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Roses Play a Role in Some Inverse Problems from Bifurcation Theory

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SUMMARY

When planar autonomous nonlinear ODE's are perturbed by two-parameter families of periodic functions, periodic solutions bifurcate from a periodic orbit. In the plane of the parameters, the bifurcation diagram is given by sectorial regions with vertices at the origin. See [6] and [7]. Our concern is the following inverse problem for certain ODE's: ***Given a bifurcation diagram, what perturbation leads to it?*** For diagrams with four or less bifurcation curves we have fairly general results. These do not include more complex diagrams. Our approach, however, proves to be helpfull in dealing with them. The n petal roses $\mathcal{R}(\alpha) = (\cos n\alpha \cos\alpha,\ \cos n\alpha \sin\alpha),\ 0 \leq \alpha < \pi,\ n = 3, 5, 7, \ldots$, play an essential role. Facts from elementary Projective Geometry are among our tools.

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1. INTRODUCTION

From the point of view of Nonlinear Analysis the early problem in bifurcation theory is: *"Given an equation depending on parameters, find the bifurcation diagram in the parameter space for some class of solutions."* We are rather concerned with the following: *"Given a bifurcation diagram, find an equation for it."*

Consider the two-dimensional nonlinear ordinary differential equation

$$(1;\lambda,\mu) \qquad x' + g(t, x, \lambda, \mu) = 0,$$

where a prime means derivative with respect to the time t; λ, μ, are real parameters and g is a sufficiently smooth time-periodic function, with period 2π. In this case we will say: *g is a 2π-periodic function.* Suppose further the equation $(1; 0, 0)$ has a 2π-periodic solution and is autonomous, that is, $g(t, x, 0, 0) = \hat{g}(x)$. For small $|(\lambda, \mu)|$, an interesting problem is the investigation on how the 2π-periodic solutions of $(1; \lambda, \mu)$ bifurcate from the one-parameter family $p_\alpha(t) := p(t + \alpha)$, $0 \leq \alpha < 2\pi$, of 2π-periodic solutions of $(1; 0, 0)$. The intriguing feature of this is that the time dependence gives to the perturbed problem a different nature from that of the unperturbed equation.

As far as we know, the paper by Loud [7] is a forerunner in this subject. However, after Loud considerable progress has been done, see [6], [9], [3; 11.2], for instance. Relevant special cases of $(1; \lambda, \mu)$ are a periodically forced Lotka-Volterra predator-prey model for two species and the second order equation for a nonlinear oscillator, with the effect of the small parameters λ, μ describing the interaction of a damping and a periodic forcing. Both of the cases will be precisely defined in the text.

The answer to these problems, which we call *Direct Problems*, is provided by bifurcation diagrams in the $\lambda\mu$-plane described by pairwise transversal curves through the origin, so that their tangent lines at the origin define a concurrent pencil. These curves are the bifurcation curves, since if $(\lambda, \mu) \neq (0,0)$ crosses one of them, the number of 2π-periodic solutions near $p_\alpha(t)$ of the corresponding perturbed equation changes by two.

Our main concern is the *Inverse Problem* that in few words is the following: "Given a diagram of the type just described and a particular equation (1;0,0) with a 2π-periodic orbit $p(t)$, how should be it perturbed in order that the 2π-periodic solutions near $p_\alpha(t)$ bifurcate according to that diagram?" Direct problems are local and their solutions are qualitative, in the sense that just the lower terms of the bifurcation curves can be effectively computed. Thus, a satisfactory answer to an inverse problem is accomplished if one shows the perturbation that leads to bifurcation curves tangent at the origin to the given ones. We point out that very often the problem of real interest is the inverse. Indeed, an instance of this is the case in which one tries to write the Equation $(1; \lambda, \mu)$ as a description of some experimentally observed behavior of a physical system. Here the word " physical" can be obviously replaced by " biological", "economical" etc.

Section 2 is included in this paper by selfcontentness reasons. It is devoted to describe precisely the direct problem and, except for few nuances, most of the results therein are known. Special attention is paid to a forced Lotka-Volterra predator-prey model and to second order equations

$$y'' = g(t, y, y', \lambda, \mu),$$

where $x = (y, y')$. Precisely, if $f_j, \tilde{f},\ j = 1,2$ are 2π-periodic functions, the case corresponding to the damped and forced oscillator,

$$g(t, y, y', \lambda, \mu) = -\tilde{g}(y) - \lambda y' + \mu \tilde{f}(t),$$

with $y\tilde{g}(y) > 0$, for $y \neq 0$; and the special one

$$g(t, y, y', \lambda, \mu) = \hat{g}(y, y') + \lambda f_1(t) + \mu f_2(t)$$

are considered.

Beginning the Section 2 we indicate a way the reader might skip part of it.

Section 3 is concerned with the Inverse Problem. As it is clear from Section 2, a closed curve $\mathcal{A}(\alpha) \neq 0$, $0 \leq \alpha < 2\pi$, in the $\lambda\mu$-plane appears in a natural way. If there are tangent lines to $\mathcal{A}(\alpha)$ passing through the origin, then the bifurcation curves are orthogonal to these lines at the origin. For a bifurcation diagram with four bifurcation curves, we will show that a curve isomorphic to the rose

$$\mathcal{R}_\delta(\alpha) = (\cos 3\alpha \cos \alpha + \delta, \cos 3\alpha \sin \alpha),$$

$0 \leq \alpha < 2\pi$, can be chosen to play the role of $\mathcal{A}(\alpha)$ in the inverse problem. In cases where no bifurcation occurs or where the bifurcation diagram has only two bifurcation curves, inverse problems are easier. They are handled at the end of the section.

Although we mostly deal with diagrams having no more than four bifurcation curves, the procedure we adopt indicates that roses

$$\mathcal{R}_{n\delta}(\alpha) = (\cos n\alpha \cos \alpha + \delta, \cos n\alpha \sin \alpha),$$

$0 \leq \alpha < 2\pi$, might be helpfull in the study of inverse problems for general bifurcation diagrams. In fact, the Theorem 3.7 is a result in this direction for a diagram with six bifurcation curves.

2. Direct Problems;

Bifurcation of Harmonic Solutions

In this section we just state the results and give the references where the proofs can be found. However, in order to show what is the closed curve $\mathcal{A}(\alpha)$ and why it comes to be so fundamental, we deduce below the bifurcation equation for the direct problem through the Liapunov-Schmidt reduction. Another reason for this is that it is a nonstandard application of the Liapunov-Schmidt method, where the role played by the phase-shift parameter α is crucial.

The reader who is aware with this specific problem might skip this part and go directly to the statement of the Theorem 2.1, its corollary and to the end of the section, where the formulae for the curves $\mathcal{A}(\alpha)$ are given for the equations under special consideration in the present paper.

Suppose $\hat{G} : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ is a C^r map, $r \geq 2$, and the two-dimensional system

$$x' = \hat{G}(x) \tag{2.1}$$

has a nonconstant 2π-periodic solution $p(t) = \big(p_1(t), p_2(t)\big)$.

Let $\Gamma := \{p(t) | 0 \leq t < 2\pi\}$ be the orbit of $p(t)$ in the x_1x_2-plane.

Let $F : \mathbf{R} \times \mathbf{R}^2 \times \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ be a C^r map, with $r \geq 2$, such that $F(t, x, \gamma) = F(t + 2\pi, x, \gamma)$, $F(t, x, 0) = 0$, for all $t \in \mathbf{R}; x, \gamma \in \mathbf{R}^2, \gamma := (\lambda, \mu)$.

In this section we will characterize the number of 2π-periodic solutions close to $p(t + \alpha)$, for some $\alpha, 0 \leq \alpha < 2\pi$, of the perturbed equation

$$x' = \hat{G}(x) + F(t, x, \gamma), \tag{2.2}$$

when γ runs over a neighborhood of $0 \in \mathbf{R}^2$. This is the *direct problem*. The 2π-periodic solutions of (2.2) are called harmonic because they have the same period of the basic solution $p(t)$. We will follow closely the approach in [9], see also [3; Chap. 11] and [4].

Let $p^{\perp}(s) = (p_2'(s), -p_1'(s))$ be a vector orthogonal to $p'(s)$, for each $s \in [0, 2\pi)$, so that each point x in a suitable neighborhood W_0 of Γ can be associated to coordinates $\langle s, \beta \rangle$, $0 \le s < 2\pi$, $|\beta| < \beta_0$, where β_0 is a sufficiently small positive constant, chosen in such a way that x can be uniquely defined by

$$x = p(s) + \beta p^{\perp}(s).$$

This defines a tubular neighborhood of Γ.

Thus, for each 2π-periodic solution $x(t)$ of (1.2) which remains in W_0, there exists a unique $\alpha, 0 \le \alpha < 2\pi$, such that

$$x(\alpha) = p(0) + \beta p^{\perp}(0),$$

that is, the value of α is chosen in such a way that the coordinates of $x(\alpha)$ are of the form $\langle 0, \beta \rangle$, with $|\beta| < \beta_0$. Such a solution can be represented as

$$x(t) = p(t - \alpha) + z(t - \alpha), \qquad z(0) \cdot p'(0) = 0. \tag{2.3}$$

For each α, $0 \le \alpha < 2\pi$, the 1-1 correspondence between solutions $x(t)$ of (2.2) and solutions $x_\alpha(t) := x(t + \alpha)$ of

$$x' = \hat{G}(x) + F(t + \alpha, x, \gamma), \tag{2.4}$$

makes our problem equivalent to studying the 2π-periodic solutions of (2.4) of the form

$$x(t) = p(t) + z(t), \qquad z(0) \cdot p'(0) = 0, \tag{2.5}$$

with small z.

If the transformation (2.5) is applied to (2.4), then

$$z' = A(t)z + \hat{F}(t, z, \gamma, \alpha),$$
$$(2.6) \qquad 0 = z(0) \cdot p'(0) = 0,$$

where $A(t) := d\hat{G}(p(t))/dx$ and

$$\hat{F}(t, z, \gamma, \alpha) := \hat{G}(p(t) + z) - \hat{G}(p(t)) - A(t)z + F(t + \alpha, p(t) + z, \gamma).$$

The function $p'(t)$ is a nontrivial 2π-periodic solution of the linear variational equation around $p(t)$,

$$(2.7) \qquad y' = A(t)y.$$

From now on we assume the hypothesis:

(H1) The space of 2π-periodic solutions of (2.7) is spanned by $p'(t)$.

Let $q(t) := (q_1(t), q_2(t))$ be a nontrivial 2π-periodic solution of the adjoint equation to (2.7),

$$(2.8) \qquad y' = -yA(t),$$

where y is a row-vector. The hypothesis (H1) implies the space of 2π-periodic solutions of (2.8) is also one-dimensional and, therefore, spanned by the solution $q(t)$. Except when the contrary is explicitly mentioned, we always assume $q(t)$ normalized in such a way that $\int_0^{2\pi} |q|^2 = 1$.

Let $Y := \{y : \mathrm{R} \to \mathrm{R}^2 | y \in C^0,\ y(t + 2\pi) = y(t), \forall t \in \mathrm{R}\}$ be endowed with the sup norm and consider its algebraic subspace $Z := \{z \in Y | z \in C^1\}$, with a C^1-norm.

If the maps $\mathcal{L} : Z \to Y$ and $\mathcal{N} : Z \times \mathbb{R}^3 \to Y$ are given by:

$$\mathcal{L}z = z' - A(t)z$$
$$\mathcal{N}(z,\gamma,\alpha) = \hat{F}(t,z,\gamma,\alpha),$$

the problem (2.6) becomes equivalent to

$$\mathcal{L}z = \mathcal{N}(z,\gamma,\alpha), \qquad z(0)\cdot p'(0) = 0. \tag{2.9}$$

In order to apply the reduction of Liapunov-Schmidt to the problem (2.9) – see [3], for instance – we need projections $\mathcal{P} : Z \to Z$, $\mathcal{Q} : Y \to Y$. These can be given by:

$$\mathcal{P}z = |p'(0)|^{-2}\left(z(0)\cdot p'(0)\right)p', \qquad z \in Z$$
$$\mathcal{Q}y = \left[\int_0^{2\pi} y(t)\cdot q(t)dt\right] q, \qquad y \in Y.$$

In this setting, the problem (2.9) is equivalent to the system of equations in Z:

$$\begin{aligned} &\mathcal{L}z = \mathcal{N}(z,\gamma,\alpha) \\ &\mathcal{P}z = 0, \end{aligned} \tag{2.10}$$

It follows from the Fredholm Alternative – according to [5], for instance – that $\mathcal{L}Z = (I - \mathcal{Q})Y$, where I denotes the identity operator. Furthermore, as a consequence of the closed graph theorem one can show the existence of a bounded linear operator $\mathcal{K} : (I - \mathcal{Q}) \longrightarrow (I - \mathcal{P})Z$ which is the pseudo-inverse of $\mathcal{L}$, that is, $\mathcal{L}\mathcal{K} = I$ in $(I - \mathcal{Q})Y$, and $\mathcal{K}\mathcal{L} = (I - \mathcal{P})Z$ in Z. A decomposition of the first equation in (2.10) in the supplementary subspaces $\mathcal{Q}Y$, $(I - \mathcal{Q})Y$, taking into account the properties of $\mathcal{K}$ combined with the second equation of (2.10), leads to:

$$\begin{aligned} &(a) \quad z = \mathcal{K}(I - \mathcal{Q})\mathcal{N}(z,\gamma,\alpha) \\ &(b) \quad 0 = \mathcal{Q}\mathcal{N}(z,\gamma,\alpha), \end{aligned} \tag{2.11}$$

The compactness of the interval $[0, 2\pi]$ implies, after a finite number of applications of the Implicit Function Theorem, the existence of a neighbohood $W \subset Z$ of 0, a neighborhood $\mathcal{V}_0$ of $(0, \alpha) \in \mathbb{R}^2 \times [0, 2\pi]$, and a C^r function $z^*(\gamma, \alpha)$, satisfying (2.11) for all $(\gamma, \alpha) \in \mathcal{V}_0$, $z^*(0, \alpha) = 0, 0 \leq \alpha \leq 2\pi$.

Thus, $(\gamma, \alpha) \in \mathcal{V}_0$ satisfies the equation (2.11)(b), with $z = z^*(\gamma, \alpha)$, if and only if, there exists a 2π-periodic solution of (2.2) in the neighborhood $W \subset Z$ of p_α, $\alpha \in [0, 2\pi]$, which can be represented by (2.3). In other words, a discussion of the existence of 2π-periodic solutions of (2.2), in the form (2.3), can be done by dicussing the solutions (γ, α) of the bifurcation equation (2.11)(b), which is now rewriten as:

$$\int_0^{2\pi} \left[q(t) \cdot \mathcal{N}(z^*(\gamma, \alpha), \gamma, \alpha)(t)\right] dt = 0. \tag{2.12}$$

By defining the 2π-periodic C^r function, $r \geq 2$:

$$\mathcal{A}(\alpha) := \int_0^{2\pi} q(t) \left[\frac{\partial F}{\partial \gamma}(t + \alpha, p(t), 0)\right] dt,$$

the Equation (2.12) can be written as

$$\mathcal{A}(\alpha) \cdot \gamma + R(\gamma, \alpha) = 0, \tag{2.13}$$

where $R(\gamma, \alpha) = O\left(|\gamma|^2\right)$, as $\gamma \to 0$.

We assume the closed curve $\mathcal{A}(\alpha) = \left(\mathcal{A}_1(\alpha), \mathcal{A}_2(\alpha)\right)$, $0 \leq \alpha < 2\pi$, satisfies the following hipotheses:

(H2) $\mathcal{A}(\alpha)$, $\mathcal{A}'(\alpha)$ are linearly independent, except possibly at finitely many values $\alpha = \alpha_i$, $i = 1, \ldots, n$, where $i \neq j$ implies $\mathcal{A}(\alpha_i)$, $\mathcal{A}(\alpha_j)$ linearly independent. Moreover, $\mathcal{A}''(\alpha)$ is never simultaneously collinear with $\mathcal{A}(\alpha)$ and $\mathcal{A}'(\alpha)$.

(H3) $\mathcal{A}(\alpha) \neq 0$, $0 \leq \alpha < 2\pi$.

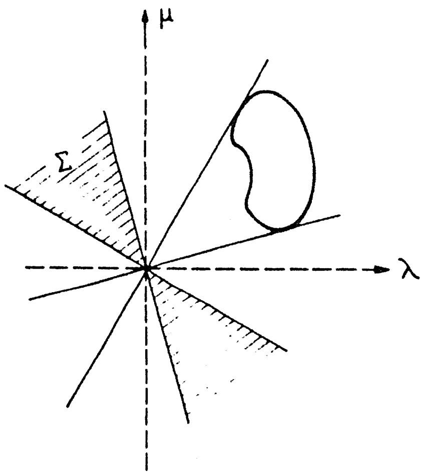

Figure 2.1

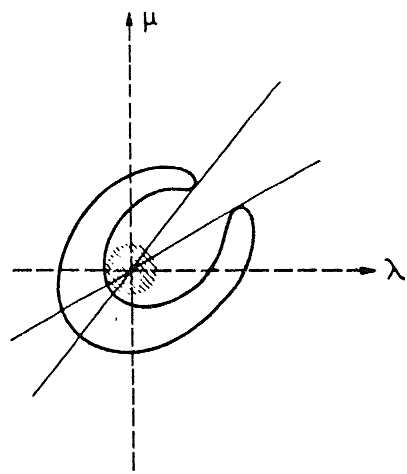

Figure 2.2

Suppose $(H1),(H2)$ are satisfied and let V a small open neighborhood of $\gamma = 0$. The set $\Sigma \subset V$ is defined as:

$$\Sigma := \{\gamma \in V \mid \gamma \cdot \mathcal{A}(\alpha) = 0, \ for \ \ some \ \ \alpha \in [0, 2\pi)\}.$$

Either $\Sigma = V$ or there are values $\alpha = \alpha_j$, $j = 1, 2$ uniquely defined by the property that the line through $\mathcal{A}(\alpha_j)$ and the origin intercept the curve $\mathcal{A}(\alpha)$ only at tangencies. In this case, the boundary of Σ in V is $(l_1 \bigcup l_2) \bigcap V$, where the lines l_1, l_2 are given by $\gamma \cdot \mathcal{A}(\alpha_j) = 0$, $j = 1, 2$.

If $\Sigma \neq V$, the set Σ^* considered in the statement of the next theorem is a perturbation of Σ, in the sense that its boundary in V is a pair of curves tangent to the lines l_1, l_2, respectively, at the origin. Otherwise, $\Sigma^* = \Sigma = V$.

THEOREM 2.1. *Suppose the hypotheses (H1–3) are satisfied, then there exists a neighborhood* $W \subset Z$ *of* p_α, $0 \leq \alpha < 2\pi$, *a ball* $V \subset \mathbb{R}^2$ *with center in* $\gamma = 0$ *and a set*

$\Sigma^* \subset V$ *such that:*

(i) If $\gamma \in \Sigma^* \backslash \partial\Sigma^*$, *there are at least two* 2π*-periodic solutions of (2.2) in* W,

(ii) If $\gamma \in V \backslash \Sigma^*$, *there are no* 2π*-periodic solutions of (2.2) in* W,

(iii) If $\gamma \in \partial\Sigma^*$, *there is a unique* 2π*-periodic solution of (2.2) in* W.

Moreover, for each α_0, $\alpha_0 \in [0, 2\pi)$, *with* $\mathcal{A}(\alpha_0)$, $\mathcal{A}'(\alpha_0)$ *linearly dependent, there exists a unique curve* $\mathcal{C}(r_0)$, *tangent to the line* $r_0 := \{\gamma | \gamma \cdot \mathcal{A}(\alpha_0) = 0\}$, *at* $\gamma = 0$, *which divides* V *in two connected sets such that the number of* 2π*-periodic solutions of (2.2) in* W *changes by two when* γ, $\gamma \neq 0$, *crosses* $\mathcal{C}(r_0)$.

A proof of Theorem 2.1 with minor adaptations can be found in [3, 11.2 - Th. 2.1] or [9, Sec. 3]. It consists basically in discussing the bifurcation equation (2.13). By studying this equation neglecting the higher order terms, $R(\gamma, \alpha)$, one can get a good insight of the problem. $\mathcal{C}(r_0)$ are the bifurcation curves. Figure 2.1 shows a specific situation where Σ is given by sectorial regions. Figure 2.2 illustrate the possibility to have $\Sigma = V$ with the existence of some bifurcation curves. The number in parenthesis indicates the number of 2π-periodic solutions in W when γ is in the corresponding region.

The following Corollary points out the possibility of do not occurring bifurcation. This is a special case where the curve $\mathcal{A}(\alpha)$ must encircle the origin.

COROLLARY. *If the hypotheses (H1–3) are satisfied with* $\mathcal{A}(\alpha)$, $\mathcal{A}'(\alpha)$ *linearly independent for all* α, $0 \leq \alpha < 2\pi$, *then there exists a neighborhood* $W \subset X$ *of* p_α *and a neighborhood* $V \subset \mathbb{R}^2$ *of* $\gamma = 0$ *such that, for each* $\gamma \in V$, *the equation (2.2) has precisely two* 2π*-periodic solutions in* W.

The equation (2.2) includes the following three particular cases that will be specially considered in the sequel:

(E1) $$y'' + \tilde{g}(y, y') + \lambda f_1(t) + \mu f_2(t) = 0, \;\; y \in \mathbf{R},$$

(E2) $$y'' + g(y) = -\lambda y' + \mu f(t), \;\; y \in \mathbf{R},$$

with $x = (y, y')$, and

(E3) $$x' = G(x) + \hat{\gamma} h(t), x \in \mathbf{R}^2,$$

where $\gamma = (\lambda, \mu) \in \mathbf{R}^2$, and

$$\hat{\gamma} := \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}.$$

The terms containing one of the functions f, f_1, f_2 and h, which are assumed to be 2π-periodic, will be referenced as defining, together with the damping term in (E2), the *perturbation part* of the corresponding equation. Thus, the unperturbed equations are obtained when all the parameters vanish.

The Equation (E3) is a perturbation of the well-known Lotka-Volterra predator-prey model for two species, that is, we assume that G is given by

$$G(x) = \begin{pmatrix} ax_1 - bx_1x_2 \\ cx_1x_2 - dx_2 \end{pmatrix},$$

where a, b, c, d are positive constants; a and d define growth and decline rates, respectively, while b and c describe predation effects. The results of this section for this specific equation are found in [9].

In the Equation (E2) g is supposed to satisfy $yg(y) > 0, \;\; y \neq 0$, so that it is the equation of a nonlinear oscillator with a damping and a 2π-periodic forcing. A first study of (E2) in the present setting is done in [6]. See [4,Th. 3.1] or [3, Chap. 11], for the Equation (E1).

In view of the importance of the curve $\mathcal{A}(\alpha)$ for the next section, we give below its formula in the special case of each of the equations (E1–3)

For (E1), we let $q = (q_1, q_2)$ be the 2π-periodic solution of the two-dimensional system associated to the adjoint equation to (E1), normalized in such a way that $\int_0^{2\pi} |q|^2 = 1$. In this case,

$$\mathcal{A}(\alpha) = -\left(\int_0^{2\pi} f_1(t+\alpha)q_2(t)dt, \int_0^{2\pi} f_2(t+\alpha)q_2(t)dt\right).$$

For the remaining equations the corresponding curves $\mathcal{A}(\alpha)$ are accomplished with the normalization of q being neglected. This implies nothing but the replacement of some quantities by a constant multiple of them.

For (E2), we take $q(t) = \left(-p_1''(t), p_1'(t)\right)$, where $\left(p_1(t), p_2(t)\right)$ defines the 2π-periodic orbit of the two-dimensional system associated to the free oscillator. In this case, if $c := \int_0^{2\pi} p_1'^2(t)dt$,

$$\mathcal{A}(\alpha) = \left(-c, \int_0^{2\pi} f(t+\alpha)p_1'(t)dt\right).$$

This shows that the bifurcation diagrams of 2π-periodic solutions of (E2) have special features. The curve $\mathcal{A}(\alpha)$ is flat and the points $(\lambda, 0)$, $\lambda \neq 0$, are always in $V \backslash \Sigma^*$. This agree with the fact a damping with no forcing term destroys periodicity and implies that Equation (E2) never fulfills the hypotheses of the Corollary to the Theorem 2.1.

For (E3), $\mathcal{A}(\alpha) = \left(\mathcal{A}_1(\alpha), \mathcal{A}_2(\alpha)\right)$ is given by:

$$\mathcal{A}_1(\alpha) = d\int_0^{2\pi} \frac{h_1(t+\alpha)}{p_1(t)}dt - cM_1,$$

and

$$\mathcal{A}_2(\alpha) = a\int_0^{2\pi} \frac{h_2(t+\alpha)}{p_2(t)}dt - bM_2,$$

where $M_j := \int_0^{2\pi} h_j(t+\alpha)dt$, $j = 1,2$, and $p(t)$ defines the 2π-periodic orbit of the unperturbed equation. It is known that the first quadrant is invariant for the Lotka-Volterra equation. Since only this quadrant is meaningful for this model, the study of the equation (E3) is restricted to it. So that one has $p_j(t) > 0$, for all t, $j = 1,2$, and, therefore, the curve $\mathcal{A}(\alpha)$ is well defined..

3. INVERSE PROBLEMS.

A bifurcation diagram in the $\lambda\mu$-plane, relative to the Theorem 2.1, is a $2n$-tuple of curves $C(r_1)\dots C(r_{2n})$, tangent at the origin to lines $r_1 \dots r_{2n}$, respectively. These curves define sectorial subsets of a neighborhood V of the origin such that, when $\gamma = (\lambda, \mu)$ is in the interior of such a subset, the number of 2π-periodic solutions of the Equation (2.2) is constant. To opposite subsets (*that is, sectorial subsets* V_1, V_2 *of* V *such that* $V \bigcap \partial V_1 \bigcap \partial V_2 = \{0\}$ *and* $\partial V_j, j = 1, 2$, *have the same tangent lines at the origin*) is associated the same constant number of solutions. This number changes by two when γ, $\gamma \neq 0$, crosses each $C(r_j)$, $j = 1, \dots, 2n$. The Figure 3.1 shows a typical diagram.

Vaguely, the inverse problem suggested by this situation is: Given such a diagram, find out a perturbed equation which fits to it. We will consider this problem related to the equations (E1–3) introduced in the last section. We mostly deal with cases where the given bifurcation diagram has four bifurcation curves. In this circumstance is possible to state satisfactory results. The cases of two bifurcation curves are much easier and we consider them at the end of this section. The cases of $6, 8, \dots$ bifurcation curves might be treated in a similar way, but it requires strong hypotheses on the bifurcation diagrams.

We now define a setting where the inverse problem can be precisely formulated. This requests some notation and definitions.

From now on, we always assume the $2n$-tuple of curves $\big(C(r_1), \dots, C(r_{2n})\big)$, which determines a bifurcation diagram satisfies the following property:

We let $\theta_i, 0 < \theta_i \leq \pi$ be the angle between r_i and the positive semi-axis 0λ, $i = 1, \dots, 2n$, in the counterclockwise direction, in such a way that $0 < \theta_1 < \dots < \theta_{2n} \leq \pi$, according to the Figure 3.1.

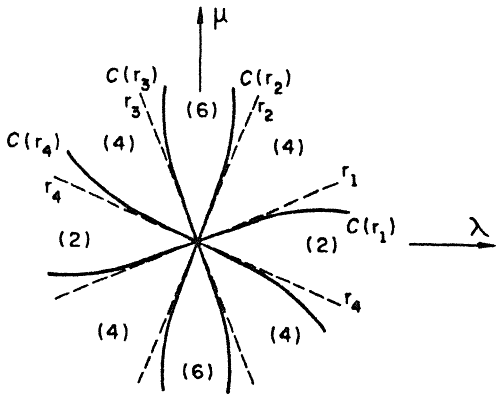

Figure 3.1

Keeping this ordering, by noncritical sectorial subsets of V we mean opposite subsets defined by two successive curves, $C(r_j), C(r_{j+1}), 1 \leq j \leq 2n$, where no bifurcation occurs. When $n = 1$ all the four sectorial subsets are noncritical.

Let us introduce a notation for a bifurcation diagram which specifies the curves $C(r_j)$, $j = 1, \ldots, 2n$, and the constant number of 2π-periodic solutions for γ in the sectorial subsets of V. Such a diagram will be denoted by a $4n$-tuple, whose first component is a number, and alternates curves $C(r_j)$ and numbers in such a way that the indexes j are in the increasing order. The numbers denote the constant number of 2π-periodic solutions for γ in the noncritical sectorial subsets of V in such a way that, when $(\lambda, \mu), \mu \geq 0$, crosses counterclockwise the curves along a sufficiently small circle, starting on the positive semi-axis 0λ, these numbers vary according to the order they occur in the $4n$-tuple.

This is easier to clarify by an example: The diagram in the Figure 3.1 will be denoted by $(2, C(r_1), 4, C(r_2), 6, C(r_3), 4, C(r_4))$.

DEFINITION 3.1. *The diagrams* $(k_1, C(r_1), \ldots, k_{2n}, C(r_{2n}))$, $(l_1, \tilde{C}(s_1), \ldots, l_{2n}, \tilde{C}(s_{2n}))$ *are said to be equivalent if*

$$k_j = l_j; \qquad r_j = s_j, \qquad j = 1, \ldots, 2n$$

Each class of equivalence of the above relation has a canonical representative $(k_1, C(r_1), \ldots, k_{2n}, C(r_{2n}))$ such that $C(r_j) = r_j$, $j = 1, \ldots, 2n$. This will be called a *canonical diagram.*

Now we are in a position to formulate precisely the inverse problem with respect to any one of the equations (E1–3):

"Given a canonical diagram $\mathcal{D} = (k_1, r_1, \ldots, k_{2n}, r_{2n})$ and a 2π-periodic solution $p(t)$ of the unperturbed equation, find functions which define the perturbation part of the concerning equation in such a way that the bifurcation diagram of the 2π-periodic solutions near p_α, provided by the Theorem 2.1, is equivalent to $\mathcal{D}$."

Before starting to deal with this problem, let us recall some elementary facts from the Projective Geometry which will be useful from now on. They are contained in the references [1] or [2], for instance.

A projective coordinate system at the point P, in the plane, is a 3-tuple of pairwise distinct lines, r_i, $i = 1, 2, 3$, concurrent at P, which we will denote by (r_1, r_2, r_3). Given a projective coordinate system at P, (r_1, r_2, r_3), and a fourth line through P, r_4, we choose the vectors e_i in the direction of r_i, $i = 1, 2, 3$, in such a way that $e_3 = e_1 + e_2$. Therefore, there exists a unique number $\rho \in \mathbf{R}$ such that the vector $e_4 := e_1 + \rho e_2$ is parallel to r_4. See the Figure 3.2. The number ρ is called the *projective coordinate of* r_4

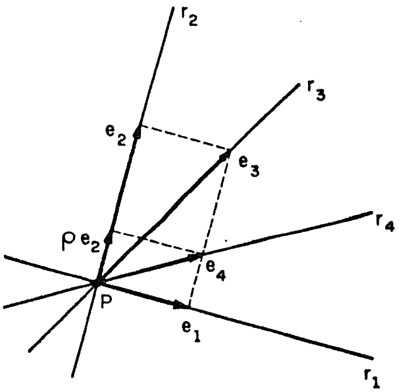

Figure 3.2

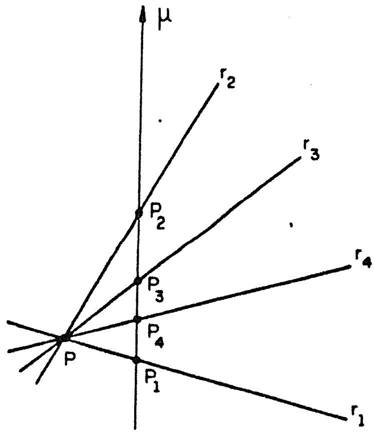

Figure 3.3

in the projective coordinate system (r_1, r_2, r_3). This is one of the six numbers associated to r_1, r_2, r_3, r_4, known as a *cross ratio* of the four lines.

REMARK. *If* $\rho \in (0,1)$, *it can be computed by the formula*

$$\rho = \frac{d(P_1, P_4)}{d(P_1, P_3)} \frac{d(P_2, P_3)}{d(P_2, P_4)},$$

where the points P_i *are the intersection of* r_i,$i = 1, 2, 3, 4$, *with any transversal line* μ *and* $d(P_i, P_j)$ *is the distance from* P_i *to* P_j. *See Figure 3.3.*

Let $r_1, \ldots, r_4$ a pencil of lines through a point P in the plane. If $e_1, \ldots, e_4$ are the vectors referenced above and θ_j is the angle between e_j and e_1, $j = 2, 3, 4$, let us choose an ordering of the lines such that $\pi > \theta_2 > \theta_3 > \theta_4 > 0$. Now, the proof of the following lemma is nothing but a straightforward computation of ρ using the above remark.

LEMMA 3.1. *Any pencil of four lines,* $r_1, \ldots, r_4$, *through a point* P *in the plane, admits* *a reordering in such a way that the projective coordinate ρ of r_4 with respect to the projective coordinate system (r_1, r_2, r_3) lies between 0 and 1.*

In order to state the main results of this paper, we need one more lemma which connects these elementary facts from Projective Geometry with the roses.

LEMMA 3.2. *Suppose $\mathcal{R}_\delta$ is the rose given by $\mathcal{R}_\delta(\beta) = (\cos 3\beta \cos\beta + \delta, \cos 3\beta \sin\beta)$, $0 \leq \beta < \pi$, $\delta \in \mathbf{R}$. Let r_1, r_2, r_3, r_4 be the pencil of lines through the origin tangent to $\mathcal{R}_\delta$, according to the Figure 3.4, if $\delta > 0$, or to the Figure 3.5, if $\delta < -1$. Then the projective coordinate ρ of r_2, with respect to the projective coordinate system (r_1, r_4, r_3), takes all the values in the interval $(0,1)$, when δ varies over $(0, \frac{9}{16})$ or $(-\frac{9}{8}, -1)$.*

PROOF: Suppose $\delta \in (0, \frac{9}{16})$. If m_i is the slope of the line r_i, $i = 3, 4$, then

$$\rho = \left(\frac{m_4 - m_3}{m_4 + m_3}\right)^2 \tag{3.1}$$

and elementary calculations give:

$$\begin{aligned} m_i &= \frac{3 - 18\cos^2\beta_i + 16\cos^4\beta_i}{(3 - 8\cos^2\beta_i)\sin 2\beta_i}, \\ \delta &= -\frac{\cos^2 3\beta_i}{16\cos^4\beta_i - 18\cos^2\beta_i + 3}; \end{aligned} \tag{3.2}$$

$$i = 3,4, \qquad \frac{4\pi}{45} < \beta_3 < \frac{\pi}{6}, \qquad \frac{\pi}{6} < \beta_4 < \frac{53\pi}{180}.$$

As a consequence of the Inverse Function Theorem, the second equation of (3.2) can be inverted and, combining it with the first one, we have m_i as a function of δ, $m_i = m_i(\delta), i = 3, 4$. Therefore, ρ is a function, obviously continuous, of δ. By noticing that $\rho(0+) = 0$ and $\rho(\frac{9}{16}-) = 1$, the lemma follows from the continuity of ρ. ◊

Our first statement concerning the inverse problem is a theorem about the equation (E3) formulated below. It is known that the first quadrant of the plane of phase is

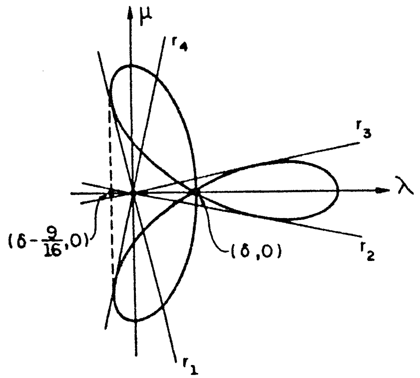

Figure 3.4

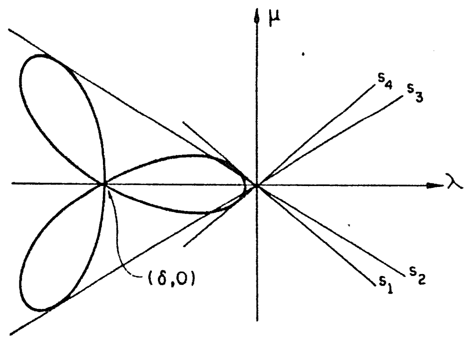

Figure 3.5

invariant for this equation and that each nonconstant orbit there is periodic and encircles the equilibrium $(\frac{d}{c}, \frac{a}{b})$. Moreover, the period is a strictly increasing function of the amplitud and this implies that our hypothesis (H1) is at least a generic property in this case.

THEOREM 3.1. *Let $\mathcal{D}$ be one of the canonical diagrams: $\mathcal{D} = (4, l_1, 2, l_2, 4, l_3, 2, l_4)$ or $\mathcal{D} = (0, l_1, 2, l_2, 4, l_3, 2, l_4)$ and suppose the hypothesis (H1) is satisfied for the unperturbed equation assotiated to (E3). Then, there exists a function $h = (h_1, h_2)$ such that the bifurcation diagram of the 2π-periodic solutions of (E3) near p_α is equivalent to $\mathcal{D}$.*

Moreover, a nonsingular matrix $M = (a_{ij}), 1 \le i, j \le 2$ can be effectively computed in such a way that h is given by $h = (\tilde{h}_1\ \tilde{h}_2)M$, where

$$\tilde{h}_1(t) = \frac{\delta}{d\int_0^{2\pi} \frac{dt}{p_1(t)} - 2c\pi} + u_2 e^{2ti} + u_{-2} e^{-2ti} + u_1 e^{ti} + u_{-1} e^{-ti},$$

$$\tilde{h}_2(t) = -v_2 e^{2ti} + v_{-2} e^{-2ti} + v_1 e^{ti} - v_{-1} e^{-ti},$$

with

$$u_n = [8\pi d\hat{p}_1(n)]^{-1}, \ v_n = i[8\pi a\hat{p}_2(n)]^{-1},$$

$n = -2, -1, 1, 2,$ *and* $\hat{p}_k(n)$ *being the nth Fourier coefficient of* $\frac{1}{p_k(t)}$ *with respect to* $e^{int}, n \in \mathbf{Z}, k = 1, 2$*:*

$$\hat{p}_k(n) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{int}}{p_k(t)} dt; \qquad n = -2, -1, 1, 2; \ k = 1, 2.$$

PROOF: We just prove for $\mathcal{D} = (4, l_1, 2, l_2, 4, l_3, 2, l_4)$, since the other case is completely similar.

Let us denote by $r_1, \ldots, r_4$ the lines through the origin, orthogonal to $l_1, \ldots, l_4$, respectively, and let ρ, $0 < \rho < 1$, be the projective coordinate of r_2 with respect to the projective coordinate system (r_1, r_2, r_3). According to the Lemma 3.2, there exists a number $\delta \in (0, \frac{9}{16})$ such that the rose

$$\mathcal{R}_\delta(\alpha) = (\cos \tfrac{3\alpha}{2} \cos \tfrac{\alpha}{2} + \delta, \cos \tfrac{3\alpha}{2} \sin \tfrac{\alpha}{2}), \ 0 \leq \alpha < 2\pi$$

has the tangent lines through the origin, $r'_1, \ldots, r'_4$, ordered according to the Figure 3.3 (with $r'_j = r_j$) in such a way that the projective coordinate of r'_2 with respect to (r'_1, r'_2, r'_3) is precisely ρ. Let us choose vectors e_j and e'_j in the direction of r_j and r'_j, respectively, $j = 1, 2, 3, 4$, in order to compute ρ by its definition. This gives an isomorphism E, unique up to homothety, such that $E(0 + e_j) = (0 + e'_j)$, *i.e.*, $Er_j = r'_j$, $j = 1, 2, 3, 4$,.

The proof is from now on a matter of routine by taking M as the matrix of the inverse isomorphism E^{-1}. Notice that we can deal with the transpose in order to keep h in the notation of row-vector. In fact, if the perturbation h is chosen as in the statement of the theorem and we use Theorem 2.1 to compute $\mathcal{A}(\alpha)$, we obtain $\mathcal{A}(\alpha) = E^{-1}\mathcal{R}_\delta(\alpha)$.

Therefore, according to the choice of E, the corresponding bifurcation curves are tangent at the origin to the given lines l_1, l_2, l_3, l_4. $\diamondsuit$

REMARKS. *(1) In order to ensure the function h is well defined, we claim that the denominator in the fraction appearing in the definition of $\tilde{h}_1$ never vanish. In fact, in the set $\mathcal{S}$ of all the continuous, positive, 2π-periodic functions f, let us denote by $V(f)$ the mean value, $V(f) := \frac{1}{2\pi}\int_0^{2\pi} f(t)dt$. Let $f \in \mathcal{S}$ satisfy $V(f)V(\frac{1}{f}) = 1$. By considering $\langle\cdot,\cdot\rangle$, the $\mathcal{L}^2$-inner product in $\mathcal{C}([0,2\pi],\mathrm{R})$, that is, $\langle f,g\rangle := \int_0^{2\pi} f(t)g(t)dt$, we have*

$$(2\pi)^2 = \left\langle \sqrt{f}, \frac{1}{\sqrt{f}} \right\rangle^2 \leq \left|\sqrt{f}\right|_2^2 \left|\frac{1}{\sqrt{f}}\right|_2^2 = \int_0^{2\pi} f(t)dt \int_0^{2\pi} \frac{1}{f(t)}dt = (2\pi)^2,$$

where $|\cdot|_2$ denotes the $\mathcal{L}^2$-norm, so that, in this case, the Schwarz Inequality reduces to an equality. Therefore, $\sqrt{f}$ is proportional to $\frac{1}{\sqrt{f}}$ and so f must be constant. Since $p_1(t)$ is not constant and it is a well-known property of the Lotka-Volterra System that $V(p_1) = \frac{d}{c}$, it follows that $V(\frac{1}{p_1}) \neq \frac{c}{d}$. $\diamondsuit$

(2) Since $p_1(t), p_2(t) \neq 0$, for all $t \in \mathrm{R}$, the Fourier coefficients mentioned in the statement of Theorem 3.1 are well defined.

(3) Beginning this section we impose an ordering of the angles θ_j between l_j and 0λ, $j = 1, \ldots, 2n$. Everything can be done again by adopting any other ordering. Therefore, Theorem 3.1 remains valid with obvious adaptations for a diagram of the type $\mathcal{D} = (2, l_1, 4, l_2, 2, l_3, 4, l_4)$. It suffices to make a convenient permutation in the indexes j of l_j, $j = 1, 2, 3, 4$, and adopt another ordering for the angles $\theta_j's$.

Theorem 3.1 has a natural counterpart for the Equation (E1). We state it below and refrain to give a proof, since it would be almost a repetition of the arguments of the Theorem 3.1. By virtue of the generality of (E1), some hypotheses are needed on the solutions of the adjoint equation. Recall that $q = (q_1, q_2)$ is the 2π-periodic solution of the first order system associated to the adjoint to (E1), with $\int_0^{2\pi} |q|^2 = 1$.

THEOREM 3.2. *Let $\mathcal{D}$ be one of the canonical diagrams: $\mathcal{D} = (4, l_1, 2, l_2, 4, l_3, 2, l_4)$ or $(0, l_1, 2, l_2, 4, l_3, 2, l_4)$, assume $\nu = \int_0^{2\pi} q_2 \neq 0$; $\hat{q}_2(n) = \frac{1}{2\pi}\int_0^{2\pi} e^{int} q_2 \neq 0$, $n = -1, -2, 1, 2$ and suppose the hypothesis (H1) is satisfied for the unperturbed equation assotiated to (E1). Then, there exist functions f_1 and f_2 such that the bifurcation diagram of (E1) is equivalent to $\mathcal{D}$.*

Moreover, a nonsingular matrix $M = (a_{ij}), 1 \leq i, j \leq 2$ can be efectively computed in such a way that $f = (f_1, f_2)$ is given by $f = (\tilde{f}_1\ \tilde{f}_2)M$, where

$$\tilde{f}_1(t) = -(\delta\nu^{-1} + u_2 e^{2ti} + u_{-2}e^{-2ti} + u_1 e^{ti} + u_{-1}e^{-ti})$$

$$\tilde{f}_2 = -(-v_2 e^{2ti} + v_{-2}e^{-2ti} + v_1 e^{ti} - v_{-1}e^{-ti}),$$

with

$$u_n = [8\pi\hat{q}_2(n)]^{-1},\ v_n = i[8\pi\hat{q}_2(n)]^{-1},\ n = -2, -1, 1, 2.$$

REMARK. *The hypotheses $\nu \neq 0$ and $\hat{q}_2^n(j) \neq 0$ are equivalent, respectively, to the following problems having no solutions:*

$$\mathcal{L}z = (0, 1), \quad z(0) \cdot p'(0) = 0$$

and

$$\mathcal{L}z = (0, \kappa_{nj}), \quad z(0) \cdot p'(0) = 0,\ j = -1, 1;\ n = 1, 2,$$

where either $\kappa_{nj}(t) := \cos njt$ or $\kappa_{nj}(t) := \sin njt$ and $\mathcal{L}$ is the linear operator defined just before (2.9).

The counterpart of these results for the equation of the oscillator (E2) has some special features and requires a more specific procedure. This is basically because in

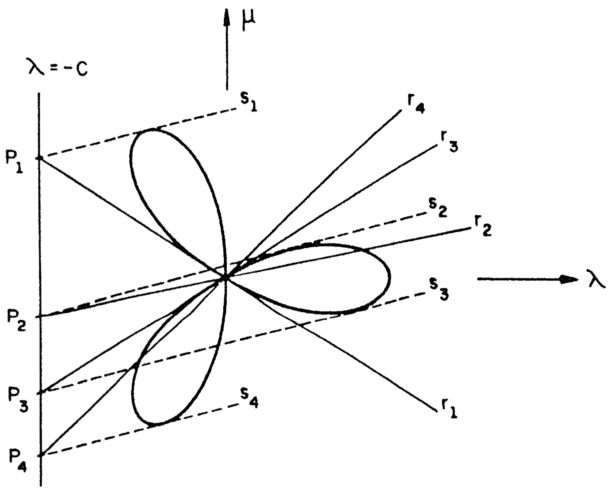

Figure 3.6

this case the closed curve $\mathcal{A}(\alpha) = (\mathcal{A}_1(\alpha), \mathcal{A}_2(\alpha))$ is always flat, $\mathcal{A}_1(\alpha) \equiv -c$, where $c := \int_0^{2\pi} p'^2(t)dt$. In view of this, the considerations below are needed before to state the main result.

Let the rose $\mathcal{R}$ be given by

(3.3) $$\mathcal{R}(\beta) = (\cos 3\beta \cos\beta, \cos 3\beta \sin\beta),\ 0 \le \beta < \pi.$$

Given a number m, $m \in (0, \sqrt{3}/3)$ consider the four lines in the $\lambda\mu$-plane, with slope m and tangent to $\mathcal{R}$ at $\mathcal{R}(\beta_i)$, $i = 1,2,3,4$:

$$s_i:\ \mu = \mathcal{R}_2(\beta_i) + m(\lambda - \mathcal{R}_1(\beta_i)),\ i = 1,2,3,4,$$

and let the points $P_i := (-c, \mu_i)$, $i = 1,2,3,4$, be the intersection of these lines with the line $\lambda = -c$, therefore,

(3.4) $$\mu_i = \mathcal{R}_2(\beta_i) - m(c + \mathcal{R}_1(\beta_i)),\ i = 1,2,3,4.$$

If r_1, r_2, r_3, r_4 are the lines drawn from P_1, P_2, P_3, P_4, respectively, through the origin, according to the Figure 3.6, it is easy to obtain from (3.4) the projective coordinate ρ of r_2, with respect to (r_1, r_4, r_3), as a continuous function $\rho(m, \beta_1, \ldots, \beta_4)$. Since $\beta_i = \beta_i(m)$, $i = 1, 2, 3, 4$, are continuous functions of m, we have $\rho = \rho(m)$ as a continuous function of m.

Moreover, $\rho(\frac{\sqrt{3}}{3}-) = 0$, since $P_4 \to P_3$, when $m \to \frac{\sqrt{3}}{3}$. Also, by noticing that $m = 0$ corresponds to the case where the lines s_1, s_2, s_3, s_4 are parallel to the λ-axis and they are distributed symmetrically with respect to this axis, according to the Figure 3.7. Therefore, the points P_1, P_2 are given by

$$P_1 = \left(-c, \frac{(3+\sqrt{33})\sqrt{30+2\sqrt{33}}}{64}\right),$$

$$P_2 = \left(-c, \frac{(-3+\sqrt{33})\sqrt{30+2\sqrt{33}}}{64}\right).$$

It is now easy to obtain $\rho(0) = \frac{69-16\sqrt{3}}{69+16\sqrt{3}}$ as a consequence of the formula (3.1).

We are now in a position to state the following

THEOREM 3.3. *Let $\mathcal{D}$ be a canonical diagram, $\mathcal{D} = (0, l_1, 2, l_2, 4, l_3, 2, l_4)$ and suppose the hypothesis (H1) is satisfied for the unperturbed equation assotiated to (E2). If $\hat{p}_1(n) := \frac{1}{2\pi}\int_0^{2\pi} e^{int} p_1'(t)dt \neq 0$, $n = -2, -1, 1.2$, and the projective coordinate ρ of l_2, with respect to (l_1, l_4, l_3), satisfies $\rho \in \left(0, \frac{69-16\sqrt{3}}{69+16\sqrt{3}}\right)$, then a 2π-periodic function f and an isomorphism*

$$E : (\lambda, \mu) \in \mathbf{R}^2 \mapsto E(\lambda, \mu) = (\tilde{\lambda}, \tilde{\mu}) \in \mathbf{R}^2$$

can be effectively computed in such a way that the bifurcation diagram of the 2π-periodic solutions of

$$x'' + g(x) = -\tilde{\lambda}(\lambda, \mu)x' + \tilde{\mu}(\lambda, \mu)f(t) \tag{3.5}$$

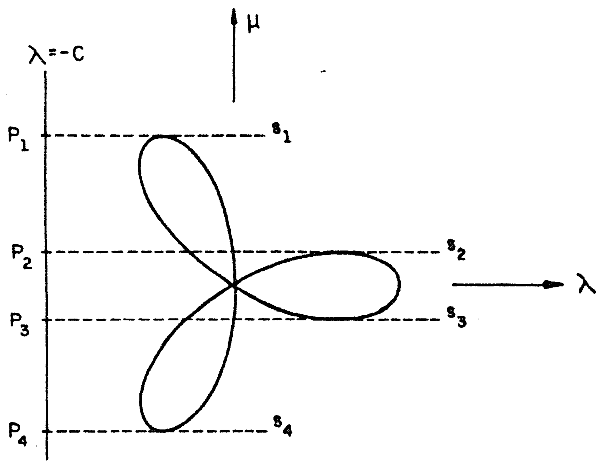

Figure 3.7

is equivalent to $\mathcal{D}$.

Moreover, f is given by

$$f(t) = -v_2 e^{2ti} + v_{-2} e^{-2ti} + v_1 e^{ti} - v_{-1} e^{-ti},$$

where $v_n = i\left[8\pi \hat{p}_1(n)\right]^{-1}$, $n = -2, -1, 1, 2$.

PROOF: Let $\rho \in \left(0, \frac{69-16\sqrt{3}}{69+16\sqrt{3}}\right)$ be the projective coordinate of the line l_2 with respect to (l_1, l_4, l_3). If $c := \int_0^{2\pi} p_1'^2(t)dt$, according to our considerations before Theorem 3.3, there is a number $m \in (0, \sqrt{3}/3)$ such that the projective coordinate of the line r_2 with respect to (r_1, r_4, r_3) in the Figure 3.6 is precisely ρ. So that, if r_j' is orthogonal to l_j at the origin, $j = 1, \ldots, 4$, there is an isomorphism E, unique up to homothety, such that $Er_j' = r_j$, $j = 1, \ldots, 4$. By using the function f defined in our statement, the proof consists now of an application of the formula which gives $\mathcal{A}(\alpha)$ for the equation (3.5)

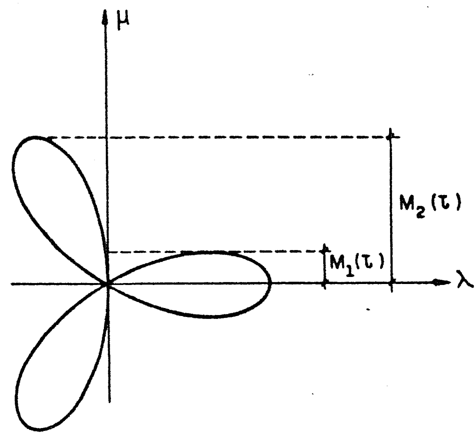

Figure 3.8

getting

$$\mathcal{A}(\alpha) = \left(-c, \mathcal{R}_2(\alpha) + m(c + \mathcal{R}_1(\alpha))\right), \ 0 \leq \alpha < 2\pi$$

where $\mathcal{R} = (\mathcal{R}_1, \mathcal{R}_2)$ is given in (3.3). Thus, the the bifurcation diagram for (3.5) in the $\tilde{\lambda}\tilde{\mu}$-plane is equivalent to $(0, r_1', 2, r_2', 4, r_3', 2, r_4')$ and, therefore, in the $\lambda\mu$-plane is equivalent to $\mathcal{D}$. ◇

REMARK. *If the restoring term g is odd, it is well-known the free oscillations are symmetric and the hypothesis $\hat{p}(n) \neq 0$ is not fulfilled. However, this condition can be accomplished by small perturbations of g.*

Theorem 3.3 can be generalized by allowing ρ to vary in intervals larger than $\left(0, \frac{69-16\sqrt{3}}{69+16\sqrt{3}}\right]$. It suffices to replace the rose $\mathcal{R}$ in the last theorem by the more general:

$$\mathcal{R}_\tau(\beta) = (\cos^\tau 3\beta \cos\beta, \cos^\tau 3\beta \sin\beta),$$

$0 \leq \beta < \pi$, $\tau = 1, 3, 5, \ldots$. These are deformations of $\mathcal{R}$ which, for convenient values of τ, in the same reasoning of Theorem 3.3 give ranges $(0, \rho_0]$ for ρ, with $\rho_0 < 1$ arbitrarily close to 1. In fact, if we let $M_1(\tau) = \max_{0<\beta<\frac{\pi}{6}}(\cos^\tau 3\beta \sin\beta)$ and $M_2(\tau) = \max_{\frac{\pi}{2}<\beta<\frac{5\pi}{6}}(\cos^\tau 3\beta \sin\beta)$, an easy calculation leads to

$$M_1(\tau) = \left(\frac{-3(\tau+1)+\sqrt{\Delta}}{2(3\tau+1)}\right)^\tau \left(\frac{3(5\tau+1)+\sqrt{\Delta}}{8(3\tau+1)}\right)^{\frac{\tau}{2}} \left(\frac{9\tau+5-\sqrt{\Delta}}{8(3\tau+1)}\right)^{\frac{1}{2}},$$

$$M_2(\tau) = \left(\frac{3(\tau+1)+\sqrt{\Delta}}{2(3\tau+1)}\right)^\tau \left(\frac{3(5\tau+1)-\sqrt{\Delta}}{8(3\tau+1)}\right)^{\frac{\tau}{2}} \left(\frac{9\tau+5+\sqrt{\Delta}}{8(3\tau+1)}\right)^{\frac{1}{2}},$$

where $\Delta = 81\tau^2 + 42\tau + 9$. See Figure 3.8.

Since

$$\lim_{\tau\to\infty} M_1(\tau) = 0 \qquad \lim_{\tau\to\infty} M_2(\tau) = \frac{\sqrt{3}}{2}. \tag{3.6}$$

one can see from (3.6) that, replacing $\mathcal{R}$ by $\mathcal{R}_\tau$, $\tau = 3, 5, \ldots$ in the definitions preceding the Theorem 3.3, the projective coordinate $\rho_\tau(m)$ of r_2 with respect to (r_1, r_4, r_3) satisfies the following properties: $\rho_\tau(0+) > \frac{69-16\sqrt{3}}{69+16\sqrt{3}}$, $\rho_\tau(0+)$ increases with τ and $\lim_{\tau\to\infty} \rho_\tau(0+) = 1$.

In view of this we can state the following extension of the Theorem 3.3

THEOREM 3.4. *Let $\mathcal{D}$ be a canonical diagram, $\mathcal{D} = (0, l_1, 2, l_2, 4, l_3, 2, l_4)$ and suppose the hypothesis (H1) is satisfied for the unperturbed equation assotiated to (E2). If $\hat{p}_1(n) := \frac{1}{2\pi}\int_0^{2\pi} e^{int} p_1'(t)dt \neq 0$, $n = -2, -1, 1, 2$, then a 2π-periodic function f and an isomorphism*

$$E : (\lambda, \mu) \in \mathbf{R}^2 \mapsto E(\lambda, \mu) = (\tilde{\lambda}, \tilde{\mu}) \in \mathbf{R}^2$$

can be effectively computed in such a way that the bifurcation diagram of the 2π-periodic solutions of

$$x'' + g(x) = -\tilde{\lambda}(\lambda, \mu)x' + \tilde{\mu}(\lambda, \mu)f(t)$$

is equivalent to $\mathcal{D}$.

Moreover, f is given by

$$f(t) = -v_2 e^{2ti} + v_{-2} e^{-2ti} + v_1 e^{ti} - v_{-1} e^{-ti},$$

where $v_n = i\left[8\pi\hat{p}_1(n)\right]^{-1}$, $n = -2, -1, 1, 2.$

Let us consider now some questions we left behind. We mostly will discuss the Equation (E3), because it is an interesting case and encloses enough generality to indicate how to solve the problem for the other equations.

The simpler inverse problem is the case where the given bifurcation diagram is empty and, for every γ in some neighborhood of $0 \in \mathbb{R}^2$, one requests that (E3) has precisely two 2π-periodic solutions near p_α, $0 \leq \alpha < 2\pi$. This corresponds to the situation described by the corollary to the Theorem 2.1, where $\mathcal{A}(\alpha)$ encircles the origin and every line through the origin is transversal to it. As it was noticed before, this case does not make sense for the nonlinear oscillator equation (E2). The theorem below [9, Corollary to Th. 2] is a solution to this problem giving a perturbation h which leads to the petal $\mathcal{A}(\alpha) = (\cos^2\frac{\alpha}{2} - \frac{1}{2}, \cos\frac{\alpha}{2}\sin\frac{\alpha}{2}), 0 \leq \alpha < 2\pi$. The proof in [9] differs from ours only by a minor aspect: this petal, which indeed is a circle, is understood here as a member of the family of roses considered all over this paper.

Keeping the notation of the Theorem 3.1 we can state:

THEOREM 3.5. *Suppose the hypothesis (H1) is satisfied for the unperturbed equation associated to (E3). Then, if the function $h(t)$ is given by*

$$(h_1(t), h_2(t)) = (u_1 e^{ti} + u_{-1} e^{-ti},\ -v_1 e^{ti} + v_{-1} e^{-ti}), \tag{3.7}$$

there exists a neighborhood $V \subset \mathbb{R}^2$ of $\gamma = 0$, such that the Equation (E3) has precisely two 2π-periodic solutions near p_α, for each $\gamma \in V$.

It is a matter of routine to compute $\mathcal{A}(\alpha)$ and see that it reduces to the circle specified above.

Keeping the same notation, let us now consider a more interesting inverse problem.

THEOREM 3.6. *Given the canonical diagram $\mathcal{D} = (0, l_1, 2, l_2)$, suppose the hypothesis (H1) is satisfied for the unperturbed equation assotiated to (E3). Then there exists a function $h = (h_1, h_2)$ such that the bifurcation diagram of 2π-periodic solutions of (E3) near p_α is equivalent to $\mathcal{D}$.*

Moreover, a nonsingular matrix $M = (a_{ij})$, $1 \leq i, j \leq 2$ can be effectively computed in such a way that h is given by $h = (\tilde{h}_1\ \tilde{h}_2)M$, with

$$\tilde{h}(t) = (\sigma + h_1(t), h_2(t)),$$

where $h = (h_1, h_2)$ is given in (3.7) and σ is a real number choosen in such a way that

$$\sigma \left| 2\pi c - d \int_0^{2\pi} \frac{dt}{p_1(t)} \right| > \frac{1}{2}.$$

PROOF: First of all we notice that the choice is possible for σ, since $2\pi c - d \int_0^{2\pi} \frac{dt}{p_1(t)} \neq 0$, according to the remark (1) following the Theorem 3.1. If we replace h by $\tilde{h}$ in the Equation (E3) and compute the corresponding curve $\tilde{\mathcal{A}}(\alpha)$, according to the formulae at the end of Section 2, we reach the petal $\tilde{\mathcal{A}}(\alpha) = (\delta + \cos^2 \frac{\alpha}{2} - \frac{1}{2}, \cos\frac{\alpha}{2} \sin\frac{\alpha}{2})$, $0 \leq \alpha < 2\pi$, with $\delta = \sigma \left(2\pi c - d \int_0^{2\pi} \frac{dt}{p_1(t)}\right)$. Let r_1', r_2' be the lines through the origin and tangent to $\tilde{\mathcal{A}}(\alpha)$ at the points $\gamma_1 = (\bar{\lambda}, -\bar{\mu}), \gamma_2 = (\bar{\lambda}, \bar{\mu})$, $\bar{\mu} > 0$, respectively, according to the Figure 3.9.

We choose the vectors $e_j' := \gamma_j - 0$, $j = 1, 2$, and, if r_1, r_2 are the lines through the origin orthogonal to l_1, l_2, respectively, we choose vectors e_1, e_2, in its direction, in such a way that the line through the origin, orthogonal to $e_1 + e_2$, lies in the sector

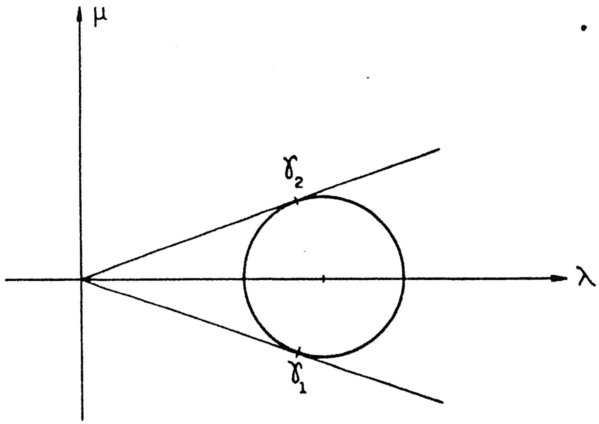

Figure 3.9

where there are two 2π-periodic solutions. Thus, there exists an isomorphism E, unique up to homothety, such that $Er_j = r'_j$, $j = 1, 2$ compatible with the orientation of these lines defined by the vectors $e_j, e'_j, j = 1, 2$.

Let us take M as the matrix of the inverse isomorphism E^{-1}. If the perturbation h is choosen as in the statement of the theorem, the proof follows now by computing $\mathcal{A}(\alpha)$ according to the formulae at the end of Section 2. We obtain $\mathcal{A}(\alpha) = E^{-1}\tilde{\mathcal{A}}(\alpha)$. Therefore, by the choice of E, the corresponding bifurcation curves are tangent at the origin to the given lines l_1, l_2. $\diamond$

A theorem similar to the Theorem 3.5 can be formulated for a diagram of the type $(2, l_1, 4, l_2)$. In this case the proof is carried out in the same way, with obvious adaptations, being the petal $\tilde{\mathcal{A}}(\alpha)$ replaced by the rose $\tilde{\mathcal{A}}(\alpha) = (\cos 3\alpha \cos\alpha + \delta, \cos 3\alpha \sin\alpha)$, with $-1 < \delta < 0$, so that one petal encircles the origin.

Similar theorems for a canonical diagram with six bifurcation curves can be stated,

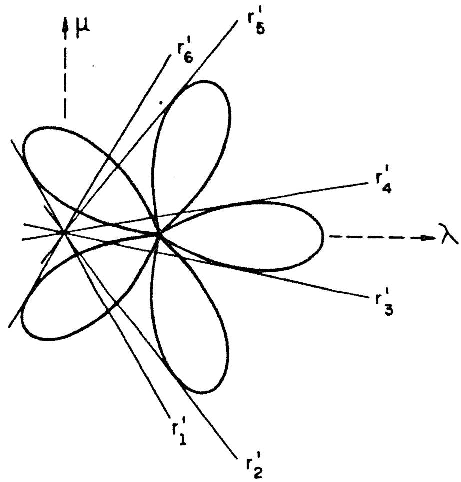

Figure 3.10

but unfortunately in this case our approach requests a somewhat strong hypothesis on the pencil $l_1, \ldots, l_6$.

This is caused by the fact that we need an isomorphism which maps the pencil $l_1, \ldots, l_6$ onto the lines through the origin, tangent to some rose with five petals. In order to formulate a sufficient condition for the fulfillment of a hypothesis of this kind, we introduce some notation. If $\mathcal{S}_\delta(\alpha) = (\cos 5\alpha \cos\alpha + \delta, \cos 5\alpha \sin\alpha)$ is a rose with five petals with the tangent lines $r_1', \ldots, r_6'$ through the origin, according to the Figure 3.10, we let ρ_2, ρ_3, ρ_4 be the projective coordinates of r_2', r_3', r_4' with respect to (r_1', r_6', r_5'), respectively. Given a canonical diagram $\mathcal{D} = (k_1, l_1, \ldots, k_6, l_6)$, the following condition on $\mathcal{D}$ is assumed:

(H4) There exists a δ such that the projective coordinates of l_2, l_3, l_4, with respect to (l_1, l_6, l_5), are ρ_2, ρ_3, ρ_4, respectively.

We still keep the notation from the Theorem 3.1 and state below a result that shows the role of the roses of five petals.

THEOREM 3.7. *Let $\mathcal{D}$ be one of the canonical diagrams:* $(4, l_1, 2, l_2, 4, l_3, 6, l_4, 4, l_5, 2, l_6)$ *or* $(0, l_1, 2, l_2, 4, l_3, 6, l_4, 4, l_5, 2, l_6)$. *Suppose $\mathcal{D}$ satisfies (H4) and the unperturbed equation associated to (E3) satisfies (H1). Then, there exists a function $h = (h_1, h_2)$ the bifurcation diagram of the 2π-periodic solutions of (E3) near p_α is equivalent to $\mathcal{D}$.*

Moreover, a nonsingular matrix $M = (a_{ij})$, $1 \leq i, j \leq 2$ can be effectively computed in such a way that h is given by $h = (\tilde{h}_1, \tilde{h}_2)M$, where

$$\tilde{h}_1(t) = \frac{\delta}{d \int_0^{2\pi} \frac{dt}{p_1(t)} - 2c\pi} + u_2 e^{3ti} + u_{-2} e^{-3ti} + u_1 e^{2ti} + u_{-1} e^{-2ti},$$

$$\tilde{h}_2(t) = -v_2 e^{3ti} + v_{-2} e^{-2ti} + v_1 e^{ti} - v_{-1} e^{-ti},$$

where δ is the number given in (H4).

A proof of Theorem 3.7 can be given by following the same reasoning of the proof of Theorem 3.1. Here we should deal with a rose with five petals. The existence of the isomorphism E which maps the lines r_j orthogonal to l_j in r'_j, $j = 1, \ldots, 6$ is ensured by the hypothesis (H4).

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