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EULER CHARACTERISTICS OF REAL VARIETIES

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Euler Characteristics of Real Varieties

J.W. Bruce*

Abstract

In this paper we show how to associate to any real projective algebraic variety $Z \subset \mathbf{R}P^{n-1}$ a real polynomial $F_1 : \mathbf{R}^n, 0 \to \mathbf{R}, 0$ with an algebraically isolated singularity, having the property that $\chi(Z) = \frac{1}{2}(1 - \deg(\operatorname{grad} F_1))$ where deg(grad F_1) is the local real degree of the gradient grad $F_1 : \mathbf{R}^n, 0 \to \mathbf{R}^n, 0$. This degree can be computed algebraically by the method of Eisenbud-Levine and Khimshiashvili [5]. The variety Z need not be smooth.

This leads to an expression for the Euler characteristic of any compact algebraic subset of $\mathbf{R}^n$, and the link of a quasihomogeneous mapping $f : \mathbf{R}^n, 0 \to \mathbf{R}^p, 0$ again in terms of the local degree of a gradient with algebraically isolated singularity.

Similar expressions for the Euler characteristic of an arbitrary algebraic subset of $\mathbf{R}^n$, and the link of any polynomial map are given in terms of the degrees of algebraically finite gradient maps. These maps do involve "sufficiently small" constants, but the degrees involved are (theoretically at least) algebraically computable.

Introduction

Let $f_1, \ldots, f_p$ be homogeneous polynomials in n variables $x_1, \ldots, x_n$ of degrees $d_1 \leq d_2 \leq \cdots \leq d_p$ respectively. The projective algebraic variety in $\mathbf{R}P^{n-1}$ given by the vanishing of $f_1, \ldots f_p$ is also given by the vanishing of the single homogeneous polynomial $\sum_{j=1}^{p} f_j^2 \|x\|^{2(d_p - d_j)} = f$ of degree $2d_p$, so in what follows we shall suppose that our real algebraic variety $Z \subset \mathbf{R}P^{n-1}$ is given by a single equation $f = 0$ of degree d . We then set $F_1 = f^2 - g$ where $g = x_1^{2d+2} + \cdots + x_n^{2d+2}$.

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We prove that F_1 has an (algebraically) isolated singularity at 0 and that the Euler characteristic of Z, $\chi(Z) = \frac{1}{2}(1 - \deg(\text{grad } F_1))$ where $\deg(\text{grad } F_1)$ is the local degree of grad $F_1 : \mathbf{R}^n, 0 \to \mathbf{R}^n, 0$. The degree of grad F_1 can be computed algebraically in terms of the signature of a certain bilinear form (cf.[5]) so the result might be viewed as a generalization of Sturm's theorem.

After completing the first part of this work the author came across a paper [8] of Szafraniec which contains similar results. He essentially uses the function $g' = \|x\|^{2d+2}$ in place of our g and obtains some very beautiful theorems (not just in the homogeneous case) which give the Euler characteristic of algebraic sets in terms of the local degree of a gradient. The advantage of the current work is that the gradient here has an algebraically isolated zero and so it can (in theory at least) be computed algebraically. The same is not necessarily true in [8]. Nevertheless we follows Szafraniec in Proposition 7 and Corollary 8 and express the Euler characteristic of a compact algebraic set in $\mathbf{R}^n$, and consequenty the real link of a quasi-homogeneous mapping, in terms of the local degree of a gradient, again with an algebraically isolated singularity. (The first of these results does, of course, imply the second: if $f : \mathbf{R}^n, 0 \to \mathbf{R}^p, 0$ is the mapping the intersection of $f^{-1}(0)$ with the unit sphere is a compact algebraic set.)

This indeed allows us to express the Euler characteristic of the link of any polynomial, as it is a compact algebraic set, in terms of the local degree of a gradient map with an algebraically isolated zero. However in this form the result is ineffective — one has to choose a "sufficiently small" radius for the intersecting sphere. To some extent we can overcome this problem (which also occurs in [8]) by a little trick. This trick, together with an idea of Szafraniec, allows us to express the Euler characteristic of any real algebraic subset of $\mathbf{R}^n$ in terms of local degrees of gradient maps, as in [8], but with the double advantage here that the maps are algebraically finite and the "sufficiently small" constants involved are algebraically computable.

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vector fields (our application rather turns the original result on its head). Her exposition of this work led me eventually to the results below.

Notation. Below $\cdot$ is the Euclidean inner product, $\| \ \|$ the norm, S_ε is the sphere (B_ε the ball) centred at 0 radius ε , S and B the unit sphere and ball.

1 Projective Case

We start with a

Lemma 1 *For all $\lambda \neq 0$, the set $F_\lambda = 0$ where*

$$F_\lambda = f^2 - \lambda^2(x_1^{2d+2} + \cdots + x_n^{2d+2})$$

is a hypersurface with an algebraically isolated singularity at 0 *. (That is* 0 *is an isolated singularity of the complex hypersurface $F_\lambda = 0$.)*

Proof. The hypersurface $g = x_1^{2d+2} + \cdots + x_n^{2d+2} = 0$ has an isolated singularity at $0 \in \mathbf{C}^n$, and so for small t the hypersurface $H_t = g - t^2 f^2 = 0$ does also. To see this note that g has no other singularities inside the unit sphere S. Let $c_1 = \min_{x\in S} \|\text{grad}\, g\| > 0$ and if $c_2 = \max_{x\in S} \|\text{grad}\, f^2\|$ we see from Rouché's principle ([6,p.112]) that the degree of grad $g/\|\text{grad}\, g\|$ equals the degree of grad $H_1/\|\text{grad}\, H_t\| : S \to S$ (and is $\mu(g)$, the Milnor number of g, $(2d+1)^n$) for $t^2 < c_1/c_2$.

Clearly grad H_t has no zeros on S , and hence only finitely many zeros inside S , [6,p.115]. In particular H_t has an isolated singularity at 0. For $s \neq 0$ the same is consequently true of $s^{-2d}t^{-2}(g - t^2 f^2)(x)$ and $s^{-2d}t^{-2}(g - t^2 f^2)(sx) = t^{-2}s^2 g(x) - f^2(x)$. Given any $\lambda \neq 0$ choose t sufficiently small and $s = t\lambda \neq 0$ to deduce the result. □

Lemma 2 *For all $\lambda \neq 0$ sufficiently small, $F_\lambda = 0$ has no real singularities inside the unit ball B except the origin, and all real spheres S_ε , $0 < \varepsilon \leq 1$, meet $F_\lambda = 0$ transversally.*

Proof. The function F_1 has an isolated singular point at 0 , so choose $\delta > 0$ with F_1 having no other critical points in B_δ . Hence neither does $\lambda^{-2d}F_1(x)$ and $\lambda^{-2d}F_1(\lambda x)$ has no critical points in $B_{\delta\lambda^{-1}}$ (all with $\lambda \neq 0$ of course). But $\lambda^{-2d}F_1(\lambda x) = F_\lambda(x)$ so if $\lambda \leq \delta$ we find F_λ has no critical points in B other than 0.

The second assertion is proved similarly. Consider the function $d(x) = \|x\|^2$ on the smooth part of the variety $F_1 = 0$. By [6,p.16] d has at most a finite number of critical values so for some $\delta > 0$ we find that d has no critical value in the interval $(0, \delta]$ and the spheres S_ε meet $F_1 = 0$ transversally for $0 < \varepsilon \leq \delta$, and hence $\lambda^{-2d} F_1 = 0$ too. One now checks that $\lambda^{-2d} F_1(\lambda x) = 0$ meets the spheres S_ε transversally for $0 < \varepsilon \leq \delta\lambda^{-1}$, so if $\lambda \leq \delta$ the spheres S_ε meet $F_\lambda = 0$ transversally for $0 < \varepsilon \leq 1$. □

Lemma 3 *Let λ be sufficiently small. If $X = \{x \in S : f(x) = 0\}$, $X_\lambda = \{x \in S : F_\lambda(x) \leq 0\}$ and $Y_\lambda = \{x \in X : F_\lambda(x) = 0\}$ then $X_\lambda - X$ is diffeomorphic to $Y_\lambda \times (0, \lambda]$.*

Proof. Consider the function $h : \mathbf{R}^n - \{0\} \to \mathbf{R}$ given by $h(x) = f^2/g$. We need to show it has only finitely many critical values when restricted to the sphere S, so we use the techniques of [6,p.16]. We find that $\operatorname{grad} h = (2f \operatorname{grad} f - f^2 \operatorname{grad} g)/g^2$ and the restriction of h to the unit sphere S has a critical point precisely when $(2g \operatorname{grad} f - f \operatorname{grad} g)$ and x are linearly dependent or $f = 0$. This determines the set of critical points as an algebraic set which consequently has only finitely many components, and each of these can be decomposed into finitely many connected smooth manifolds. Clearly h is critical when restricted to each manifold, and hence constant. So $h \mid S$ has only finitely many critical values. In particular $(0, \lambda]$ has an open neighbourhood in $(0, \infty)$ of regular values of h for λ sufficiently small. So $h : X_\lambda \setminus X \to (0, \lambda]$ is a proper submersion (since it has compact fibres, $h^{-1}(\mu) = Y_\mu$). The result now follows. □

Before stating our main theorem we need a fact concerning the local degree of $\operatorname{grad} F_\lambda$.

Lemma 4 *For $\lambda > 0$ sufficiently small the degree of*

$$G_\lambda = \operatorname{grad} F_\lambda / \|\operatorname{grad} F_\lambda\| : S \to S$$

is related to the Euler characteristic of the set $(F_\lambda \geq 0) \cap S$ by

$$\chi((F_\lambda \geq 0) \cap S) = 1 + (-1)^{n-1} \deg G_\lambda .$$

Proof. We shall use some ideas and a Lemma of Milnor [6,p.61]. We first check three facts concerning G_λ.

(i) Every fixed point of G_λ lies in $(F_\lambda > 0) \cap S$. For if $G_\lambda(x) = x$ then $\operatorname{grad} F_\lambda(x) \cdot x > 0$ so $2d(f^2 - \lambda^2 g) - 2\lambda^2 g = 2dF_\lambda - 2\lambda^2 g > 0$ and $F_\lambda > 0$.

(ii) We next check that if $x \in (F_\lambda \geq 0) \cap S$ then $G_\lambda(x) \neq -x$ for λ sufficiently small. In fact we first show that $G_1(x) \neq -x/\|x\|$ for $x \in (F_1 \geq 0) \cap B_\varepsilon - \{0\}$, ε sufficiently small. For otherwise by the curve selection lemma [6,p.25] we can find a real analytic curve $p : [0, \delta) \to \mathbf{R}^n$ with $p(0) = 0$, grad $F_1(p(t)) = \mu_t p(t)$, $\mu_t < 0$, $F_1(p(t)) \geq 0$. So $\frac{d}{dt} F_1(p(t)) =$ grad $F_1(p(t)) \cdot p'(t) = \mu_t p(t) \cdot p'(t) = \mu_t \frac{d}{dt} \|p(t)\|^2 < 0$ for t small. So $F_1(p(t))$ is decreasing and as $F_1(p(0)) = 0$ and $F_1(p(t)) \geq 0$ we deduce that $F_1(p(t)) = 0$. But $F_1 = 0$ meets small spheres transversally and we have a contradiction. Now if $G_\lambda(x) = -x$ for $x \in S$, $F_\lambda(x) > 0$ we see that $F_1(\lambda x) = \lambda^{2d} F_\lambda(x) \geq 0$ and $G_1(\lambda x) = G_\lambda(x) = -x$. So if $0 < \lambda \leq \varepsilon$ no point of $F_\lambda \geq 0$ is mapped by G_λ to its antipode.

(iii) Finally note that if $x \in (F_\lambda = 0) \cap S$ and $n(x)$ is tangent to the sphere, normal to $(F_\lambda = 0) \cap S$ and points into $F_\lambda \geq 0$, the inner product $G_\lambda(x).n(x)$ is a positive multiple of the derivative of F_λ in the $n(x)$ direction and consequently is positive.

We have thus verified the three hypotheses of Lemma 7.4 of [6] and deduce that

$$\chi((F_\lambda \geq 0) \cap S) = 1 + (-1)^{n+1} \deg(G_\lambda). \quad \square$$

(Other proofs of this result have been given by Arnold [1] and Wall [9]. Indeed these authors prove a more general result: $\chi((f \geq 0) \cap S_\varepsilon) = 1 + (-1)^{n+1} \deg \operatorname{grad} f$, for any real f with an isolated singularity, where S_ε is a Milnor sphere. The proof given here generalizes to give this result too).

Theorem 5 *Let f be as above. Then if $X = f^{-1}(0) \cap S$ we have $\chi(X) = 1 - \deg(\operatorname{grad} F_1)$ where $\deg(\operatorname{grad} F_1)$ is the local degree at 0 of* grad F_1 .

Proof. Throughout we use singular homology with integer coefficients. Now by Lefschetz duality [7,p.297] we have $H_k(S \setminus X) \cong H^{n-k-1}(S, X)$ so $\chi(S \setminus X) = \sum (-1)^k \beta_k(S \setminus X) = \sum (-1)^k \beta_{n-k-1}(S, X) = (-1)^{n+1} \chi(S, X) = (-1)^{n+1} (\chi(S) - \chi(X)) = (-1)^n \chi(X) + ((-1)^{n+1} + 1)$ (using the universal coefficient theorem [7,p.244]). On the other hand $S \setminus X = (S \cap (F_\lambda \leq 0) \setminus X) \cup ((F_\lambda \geq 0) \cap S$ so $\chi(S \setminus X) = \chi((F_\lambda = 0) \cap S) + \chi((F_\lambda \geq 0) \cap S) - \chi((F_\lambda = 0) \cap S) = \chi(F_\lambda \geq 0) = 1 + (-1)^{n+1} \deg(G_\lambda)$ (by Lemmas 3 and 4). Thus $\chi(X) = (-1)^n ((-1)^n + (-1)^{n+1} \deg(G_\lambda) = 1 - \deg(\operatorname{grad} F_\lambda)$. Using Lemma 2 we note that $\deg(\operatorname{grad} F_\lambda) = \deg(\operatorname{grad} F_1)$ and the result is proved. $\square$

Corollary 6 *Let f be as above. Then the Euler characteristic of the real projective variety $V_f = \{ x \in \mathbf{R}P^n : f(x) = 0 \}$ is $\frac{1}{2}(1 - \deg(\operatorname{grad} F_1))$.*

Note in particular that Corollary 6 and Eisenbud-Levine/Khimshiashvili allows one to compute algebraically the number of points of a real zero dimensional variety. One can use results of [2], [4] to do this too, but only in the cases when all of the points are simple.

2 Affine Case

We now follow [8,Lemma 2].

Proposition 7 *Let $f_1, \ldots, f_p : \mathbf{R}^n \to \mathbf{R}$ be polynomials of degree $\leq d$, and suppose that the equations $f_1(x) = \cdots = f_p(x) = 0$ determine a compact subset X of $\mathbf{R}^n$. Setting*

$$h_i(x_0, x_1, \ldots, x_n) = x_0^{d+1} f_i(x_1/x_0, \ldots, x_n/x_0), \ 1 \leq i \leq p,$$

$$H = h_1^2 + \cdots + h_p^2 - (x_0^{2d+4} + \cdots + x_n^{2d+4})$$

then H has an algebraically isolated singular point at 0 and $\chi(X) = \frac{1}{2}((-1)^n - \deg(\operatorname{grad} H))$.

Proof. Each h_i is homogeneous of degree $d+1$ so we can apply Theorem 5 to deduce that $\chi(Y) = 1 - \deg(\operatorname{grad} H)$ where $Y = S^n \cap (h_1 = \cdots = h_p = 0)$. Now the sets X and $Y_\pm = \{ x \in S^n : \pm x_0 > 0, h_1(x) = \cdots = h_p(x) = 0 \}$ are homeomorphic and Y is the disjoint union $Y_+ \cup Y_- \cup S^{n-1}$. The result now follows. □

Corollary 8 *If $f : \mathbf{R}^n, 0 \to \mathbf{R}^p, 0$ is a quasihomogeneous mapping there is a polynomial function $H : \mathbf{R}^n, 0 \to \mathbf{R}, 0$ with an algebraic isolated singularity satisfying $\chi(\text{link of } f = 0) = \frac{1}{2}((-1)^n - \deg(\operatorname{grad} H))$.*

Proof. The link of f is the compact algebraic set $f = \|x\|^2 - 1 = 0$, so we can apply Proposition 7. □

See [2] and especially [4] for a much more effective expression in the case when $p = n - 1$ and f is of finite singularity type.

Of course Corollary 8 holds for any polynomial mapping f, but the sphere intersecting $f = 0$ has to be "sufficiently small" and so H depends on the choice of a suitably small constant. To some extent one can circumvent the problem using the following Lemma.

Lemma 9 *Let $f = \mathbf{R}^n \times \mathbf{R}, 0 \to \mathbf{R}^n, 0$ be a polynomial map, with $f(0,t) = f_t(0) = 0$ and $f_t : \mathbf{R}^n, 0 \to \mathbf{R}^n, 0$ algebraically finite for $t \in \mathbf{R}$ small. Suppose that for $0 < t < \delta$ the dimension of the local algebra $Q_t = \varepsilon(n)/f_t^* m(n) \cdot \varepsilon(n)$ is constant. Then the real degree of f_t is also constant for t in this interval. (Here $\varepsilon(n)$ is the ring of germs of functions $\mathbf{R}^n, 0 \to \mathbf{R}$ and $m(n)$ its maximal ideal.)*

Proof. One can prove this in various ways. For example we know that the real degree of f_t is the signature of a quadratic form on Q_t [5]. So we have a continuous 1-parameter family of quadratic forms on some vector space V. Each of these forms in non-degenerate [5] and so the signature cannot change in the family. □

Note that one can easily outline an algorithm for computing a suitable value of δ above if we have some upper bound for the dimension of the local algebras Q_t, say k. For if $\dim_{\mathbf{R}} Q_t \leq k$ then using Nakayama's Lemma one can show that $f_t^* m(n) \cdot \varepsilon(n) \supset m^k(n)$. One can then write down a finite number of polynomial vectors $v_1(t), \ldots, v_N(t)$ which span $f_t^* m(n) \cdot \varepsilon(n)/m^k(n)$. There is a maximum value (as t varies) for the dimension of this quotient, say ℓ, and the condition that this is attained is a polynomial inequality $q(t) \neq 0$ in t, which arises by considering $\ell \times \ell$ minors in the matrix whose columns are the v_i. This same inequality is the condition for the quotient Q_t to have minimal dimension, and we choose δ with no point of $(0, \delta)$ satisfying the given inequality. (Clearly this procedure is lengthy and rapidly becomes impractical. Our aim however is to reduce each step in the computation of the Euler characteristic to a problem in linear algebra. The determination of the required interval $(0, \delta)$ given $p(t)$ is possible using Sturm's Theorem!)

For our final application we follow Szafraniec [8,Theorem 2].

Theorem 10 *Let $f_1, \ldots, f_p : \mathbf{R}^n \to \mathbf{R}$ be polynomials and consider the variety $Z = \{x : f_1(x) = \cdots = f_p(x) = 0\}$. Then we can find functions $G : \mathbf{R}^{n+1}, 0 \to \mathbf{R}, 0, H : \mathbf{R}^n, 0 \to \mathbf{R}, 0$ with algebraically isolated singularities, and express $\chi(Z)$ in terms of the local degrees of* grad H *and* grad G.

Proof. Set $f = f_1^2 + \cdots + f_p^2$. We may assume that $f(0) \neq 0$ and define $F : \mathbf{R}^n \to \mathbf{R}$ by $F(x_1, \ldots, x_n) = \|x\|^{2d+2} f(x/\|x\|^2)$ where the degree of f is $\leq d$.

Clearly F is a polynomial, $F(0) = 0$, and $(F = 0)$ is the one point compactification of Z . By Proposition 7 we can express $\chi(F = 0)$ in terms of the degree of an algebraically finite map.

Now if $\varepsilon > 0$ then

$$\chi(F = 0) = \chi((F = 0) \cap B_\varepsilon) + \chi((F = 0) \cap (\mathbf{R}^n \setminus B_\varepsilon)) - \chi((F = 0) \cap S_\varepsilon).$$

If ε is sufficiently small then $(F = 0) \cap B_\varepsilon$ is a cone so $\chi((F = 0) \cap B_\varepsilon) = 1$, while $\chi((F = 0) \cap \mathbf{R}^n - B_\varepsilon)) = \chi((F=0)-\{0\}) = \chi(Z)$. So we need to compute $\chi((F = 0) \cap S_\varepsilon)$, but this can be expressed in the required form by Proposition 7 also. □

Remark 11 *Lemma 9 and the discussion following gives a procedure for determining* $\chi((F = 0) \cap S_\varepsilon)$ *for* ε *"sufficiently small".*

References

[1] Arnold V.I., *Index of a singular point of a vector field, the Petrovski-Olenik inequality, and mixed hodge structures*, Funct. Anal. Apl. 12 (1978), 1-11.

[2] Aoki K., Fukuda T. and Nishimura T., *On the number of branches of the zero locus of a map germ* $\mathbf{R}^n, 0 \to \mathbf{R}^{n-1}, 0$, Topology and Computer Science, 347-363 (1987).

[3] Bruce J.W., *Branches of algebraic curves*, Preprint, Newcastle.

[4] Damon J.N., *On the number of branches for real and complex weighted homogeneous curve singularities*, Preprint, Warwick (1989).

[5] Eisenbud D. and Levine H., *An algebraic formula for the degree of a* C^∞ *map-germ*, Ann. Math., 106, 19-38 (1977).

[6] Milnor J.W., *Singular points of complex hypersurfaces*, Princeton Univ. Press (1968).

[7] Spanier E., *Algebraic Topology*, McGraw-Hill, 1966.

[8] Szafraniec Z., *On the Euler characteristic of analytic and algebraic sets*, Topology, vol.25, 411-414 (1986).

[9] Wall C.T.C., *Topological invariance of Milnor number, mod.2*, Topology, vol.22, 345-350 (1983).

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