intersection of the level surfaces with a family of planes. From a qualitative viewpoint, the problem of tomography can be formalized as follows: The *object* $\mathcal{O}$ to be studied is a compact 3-dimensional manifold, with boundary and corners; the *inner structure* is a differentiable function $f : \mathcal{O} \to \mathbf{R}$. We want to determine f, by studying its restriction to the family of cuts defined by $p^{-1}(c)$, where $p : \mathcal{O} \to \mathbf{R}$ is the *microtome*. Two inner structures f and f' are said to be (qualitatively) indistinguishable relatively to the cuts p=constant if, up to a reparametrization (that is, changing p by $k o p$, where k is a homeomorphism), there exist a homeomorphism

$$G_c : p^{-1}(c) \to p^{-1}(c), \quad \text{for all} \quad c \in \mathbf{R} \, ,$$

which interchange the level surfaces of f with the ones of f' . We can also express this fact by the following commutative diagram:

$$\begin{array}{ccc} p^{-1}(c) & \xrightarrow{f} & \mathbf{R} \\ G_c\downarrow & & \downarrow h_c \\ p^{-1}(k(c)) & \xrightarrow{f'} & \mathbf{R} \end{array}$$

where h_c and G_c are homeomorphisms, for all c . Or, in other words, by the commutativity of the diagram:

$$\begin{array}{ccccc} \mathcal{O} & \xrightarrow{(f,p)} & \mathbf{R}\times\mathbf{R} & \xrightarrow{\pi} & \mathbf{R} \\ G\downarrow & & \downarrow H & & \downarrow k \\ \mathcal{O} & \xrightarrow{(f',p)} & \mathbf{R}\times\mathbf{R} & \xrightarrow{\pi} & \mathbf{R} \end{array} \qquad (1)$$

where G, H and k are homeomorphisms,

$$H(u,v) = (h_v(u), k(v)) \, , \quad G_c = G_{|p^{-1}(c)} \, ,$$

and π is the projection on the second factor.

This diagram defines an equivalence relation on the set of functions $f : \mathcal{O} \to \mathbf{R}$. Therefore, the problem of tomography can now be placed as follows: Does the class of f with respect to this equivalence relation determine f (qualitatively)? Or, what kind of information on the class of f can we obtain from this equivalence relation?

In [2], the first two authors considered the local aspects of this problem. In §2, we recall these results; in §3 we study the global aspects of tomography.

Next, we give the basic vocabulary used in this paper.

Definition 1.1 *Let $p : \mathcal{O} \to \mathbf{R}$ be a fixed differentiable function. f and $f' : \mathcal{O} \to \mathbf{R}$ are p-equivalent, and we denote $f \sim_p f'$, if there exists a commutative diagram (1), where G, H and k are diffeomorphisms isotopic to the identity. f and f' are equivalent, and we write $f \sim f'$, if the following diagram commutes*

$$\begin{array}{ccc} \mathcal{O} & \xrightarrow{f} & \mathbf{R} \\ G\downarrow & & \downarrow h \\ \mathcal{O} & \xrightarrow{f'} & \mathbf{R} \end{array}$$

where G and h are diffeomorphisms isotopic to the identity.

The concept of equivalence is the natural notion when we want to study the "qualitative" behavior of mappings from $\mathcal{O}$ to $\mathbf{R}$. The notion of p-equivalence was introduced by the first time in [3], with no references to the problem of tomography. In the specific situation where $\mathcal{O} = V \times \mathbf{R}$ and p is the canonical projection to $\mathbf{R}$, the concept of p-equivalence coincides with the notion of a one-parameter unfolding in the sense of R. Thom [9].

Given $f : \mathcal{O} \to \mathbf{R}$, we denote by $[f]_p$ and $[f]$, the classes of p-equivalence and equivalence, respectively. With this notation,we can say that the problem of tomography consists in determining all the classes $[g]$ that meet a given class $[f]_p$.

In this paper we shall be assuming that $\mathcal{O}$ is a 3-dimensional manifold, and $p : \mathcal{O} \to \mathbf{R}$ is a Morse function (i.e. p has only non-degenerated singularities, with simple critical values). The more general situation, where $\mathcal{O}$ has boundary and corners, or even is higher dimensional was considered by the first two authors in [1].

2 Local tomography

2.1 The generic situation

Let $p : \mathcal{O} \to \mathbf{R}$ be a fixed Morse function.

Definition 2.1.1 *A fold point* ([4]) *$s \in \mathcal{O}$ of the mapping $(f,p) : \mathcal{O} \to \mathbf{R}\times\mathbf{R}$ is a p-transverse fold (respectively p-tangent) if s is a regular point of p (respectively s is a singular point of p).*

Analogously, a cusp point ([4]) $s \in \mathcal{O}$ *of the mapping* (f,p) *is called a p-transverse cusp when* p *is regular at* s .

Theorem 2.1.2 *The following properties are generic* ([1]) *for* $f : \mathcal{O} \to \mathbf{R}$ *:*

- *(i) if* s *is a singular point of* $(f,p) : \mathcal{O} \to \mathbf{R}^2$ *, then* s *is a p-transverse fold, or a p-tangent fold, or a p-transverse cusp.*
- *(ii) in the inverse image by* (f,p) *of a point in* $\mathbf{R}^2$ *, there are at most two singular points of* (f,p) *. Furthermore, the double singular points must be p-transverse folds intersecting transversally.*
- *(iii) the images of p-tangent folds, cusps, and the double crossings in the images of fold lines are two by two on distinct horizontal lines in* $\mathbf{R}^2$.

The proof of this theorem (see [1],[3]) uses classical methods of transversality. In fact, the set of mappings f satisfying *(i), (ii)* and *(iii)* is open and dense in $C^\infty(\mathcal{O},\mathbf{R})$, the space of C^∞–mappings from $\mathcal{O}$ to $\mathbf{R}$, with the Whitney topology. Such mappings f are *p-stable* in the sense that if f' is close enough of f , then f and f' are p-equivalence (this is one of the main results in [1] and [3]; the proof uses the equivalence between the p-stability of f and the stability of the diagram $D : \xrightarrow{(f,p)} \xrightarrow{\pi}$). We denote by $\mathcal{P}$ the set of mappings f satisfying *(i), (ii)* and *(iii)*, and according to the philosophy introduced by R. Thom, we shall restrict our attention to the mappings in $\mathcal{P}$. For any $f \in \mathcal{P}$, the qualitative behavior of the family of cuts $(f_{|p^{-1}(c)})_c$ does not change either by small perturbations of f , or of the microtome p .

2.2 In the neighbourhood of a fold

Let us assume that $s \in \mathcal{O}$ is a *p-tangent fold*; it follows from a result of Wan ([10]) that we can choose local coordinates (x,y,z) in the neighbourhood of s , such that

$$p(x,y,z) = \lambda(x) \pm y^2 \pm z^2 \ , \quad f(x,y,z) = x \ ;$$

making a coordinate change in the x variable, followed by a diffeomorphism in the target of f , it is possible to reduce to:

$$p(x,y,z) = \pm x^2 \pm y^2 \pm z^2 \ , \quad f(x,y,z) = x \ .$$

This case is not very interesting, since f is non-singular.

If s is a *p-transverse fold*, by the same result of Wan [9], we get

$$p(x,y,z) = x \ , \quad f(x,y,z) = \lambda(x) \pm y^2 \pm z^2 \ ,$$

where λ is arbitrary. The function f has a regular point, a non-degenerated critical point, or a degenerated critical point at s , as λ is regular, has a non-degenerated critical point, or is degenerated at 0 ; however all those cases are p-equivalent, since the local diffeomorphism $(u,v) \longrightarrow (u - \lambda(v), v)$ in the target of (f,p) reduces them to the unique normal form:

$$p(x,y,z) = x \ , \quad f(x,y,z) = \pm y^2 \pm z^2 \ .$$

Notice that the critical points of f are necessarily p-transverse folds or p-transverse cusps. However, the above reduction shows that *the tomography does not distinguish among the p-transverse folds points the ones that are critical points of* f .

There are two types of fold lines [6]: the definite fold $(x, \pm(y^2 + z^2))$, and the indefinite fold $(x, \pm(y^2 - z^2))$. In fact, we can refine a little this classification by noticing that the index of the second intrinsic derivative ([4])

$$(f,p)^{(2)}(s) : \ker(d(f,p))_s \longrightarrow \frac{\mathbf{R}^2}{Im(d(f,p))_s}$$

remains the same along the p-transverse fold line, and is invariant with respect to p-equivalence. More precisely to each p-transverse fold line is associated an index which can be 0 , 1 or 2 . The meaning of this index appear on the image of the fold line in $\mathbf{R}^2$: the image of (f,p) folds on a unique side in a neighbourhood of a point on a fold line of index 0 , or 2 , and (f,p) is locally surjective on the lines of index 1 ; the lines with index 0 have the image of (f,p) to their right, and the lines with index 2 have the image of (f,p) to their left (see Figure 1). The indices 0 and 2 are interchanged when a p-tangent fold is crossed.

The critical points of f also can be seen on the image of the fold line: they correspond to points with critical tangents. We can compute the index of a non-degenerate critical point of f as follows: according to the concavity of the fold line, we add 0 or 1 to the index of the curve to obtain the index of f (this follows by reasoning on the local models).

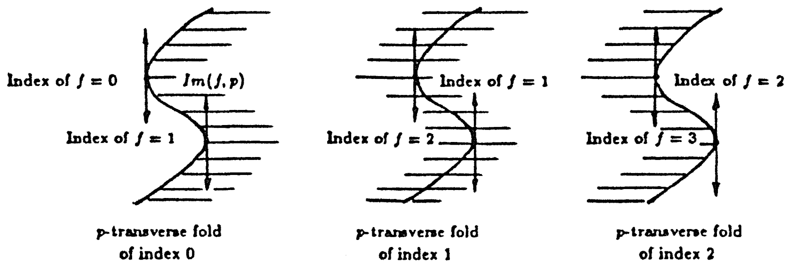


Figure 1

2.3 In the neighbourhood of a cusp

If we assume that the point $s \in \mathcal{O}$ is a p-transverse cusp, it follows from Wan ([10]) that we can choose coordinates around s, such that (f,p) is equivalent to

$$p(x,y,z) = x \ , \quad f(x,y,z) = \lambda(x) + z^3 + xz \pm y^2 \ ,$$

where $\lambda : (\mathbf{R},0) \to (\mathbf{R},0)$ is arbitrary. With respect to p-equivalence we get simply

$$p(x,y,z) = x \ , \quad f(x,y,z) = z^3 + xz \pm y^2 \ .$$

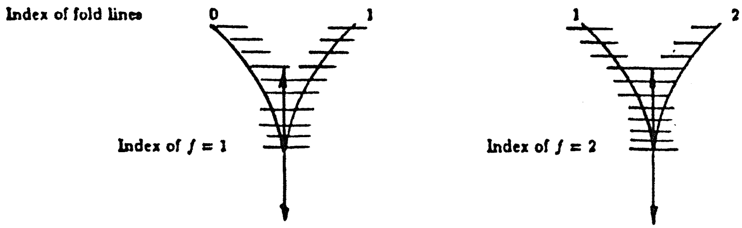
Therefore, also in this case, the tomography does not give the information whether s is a singular point of f or not. However, in case s is singular, it is necessarily non-degenerate.

From the above normal form,it follows that the indices of the two branches of the fold line meeting at $(f,p)(s)$ differ by one. In case s is a singular point of f, we can also determine its index (see Figure 2).

3 Global tomography

Let $\mathcal{M}$ denote the set of Morse functions in $C^\infty(\mathcal{O},\mathbf{R})$.

The local study of p-equivalence shows that the technic of tomography doesn't determine exactly the inner structure of f. In fact, a given class



Index of vertical cusps

Figure 2

$[f]_p$ with respect to p-equivalence can meet several classes of $[g]$ with respect to equivalence. The problem of determining all those classes is very difficult, since we don't know how to describe conveniently the quotient $C^\infty(\mathcal{O},\mathrm{R})/\sim$, or even $\mathcal{M}/\sim$ (see[5]). In this paragraph we shall give some results in this direction.

3.1 Reduction to a problem of plane curves

Definition 3.1.1 *Let f , $f' \in C^\infty(\mathcal{O},\mathrm{R})$. We say that f and f' are p-strongly equivalent if there exists a commutative diagram*

$$\begin{array}{ccc} \mathcal{O} & \xrightarrow{(f,p)} & \mathrm{R}\times\mathrm{R} \\ & \overset{(f',p)}{\searrow} & \downarrow H \\ & & \mathrm{R}\times\mathrm{R} \end{array}$$

where H is an orientation preserving diffeomorphism in R^2, of the form $H(u,v) = (h_v(u),v)$ (hence, H leaves invariant all horizontal lines).

Notice that this diagram is a particular form of diagram (1), where we take the diffeomorphisms G and k as the identity.

Lemma 3.1.2 *If f' is p-equivalent to f , then there exist g such that g is equivalent to f' , and g is p-strongly equivalent to f .*

Proof. With the notation of Definition 1.1, it suffices to take $g = f' \circ G$. □

Let $f_0 \in \mathcal{P}$ be fixed. The above lemma reduces the problem of finding all $[g]$, $g \in \mathcal{M}$, such that $[g] \cap [f_0]_p \neq \emptyset$, into the following: finding all $[g]$, $g \in \mathcal{M}$, such that $[g]$ meets the orbit of f_0 with respect to p-strong equivalence.

We denote by $\mathcal{C}$ the class of f_0 with respect to the p-strong equivalence.

Given any differentiable function $F : \mathcal{O} \to \mathbf{R}^2$ we denote by ΣF the singular set of F , and by SF the set $F(\Sigma F)$.

Notice that for all $f \in \mathcal{C}$, $\Sigma(f,p) = \Sigma(f_0,p)$, and $S(f,p)$ can be deformed into $S(f_0,p)$ by a diffeomorphism in $\mathbf{R}^2$, preserving the horizontal lines.

Lemma 3.1.3 *If f and g are in $\mathcal{C} \cap \mathcal{M}$, and $S(f,p) = S(g,p)$, then f and g are equivalent.*

Proof. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{O} & \xrightarrow{(f,p)} & \mathbf{R}^2 \\ Id \downarrow & & \downarrow H \\ \mathcal{O} & \xrightarrow{(g,p)} & \mathbf{R}^2 \end{array} ,$$

where H is an orientation preserving diffeomorphism of the form: $(u,v) \to (h(u,v),v)$. It folows that H fixes the set $S(f,p) = S(g,p)$.

Let H_t be defined by $(u,v) \to (u + t(h(u,v) - u), v)$. Then H_t is an isotopy between the identity $(= H_0)$, and H $(= H_1)$, that fixes the set $S(f,p)$. Let f_t denote the first component of $H_t(f,p)$. Then, $(f_t)_{t \in [0,1]}$ is a family of functions, depending continuously on t , with $f_0 = f$, $f_1 = g$ and $S(f_t,p) = S(f,p)$.

It follows from §2 that we can see on the image of the fold line $S(f_t,p)$ it f_t is a Morse function or not: the critical points of f_t correspond to points of $S(f_t,p)$ with vertical tangent; they are non-degenerated if the contact of $S(f_t,p)$ with this vertical line is quadratic, the critical values are simple if there is no double vertical tangencies on $S(f_t,p)$.

Since f is a Morse function and $S(f_t,p) = S(f,p)$, $\forall t \in [0,1]$, f_t is also a Morse function, $\forall t \in [0,1]$. The stability of f_t and the connectivity of $[0,1]$ give the result. □

It follows from the above lemma that $S(f,p)$ plays a fundamental role in determining the class of f . To obtain more precise results, we should be able to classify all possible "forms" of the set $S(f,p)$, when $f \in \mathcal{C} \cap \mathcal{M}$.

Definition 3.1.4 *Let S and S' be two finite families of differentiable plane curves having with the vertical lines only quadratic and simple contacts. We say that S and S' have the same form if there exists an isotopy $(H_t)_{t\in[0,1]}$ of the identity in $\mathbf{R}^2$ with*

$$H_t : (u,v) \longrightarrow (h_t(u,v),v)$$

being differentiable, such that $H_1(S) = S'$, and for all $t \in [0,1]$, the curve $H_t(S)$ has only quadratic and simple contacts with the vertical lines. The vertical cusps are admited as having quadratic contact with the vertical lines.

Theorem 3.1.5 *Let f and g be in $\mathcal{C} \cap \mathcal{M}$. If $S(f,p)$ and $S(g,p)$ have the same form, then f and g are equivalent.*

Proof. Let H_t be the isotopy given by Definition 3.1.4 that transforms $S(f,p)$ into $S(g,p)$. Let us denote by f_t the first component of $H_t \circ (f,p)$. Then, $f_t \in \mathcal{C} \cap \mathcal{M}$, and the same argument as in Lemma 3.1.3 shows that f_1 is equivalent to f . Now, $S(f_1,p) = S(g,p)$, and the conclusion follows from Lemma 3.1.3. □

In Figure 3 below we give an example of two curves that do not have the same form, but that could be the images of the singular sets $S(f,p)$ and $S(g,p)$ for two p-strongly equivalent Morse functions f and g . This example shows that Theorem 3.1.5 does not have an evident reciprocal.

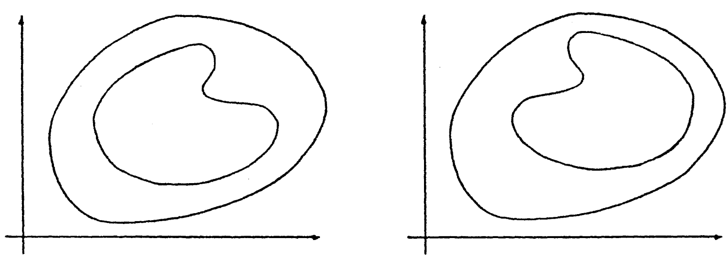


Figure 3

3.2 Elements of classification

With the same notations of 3.1, let $f_0 \in \mathcal{P} \cap \mathcal{M}$, and let C denote its class with respect to p-strong equivalence. While f runs over C , $S(f,p)$ runs over the set of families of curves $H(S(f_0,p))$, where H is a diffeomorphism of the form $(u,v) \mapsto (h(u,v),v)$. The "form" of such a family depends only on the contacts with the vertical lines. In the following proposition we show that there exists a path from $S(f_0,p)$ to $H(S(f_0,p))$ that is generic with respect to these contacts.

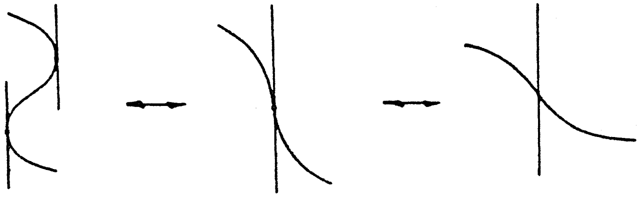
Proposition 3.2.1 *With the above notation, there exists a path $H_t(u,v) = (h_t(u,v),v)$, $t \in [0,1]$, such that*

$$H_0(S(f_0,p)) = S(f_0,p)$$

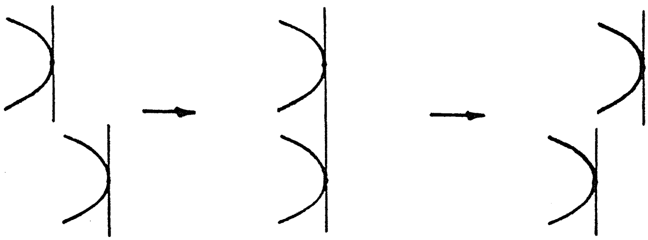
$$H_1(S(f_0,p)) = S(f,p)$$

and $H_t(S(f_0,p))$ has simple quadratic contacts with the vertical lines for all $t \in [0,1]$, except for a finite number of t's where only the following bifurcations occur

(a) third order contact



(b) double quadratic contact with a vertical line



Proof. Working with normal forms, we can show that the mapping:

$$\mathbf{R}[u,v] \times [0,1] \times \mathcal{O}^{(s)} \longrightarrow {}_sJ^2(\mathcal{O},\mathbf{R}) \quad (s = 1,2,3)$$

$$(k,t,x) \longmapsto {}_sj^2[(1-t)f_0 + t(k+h_1)(f_0,p)](x)$$

is a submersion, where as usual, $\mathcal{O}^{(s)} = \{(x_1,\ldots,x_s) \in \mathcal{O} \times \ldots \times \mathcal{O} / x_i \neq x_j ,\ \forall\, i \neq j\}$ and ${}_sJ^2(\mathcal{O},\mathbf{R})$ is the s-fold 2-jet bundle.

Thus, for almost all h in the set $C^\infty(\mathbf{R}^2,\mathbf{R})$, $\frac{\partial h}{\partial u} > 0$, the mapping

$$[0,1] \times \mathcal{O}^{(s)} \longrightarrow {}_sJ^2(\mathcal{O},\mathbf{R})$$

$$(t,x) \longmapsto {}_sj^2[(1-t)f_0 + th(f_0,p)](x)$$

is transverse to the ${}_s\mathcal{A}^2$-orbits in ${}_sJ^2(\mathcal{O},\mathbf{R})$. Hence, there exists a generic set $\mathcal{H} \subset C^\infty(\mathbf{R}^2,\mathbf{R})$ such that for all $h \in \mathcal{H}$, $\frac{\partial h}{\partial u} > 0$ the path $F_h(t,x) = (1-t)f_0 + th(f_0,p)$ is $\mathcal{A}$−versal in the neighbourhood of any $t_0 \in [0,1]$. According to [7], this stablish the result. □

As a consequence of Theorem 3.1.5 and Proposition 3.2.1, it is sufficient to analyse what happens in the neighbourhood of the bifurcations. The bifurcation *(a)* corresponds to a cancelation or a creation of a pair of singularities of indices i and $i+1$ ($i = 0,1,2$), and the bifurcation *(b)* corresponds to a rearrangement (see [8]). In principle, if we know the class of f before the bifurcation, we also know its class after the bifurcation ([5]).

A natural procedure will be to identity in C those Morse functions that are minimal, in the sense that they have the minimum number of critical points. Then, all the others could be obtained by the process of creation (without limitation), or rearrangement (more difficult to describe).

If we assume the natural hypothesis that the "microtome" p is the simplest possible, the following result shows that the simplest function in C is equivalent to p. Thus, a given f in C will be either equivalent to p or to any Morse function obtained from p by creation or rearrangement of singularities.

Proposition 3.2.2 *In the set C there exists a Morse function equivalent to the "microtome" p.*

Proof. The proof is very simple.

Let H_α be defined by $H_\alpha(u,v) = (\alpha u + v, u)$, $\alpha > 0$. Then $H_\alpha(f_0, p) = (\alpha f_0 + p, p)$, and $\alpha f_0 + p \in C$, $\forall \alpha > 0$. Since p is stable, for all sufficiently small α, $\alpha f_0 + p$ is equivalent to p.

3.3 Reduced Normal Form

Let $f \in C \cap \mathcal{M}$. We will describe a method to produce in C a representative g which is equivalent to p. Assuming, as before, that p is the simplest possible, we call $S(g,p)$ the reduced normal form.

We consider $S(f,p)$; on each horizontal line there exists at most a cusp point, or a p-tangent fold, or a double crossing.

By a diffeomorphism $(u,v) \to (h(u,v),v)$, we can move all singularities to the diagonal $\Delta = \{(x,x), x \in \mathbb{R}\}$ in $\mathbb{R}^2$. The set $S(g,p)$ obtained is a planar graph with its vertices on Δ; the vertices are of degree 2 if they correspond to cusp points or p-tangent folds, of degree 4 if they correspond to double-crossings. Furthermore, the edges of a vertex of degree 2 are on the same side of the horizontal line that passes through this vertex; in a vertex of degree 4 two edges are above and two are below the horizontal line. Thus, by a convenient isotopy $(H_t)_{t\in[0,1]}$, with $H_t(u,v) = (h_t(u,v),v)$, we can transform all the edges in such a way that all the vertical tangents are moved to a sufficiently small neighbourhood of the p-tangent folds, and the resulting function (that we still denote by g) has the same number of singularities as p (and with the same indices). (Prop. 3.2.1). This procedure is illustrated in Figure 4 below.

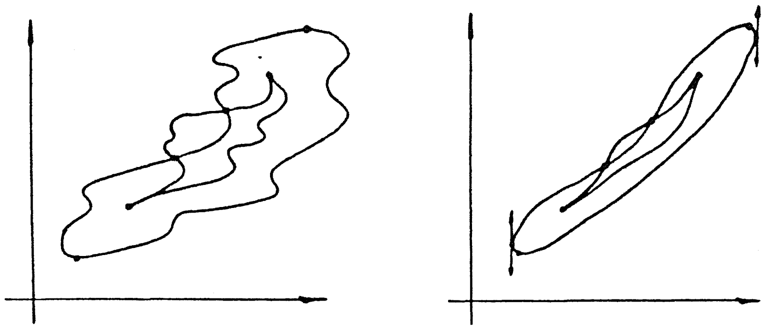


Figure 4

In fact, we can prove that g and p are equivalent:

Let $q_\theta : \mathbf{R}^2 \to \mathbf{R}$ be the orthogonal projection onto the line through the origin making angle θ with the x-axis. We have

$$q_0 \circ (g, p) = g \ , \quad q_{\pi/2} \circ (g, p) = p \ .$$

Without loss of generality, in the above reduction we can assume that each vertex v corresponding to a horizontal tangent has a neighbourhood U_v such that the contacts between $S(g,p)\,|_{U_v}$ and lines of direction $\pi/2 + \theta$ (kernel of q_θ) are quadratic and simple. Furthermore, outside U_v, $S(g,p)$ is C^1-close to the diagonal Δ, hence transversal to the lines $\pi/2 + \theta$.

Then, $q_\theta \circ (g, p)$ is a Morse function for any $\theta \in [0, \pi/2]$, and thus equivalent to $q_{\pi/2} \circ (g, p) = p$, for all $\theta \in [0, \pi/2]$.

Final Remark

It seems possible to use the "reduced normal form" to give a combinatorial description of the form $S(f,p)$. This process would allow a better description of all classes $[f]$ that meet a given class C.

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NOTAS DO ICMSC/USP

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