UNIVERSIDADE DE SÃO PAULO
CAMPUS DE SÃO CARLOS
INSTITUTO DE CIÊNCIAS MATEMÁTICAS DE SÃO CARLOS

I.C.M.S.C.

A note on the orthogonal reparametrization in Bayesian inference for the extreme value distribution

RODRIGUES, J. & LOUZADA NETO, F.

Nº 64

Notas do ICMSC - USP

ISSN 0103-2577

A note on the orthogonal reparametrization in Bayesian inference for the extreme value distribution

RODRIGUES, J. & LOUZADA NETO, F.

Nº 64

São Carlos (SP)
1990

A note on the orthogonal reparametrization in Bayesian inference for the extreme value distribution

Josemar Rodrigues
ICMSC-USP CP-668-13560
São Carlos

Francisco Louzada Neto
UFSCar CP-676
São Carlos

1 Summary

Consider the problem of inference about the scale parameter φ in presence of location parameter ϕ under the extreme value distribution. In a Bayesian framework, the marginal posterior and the profile posterior may be of interest. In this paper, using the Jeffreys prior density and the Laplace approximation, we show that these posteriors are invariant under the orthogonal reparametrization (Cox & Reid, 1987) adopted for the location parameter ϕ. Also, from the data generated by the extreme value distribution, we show that the marginal and profile posteriors do not give conflicting information about the scale parameter φ.

Key words: marginal posterior; profile posterior; orthogonality; reparametrization.

2 Introduction

Consider the situation where a random sample $X_1, \ldots, X_n$ is taken from the extreme value distribution given by

$$f(x/\varphi,\phi) = \frac{1}{\varphi}\exp\{\frac{x-\phi}{\varphi} - e^{\frac{x-\phi}{\varphi}}\}, \quad (1)$$
$$-\infty < x < \infty, \quad -\infty < \phi < \infty, \quad 0 < \varphi < \infty.$$

The extreme value distribution is one of the most useful survival models for the reliability theory. In general, this distribution presents some inaccuracy of the normal approximations for the maximum likelihood estimators of φ and ϕ, specially for small and moderate samples sizes (Achcar,1990).

Once $X = (X_1, \ldots, X_n)$ is observed [call the observed value $x = (x_1, \ldots, x_n)$], then all information about (φ, ϕ) is contained in the log likelihood function

$$l(\varphi,\phi) = -n\log(\varphi) + \frac{1}{\varphi}\sum_{i=1}^{n}(x_i - \phi) - \sum_{i=1}^{n}\exp\{\frac{x_i-\phi}{\varphi}\}. \quad (2)$$

But if the only parameter of interest is φ, then it is desirable to remove the nuisance parameter ϕ from (2). Let us consider the problem from a Bayesian viewpoint. Suppose that the Jeffreys prior

$$\pi(\varphi,\phi) = \{\text{deter} I(\varphi,\phi)\}^{\frac{1}{2}} \propto \frac{1}{\varphi^2} \quad (3)$$

is assigned to (φ, ϕ), where the Fisher information matrix is given by

$$I(\varphi,\phi) = \begin{bmatrix} \frac{n}{\varphi^2} & \frac{n}{\varphi^2}(1+\psi(1)) \\ \frac{n}{\varphi^2}(1+\psi(1)) & \frac{n}{\varphi^2}(1+\psi(2)+\psi^2(2)) \end{bmatrix}, \quad (4)$$

with the digamma function $\psi(k) = \frac{d\Gamma(k)}{dk}$, [see Box & Tiao, 1973 and Lawless,1982]. Then inferences about φ are typically based on the marginal posterior

$$\pi(\varphi/x) \propto \frac{1}{\varphi^2}\int_{-\infty}^{\infty} e^{l(\varphi,\phi)}d\phi. \quad (5)$$

An alternative method of removing the nuisance parameter ϕ is the profile posterior given by

$$\pi_p(\varphi) \propto \frac{1}{\varphi^2}e^{l(\varphi,\hat{\phi}_\varphi)}, \quad (6)$$

where for a fixed φ, $\hat{\phi}_\varphi$ is the maximizing value of $l(\varphi,\phi)$.

To a Bayesian, the marginal posterior (5) is the preferred way of removing the nuisance parameter. The profile posterior (6), however, can have computational advantages over (5), since maximization is easier than integration over ϕ. For fixed φ, expanding $l(\varphi,\phi)$ in a second order Taylor's series about $\hat{\phi}_\varphi$, we have the Laplace approximation

$$\pi(\varphi/x) \approx k\frac{1}{\varphi^2}e^{l(\varphi,\hat{\phi}_\varphi)} \mid J(\varphi,\hat{\phi}_\varphi) \mid^{-\frac{1}{2}}, \tag{7}$$

where k is a constant and $J(\varphi,\phi) = -\frac{\partial^2 l(\varphi,\phi)}{\partial^2 \phi}$. For a general discussion of the use of the Laplace method in Bayesian analysis, see Tierney and Kadane (1986). In this paper, we show that (5) and (6) are invariant under the orthogonal reparamertrization (Cox & Reid, 1987) adopted to ϕ. Also, by simulation will be shown that (5) and (6) do not give conflicting inferences about φ.

3 The orthogonal reparametrization of the nuisance parameter ϕ

For a fixed value of φ, by differentiating (2) with respect to ϕ and equating to zero, we get, after algebraic manipulation

$$\hat{\phi}_\varphi = \log\{\frac{\sum_{i=1}^n \exp(\frac{x_i}{\varphi})}{n}\}. \tag{8}$$

Then the log-profile posterior is given by

$$\begin{aligned}\log\pi_p(\varphi) &\propto -(n+2)\log(\varphi) + \frac{1}{\varphi}\sum_{i=1}^n(x_i - \hat{\phi}_\varphi) - \sum_{i=1}^n \exp\{\frac{x_i - \hat{\phi}_\varphi}{\varphi}\} \\ &= -(n+2)\log(\varphi) + \frac{1}{\varphi}\sum_{i=1}^n(x_i - \hat{\phi}_\varphi) - n \end{aligned} \tag{9}$$

and the log-marginal posterior [from (7)] by

$$
\begin{aligned}
\log\pi(\varphi/x) &\approx \log(k) - 2\log(\varphi) + l(\varphi, \hat{\phi}_\varphi) - \frac{1}{2}\log\{\frac{\sum_{i=1}^n \exp[\frac{x_i - \hat{\phi}_\varphi}{\varphi}]}{\varphi^2}\} \\
&= \log(k) - 2\log(\varphi) + l(\varphi, \hat{\phi}_\varphi) - \frac{1}{2}\log(\frac{n}{\varphi^2}) \\
&\propto \log\pi_p(\varphi) - \frac{1}{2}\log(\frac{n}{\varphi^2}) \qquad (10)
\end{aligned}
$$

As discussed by Cox & Reid (1987), the orthogonal parametrization is a parametrization which diagonalize the Fisher matrix (4). Therefore, we transform the origin parameters (φ, ϕ) in model (2) to the orthogonal parameters (φ, λ). The orthogonal parameter λ is obtained by solving the differential equation

$$
\begin{aligned}
a\partial\varphi &= \partial\phi \quad \text{where} \qquad (11) \\
a &= -(1 + \psi(1)).
\end{aligned}
$$

This differential equation gives a one-to-one transformation

$$\lambda = \phi - a\varphi. \qquad (12)$$

Then, the log-likelihood function in the new parametrization is

$$l(\varphi, \lambda) = n\log(\varphi) + \frac{1}{\varphi}\sum_{i=1}^{n}(x_i - a\varphi - \lambda) - \sum_{i=1}^{n}\exp\{\frac{x_i - a\varphi - \lambda}{\varphi}\}. \qquad (13)$$

For a fixed value of φ, by differentiating (13) with respect to λ and equating to zero, we obtain

$$\hat{\lambda}_\varphi = \hat{\phi}_\varphi - a\varphi. \qquad (14)$$

Let us denote $\pi_p^\perp(\varphi)$ and $\pi^\perp(\varphi/x)$ the profile and marginal posteriors based on (13), respectively. Since,

$$
\begin{aligned}
l(\varphi, \hat{\phi}_\varphi) &= l(\varphi, \hat{\lambda}_\varphi), \qquad (15) \\
\frac{\partial^2 l(\varphi, \phi)}{\partial^2 \phi}\Big|_{\phi=\hat{\phi}_\varphi} &= \frac{\partial^2 l(\varphi, \lambda)}{\partial^2 \lambda}\Big|_{\lambda=\hat{\lambda}_\varphi}
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\pi_p(\varphi) &= \pi_p^\perp(\varphi), \qquad (16) \\
\pi(\varphi/x) &= \pi^\perp(\varphi/x). \qquad (17)
\end{aligned}
$$

Note that $\pi^{\perp}(\varphi/x)$ is just the approximate conditional likelihood introduced by Cox&Reid (1987) without the Jeffreys prior. We conclude from a bayesian point of view that, based on the extreme value distribution with Jeffreys prior for (φ, ϕ) and (φ, λ), we do not gain any information about φ when replacing ϕ for λ. Also, Figure 1 shows for a data generated by (1) with $\varphi = 0.5, \phi = 2$ and $n = 100$, that the marginal posterior $\pi(\varphi/x)$ and the profile posterior $\pi_p(\varphi)$ give approximately no conflicting inference about φ.

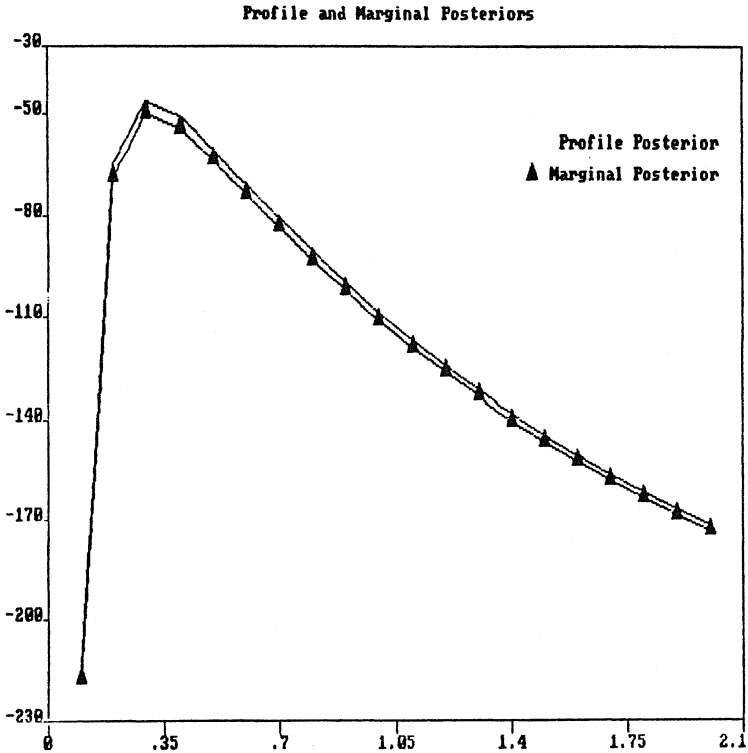

Figure 1 : Profile and Marginal Posteriors

References

[1] Achcar, J. (1990). An Useful Reparametrization for the Extreme Value Distribution, Notas do ICMSC-USP, no.58.

[2] Cox, D.R., and Reid, N. (1987). Parameter Orthogonality and Approximate Conditional Inference, Journal of the Royal Statistical Society, Ser. B, 49, 1-39.

[3] Box, G.E.P. and Tiao, G.C. (1973). Bayesian Inference in Statistical Analysis, Adison-Wesley.

[4] Lawless, J.F. (1982). Statistical Models and Methods for lifetime Data, John Wiley & Sons.

[5] Tierney, L., and Kadane, J.B. (1986). Accurate Approximations for Posterior Moments and Marginal Densities, Journal of the American Statistical Association, 81, 82-86.

NOTAS DO ICMSC/USP

Nº 63 - RODRIGUES, S.R.; MONARD, M.C. - Implementação lógica de um módulo coletor de dados para a construção de sistemas especialistas

Nº 62 - MONARD, M.C.; NICOLETTI, M.C. - Método sintático de prova de teorema algorítmo de Wang

Nº 61 - GIONGO, M.A.; TÁBOAS, P.Z. - Roses play a role in some inverse problems from bifurcation theory

Nº 60 - MORABITO, R.N.; ARENALES, M.N.; ARCARO, V.F. - An and-or-graph representation for two-dimensional cutting problems

Nº 59 - ACHCAR, J.A. - A bayesian approach to reparametrization of the exponential distribution with type I censored data

Nº 58 - ACHCAR, J.A. - An useful reparametrization for the extrem value distribution

Nº 57 - MASIERO, P.C.; et al - Um ambiente de desenvolvimento baseado na abordagem operacional

Nº 56 - ANDRADE, E.X.L.; BRACCIALI, C.F. - Um método, assemelhado ao de Francis, para determinação de auto-valores de matrizes

Nº 55 - BRUCE, J.W. - Euler characteristics of real varieties

Nº 54 - FURKOTTER, M.; RODRIGUES, H.M - Symmetry and bifurcation to 2"-periodic solutions on nonlinear second order equations with 2"/m-periodic forcings

Nº 53 - RIEGER, J.H.; RUAS, M.A.S. - Classification of A-simple germs from K^n to K^2