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Symmetry and bifurcation to 2π - periodic solutions on nonlinear second order equations with 2π/m-periodic forcings

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Symmetry and Bifurcation to 2π-Periodic Solutions of Nonlinear Second Order Equations with $2\pi/m$-Periodic Forcings

by

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Abstract

Consider the equation $\ddot{u}+u = g(u,p)+\mu f(t)$ where p,μ are small parameters, f is an even continuous $2\pi/m$-periodic function, $m \geq 2$ is an integer and g is an odd smooth nonlinear function of u. The main result is that, under certain conditions, the small 2π-periodic solutions maintain some symmetry properties of the forcing function $f(t)$, when $\mu \neq 0$. Other interesting results describe the changes of the number of such solutions as p and μ vary in a small neighborhood of the origin. The main tool used in this work is the Liapunov-Schmidt Method.

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Symmetry and Bifurcation to 2π-Periodic Solutions of Nonlinear Second Order Equations with $2\pi/m$-Periodic Forcings

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1 Introduction

We consider the equation

$$\ddot{u} + u = g(u,p) + \mu f(t) \tag{1.1}$$

where p, μ are small parameters, f is an even continuous $2\pi/m$-periodic function, g is an odd function of u, sufficiently smooth, and $m \geq 2$ is an integer.

Our main results are, under certain conditions on g and f, that the small 2π-periodic solutions of (1.1) maintain some symmetry properties of the forcing term $f(t)$, when $\mu \neq 0$. We also find the bifurcation curves and describe the changes of the number of such solutions, as (p,μ) varies in a small neighborhood of the origin. A conjecture which was stated in Fürkotter-Rodrigues [2] is proved.

Hale-Rodrigues [5] studying Duffing's Equation $\ddot{u} + u = pu - u^3 + \mu \cos t$, showed that the only small 2π-periodic solutions are even functions of t, if $\mu \neq 0$. They also stated the same result for a general even forcing function with minimal period 2π under the condition $\int_0^{2\pi} f(s) \cos s \, ds \neq 0$.

Rodrigues-Vanderbauwhede [6], generalized this result for equations like (1.1) where f satisfies the former hypothesis and $g(u,p) = 0(|pu| + u^2)$ as (u,p) goes to $(0,0)$. They also presented an abstract version for equations in Banach spaces. Vanderbauwhede [7] also considers problems related to the above ones in an abstract form.

Fürkotter-Rodrigues [2] considered the case in which f is π-periodic, that is $m = 2$.

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In this work we emphasize the case where f is $2\pi/m$- periodic for $m \geq 3$, but we also make some comments about the case $m = 2$ because, in some respects, the techniques presented here are different from the ones of the above work. The main features of this papers are to find a set of small 2π-periodic solutions of (1.1) and to prove that these are the only feasible solutions.

In §2, using Liapunov-Schmidt Method, we show that symmetries in (1.1) imply symmetries in the solutions of the auxiliary equation. We call special attention to Theorem 2.1 which plays a central role in this work.

In §3, under certain conditions on $g(u,p)$ and on f, we prove that if $u(t)$ is a small 2π-periodic solution of (1.1) then there exists k, $-m/2 < k \leq m/2$, such that $u(t + k\pi/m)$ is even in t, for (p, μ) small and $\mu \neq 0$. This is stated in Theorem 3.3, where it is required that a certain coefficient, $\rho = \rho(g, f)$, is nonzero.

Our results indicate that the bifurcation equation are more degenerate when more symmetries are present in (1.1).

At the end of §3 we give some examples.

In §4 we prove that the condition $\rho \neq 0$ is generic. It is also proved that ρ only depends on the coefficientes of the Taylor expansion of $g(u, 0)$, around $u = 0$, up to the order $m + 1$ if m is even and up to the order m if m is odd.

An application which can be reduced to (1.1) is the equation $\ddot{v} + \omega_0^2 v + G(v) = \sigma f(\omega t)$ where $G(v)$ is $O(v^2)$ as v tends to zero, f is an even $2\pi/m$-periodic function and we look for $2\pi/\omega$-periodic solutions fo ω close to ω_0. This includes the pendulum equation and many other mechanical and electrical oscillators. If we let $u(t) \stackrel{\text{def}}{=} v(t/\omega)$ and $\omega_0^2/\omega^2 = 1 - p$ we get an equation like (1.1).

We thanks a referee for pointing out that this problem could also be treated by using a complexification technique and representation of group $\mathbf{Z}_m$, such as appears in Golubitisky, Stewart and Shaeffer [3].

A similar approach is used by Vanderbauwhede [8], where he considers a related problem.

We point out that if we suppose that $f(t + \pi/m) = -f(t)$ for m even, the conditions of this work no longer hold and the bifurcation equations become more degenerate. This is a harder problem which will be presented in a future work. The case $m = 2$ is treated in Fürkotter-Rodrigues [2].

2 The Auxiliary Equation

Consider the equation (1.1)

$$\ddot{u} + u = g(u,p) + \mu f(t)$$

where (p,μ) varies in a small neighborhood of the origin, and the following hypothesis:

(A) f is a real $2\pi/m$-periodic, even function, continuous on R and $m \geq 2$ is an integer.

(B) g is a C^∞ real function defined in a neighborhood of $(u,p) = (0,0)$, odd in u and $g(u,p) = pu + \alpha u^3 + \beta u^5 + O(|pu^3| + |u|^7)$, as (u,p) goes to $(0,0)$.

Let $\mathcal{P}$ be the space of all 2π-periodic real functions, continuous on R, with the norm $\|w\| = \sup_{0 \leq t \leq 2\pi} |w(t)|$ and let $\mathcal{P}^{(2)}$ be the space of all 2π-periodic real functions, with second derivative continuous on R, with the norm $\|w\| = \sup\{|w^{(j)}(t)|,\ 0 \leq t \leq 2\pi,\ j = 0,1,2\}$.

On these spaces we consider the projection

$$(2.1) \qquad (Pw)(t) \stackrel{\text{def}}{=} \frac{\cos t}{\pi} \int_0^{2\pi} w(s) \cos s \, ds + \frac{\sin t}{\pi} \int_0^{2\pi} w(s) \sin s \, ds \ .$$

The Fredholm Alternative implies that the equation $\ddot{u} + u = h(t)$, with h in $\mathcal{P}$, has a solution in $\mathcal{P}^{(2)}$ if and only if $Ph = 0$. Moreover, if $Ph = 0$ then there exists a unique solution $u(t)$ in $\mathcal{P}^{(2)}$ such that $Pu = 0$. We indicate this solution by $\mathcal{K}h$. From the variation of constants formula, we obtain,

$$(2.2) \quad \mathcal{K}h = (I - P)[-\cos(\cdot) \int_0^{(\cdot)} h(s) \sin s \, ds + \sin(\cdot) \int_0^{(\cdot)} h(s) \cos s \, ds] \ .$$

Following the usual procedure of Liapunov-Schmidt Method, the problem of finding a 2π-periodic solution $u(t)$ of (1.1) is reduced to that of finding a solution w in $\mathcal{P}^{(2)}$ of the following equations:

$$(2.3) \quad \begin{array}{ll} \text{(a)} & w = \mathcal{K}(I - P)[g(r\cos(\cdot - \phi) + w, p) + \mu f(\cdot)] \\ \text{(b)} & P[g(r\cos(\cdot - \phi) + w, p) + \mu f(\cdot)] = 0 \end{array}$$

where $u(t) = r\cos(t-\phi) + w(t)$, $r \in R$ and $\phi \in (-\pi/2, \pi/2]$.

The equations (a) and (b) are called the auxiliary and the bifurcation equation, respectively. It follows from the implicit function theorem that the equation (2.3.a) has a unique small solution, for (p,μ) in a small neighborhood of the origin. We denote this solution by $w^*(r,\phi,p,\mu)(t)$. If we substitute in (2.3.b) we obtain the following equivalent system of equations:

$$\text{(a)}\quad F(r,\phi,p,\mu) \stackrel{\text{def}}{=} \frac{1}{\pi}\int_0^{2\pi} g(r\cos s + w^*(r,\phi,p,\mu)(s+\phi), p)\cos s\, ds = 0$$

(2.4)

$$\text{(b)}\quad G(r,\phi,p,\mu) \stackrel{\text{def}}{=} \frac{1}{\pi}\int_0^{2\pi} g(r\cos s + w^*(r,\phi,p,\mu)(s+\phi), p)\sin s\, ds = 0$$

The following lemma gives informations about some symmetries and estimates of w^*.

Lemma 2.1 *If hypothesis (A) and (B) are satisfied, then the solution w^* of (2.3.a) has the following properties:*

(2.5) *$w^*(0,\phi,p,\mu)(t)$ is an even $2\pi/m$-periodic function of t and is independent of ϕ.*

(2.6) *$w^*(r,k\pi/m,p,\mu)(t+k\pi/m)$ is even in t for $-m/2 < k \le m/2$.*

(2.7) *$w^*(r,\phi,p,0)(t+\phi)$ is even in t.*

(2.8) *$w^*(0,\phi,p,\mu) = \mu\mathcal{K}f + 0(|p\mu| + |\mu^3|)$, as $(p,\mu) \to (0,0)$.*

(2.9) *$w^*(r,\phi,p,\mu) = w^*(0,\phi,p,\mu) + rS(r,\phi,p,\mu)$ where $S(r,\phi,p,\mu) = 0(r^2 + |\mu|)$, as $(r,p,\mu) \to (0,0,0)$.*

If m is even then the following properties hold:

(2.10) $w^*(r,\phi,p,\mu)(t) = w^*(-r,\phi,p,\mu)(t-\pi)$

(2.11) $w^*(r,\phi,p,\mu)(t) = -w^*(r,\phi,p,-\mu)(t-\pi)$

If $m = 3$, the following hold:

(2.12) $w^*(r,-\pi/3,p,\mu)(t-\pi/3) = w^*(r,\pi/3,p,\mu)(t+\pi/3)$

(2.13) $w^*(-r,\pi/3,p,\mu)(t-2\pi/3) = w^*(r,0,p,\mu)(t)$

Proof. Properties (2.5), (2.6), (2.7), (2.10) to (2.13) follow essentialy from the fact that the auxiliary equation is invariant under certain transformations. Properties (2.8) and (2.9) can be proved in a natural way. □

Let $\mathcal{P}_{2\pi\times 2\pi} = \{f : R \times R \to R : f(t+2\pi,\phi) = f(t,\phi) = f(t,\phi+2\pi)$ for every $(t,\phi) \in R \times R$, f continuous$\}$ with the sup norm.

Let m and n be positive integers, with $m \geq 2$ and $n \leq m-1$. If n is even we define $\mathcal{F}_n$ as the space of the functions $y \in \mathcal{P}_{2\pi\times 2\pi}$, such that $y(t,\phi)$ can be written into the form:

$$\sum_{j=0}^{n/2}[a_{n,2j}(t)\cos 2j(t-\phi) + b_{n,2j}(t)\sin 2j(t-\phi)] \tag{2.14}$$

where $a_{n,2j}$ is an even $2\pi/m$-periodic function and $b_{n,2j}$ is an odd $2\pi/m$-periodic function, for $j = 0,\ldots,n/2$.

If n is an odd integer, we define $\mathcal{F}_n$ as the space of the functions $y \in \mathcal{P}_{2\pi\times 2\pi}$, such that $y(t,\phi)$ can be written into the form:

$$\sum_{j=0}^{(n-1)/2} [a_{n,2j+1}(t)\cos(2j+1)(t-\phi) + b_{n,2j+1}(t)\sin(2j+1)(t-\phi)] \tag{2.15}$$

where $a_{n,2j+1}$ is an even $2\pi/m$-periodic function and $b_{n,2j+1}$ is an odd $2\pi/m$-periodic function for $j = 0,1,\ldots,(n-1)/2$.

Remark 2.1 *In Lemma 2.2, 2.3, 2.4 and Theorem 2.1 we allow ϕ to vary in R. To avoid picking up twice the same solution, in §3 we restrict ϕ to $(-\pi/2,\pi/2]$.*

Lemma 2.2 *$\mathcal{F}_n$ is closed in $\mathcal{P}_{2\pi\times 2\pi}$.*

Proof. Let us assume first that n is even and that y is in $\mathcal{F}_n$. Then it has the form (2.14).

It is possible to prove that

$$a_{n,2j}(t) = \frac{1}{\pi}\int_0^{2\pi} y(t,\phi)\cos 2j(t-\phi)d\phi$$

and

$$b_{n,2j}(t) = \frac{1}{\pi}\int_0^{2\pi} y(t,\phi)\sin 2j(t-\phi)d\phi\ .$$

From this fact it follows that if a sequence $y_k \in \mathcal{F}_n$ converges in $\mathcal{P}_{2\pi\times 2\pi}$, then its limit is in $\mathcal{F}_n$.

If n is odd the proof is similar. □

Lemma 2.3 *If q_i, n_i, β are positive integers, $y_i \in \mathcal{F}_{n_i}$ $i = 1, \dots, \beta$ and $\alpha = \sum_{i=1}^{\beta} q_i n_i \leq m-1$, then $\prod_{i=1}^{\beta} y_i^{q_i} \in \mathcal{F}_\alpha$.*

The next lemma plays an important role in this work.

Lemma 2.4 *If m and n are positive integers, with $m \geq 2$, $n \leq m-1$, and $f \in (I-P)\mathcal{F}_n$ then $\mathcal{K}f$, that is the function $(t,\phi) \mapsto \mathcal{K}f(\cdot,\phi)(t)$, belongs to $(I-P)\mathcal{F}_n$.*

Proof. Let us suppose first that n is even.

Our purpose is to prove that there exist coefficients $a_{n,2j}$, $b_{n,2j}$ in such a way that a function $x(t)$ given by (2.14) is a solution of $\ddot{x} + x = f(t,\phi)$ and $x \in (I-P)\mathcal{F}_n$. Since $\mathcal{K}f(\cdot,\phi)$ is the unique 2π-periodic solution which belongs to the range of $I-P$, it will follow that $x = \mathcal{K}f(\cdot,\phi)$.

If we substitute (2.14) into $\ddot{x} + x = f(t,\phi)$, where $f(t,\phi) \stackrel{\text{def}}{=} \sum_{j=0}^{n/2} [A_{n,2j}(t) \cos 2j(t-\phi) + B_{n,2j}(t) \sin 2j(t-\phi)]$ and equate coefficients, we obtain the equivalent system:

$$\ddot{a}_{n,2j} - (4j^2-1)a_{n,2j} + 4j\dot{b}_{n,2j} = A_{n,2j}$$

$$\ddot{b}_{n,2j} - (4j^2-1)b_{n,2j} + 4j\dot{a}_{n,2j} = B_{n,2j}$$

for $j = 0, \dots, n/2$.

Now, if we let $y_1 = a_{n,2j}$, $\dot{y}_1 = y_2$, $y_3 = b_{n,2j}$, $\dot{y}_3 = y_4$, $y = \mathrm{col}(y_1, y_2, y_3, y_4)$, we obtain the equivalent equation:

$$\dot{y} = C_j y + F_j$$

where $F_j = \mathrm{col}(0, A_{n,2j}, 0, B_{n,2j})$ and

$$C_j = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 4j^2-1 & 0 & 0 & -4j \\ 0 & 0 & 0 & 1 \\ 0 & 4j & 4j^2-1 & 0 \end{bmatrix}$$

for $j = 0, \dots, n/2$.

The eigenvalues of C_j are $\pm(2j+1)i$ and $\pm(2j-1)i$.

If m is even the system $\dot{y} = C_j y$ is noncritical with respect to $\mathcal{P}_{2\pi/m}$, the space of $2\pi/m$-periodic continuous functions. The same holds if m is odd

$0 \leq j \leq n/2 \leq (m-1)/2$ or $0 \leq j \leq n/2 < (m-1)/2$. If $j = n/2 = (m-1)/2$ the system $\dot{y} = C_j y$ is critical with respect to $\mathcal{P}_{2\pi/m}$.

For the noncritical cases the equation $\dot{y} = C_j y + F_j$ has a unique $2\pi/m$-periodic solution $y(t) = \mathrm{col}(y_1(t), y_2(t), y_3(t), y_4(t))$. Since $z(t) \stackrel{\text{def}}{=} \mathrm{col}(y_1(-t), -y_2(-t), -y_3(-t), y_4(-t))$ is also a $2\pi/m$-periodic solution it follows that $y_1(t)$ must be even and $y_3(t)$ must be odd functions of t.

For $j = n/2 = (m-1)/2$ we have a critical case. Following Hale [3, p.275] we have that

$$\phi(t) \stackrel{\text{def}}{=} \begin{bmatrix} \cos mt & \sin mt \\ -m \sin mt & m \cos mt \\ \sin mt & -\cos mt \\ m \cos mt & m \sin mt \end{bmatrix}$$

is a matrix whose columns form a basis of the space of $2\pi/m$-periodic solutions of $\dot{y} = C_j y$, and

$$\Psi(t) = \begin{bmatrix} -(m-2)\sin mt & \cos mt & (m-2)\cos mt & \sin mt \\ (m-2)\cos mt & \sin mt & (m-2)\sin mt & -\cos mt \end{bmatrix}$$

is a matrix whose rows form a basis of the space of $2\pi/m$-periodic solutions of the adjoint equation, $\dot{z} = -zC_j$.

If we define projections $\bar{P}, \bar{Q}$ as in Hale [3, p.276, (2.5)], we obtain

$$\bar{P}f = \Phi(\cdot)[\int_0^{2\pi/m} \Phi'(t)\Phi(t)dt]^{-1} \int_0^{2\pi/m} \Phi'(t)f(t)dt$$

$$\bar{Q}f = \Psi'(\cdot)[\int_0^{2\pi/m} \Psi(t)\Psi'(t)dt]^{-1} \int_0^{2\pi/m} \Psi(t)f(t)dt$$

$Pf(\cdot,\phi) = 0$ implies that $\bar{Q}F_j = 0$, for $j = (m-1)/2$.

Then the equation $\dot{y} = C_j y + F_j$ has a unique $2\pi/m$-periodic solution $y(t)$ such that $(\bar{P}y)(t) \equiv 0$.

If $z(t) \stackrel{\text{def}}{=} \mathrm{col}(y_1(-t), -y_2(-t), -y_3(-t), y_4(-t))$ then $z(t)$ is also a solution of the same equation with $(\bar{P}z)(t) \equiv 0$. This implies that $y_1(t)$ is even and $y_3(t)$ is odd.

The above informations will provide a solution $x(t)$ of $\ddot{x} + x = f(t,\phi)$, and the condition $\bar{P}y = 0$ implies that $Px = 0$. This shows that $x \in (I-P)\mathcal{F}_n$ and that $x(t) = \mathcal{K}f(\cdot,\phi)(t)$.

The case n odd is similar.

Lemma 2.5 *Let X be a Banach space and $I \subset R$ an interval. Let $\xi : I \to X$ and $g : X \to X$ be functions with continuous derivatives up to the order n. Let $H = g \circ \xi$. Then for $n \geq 1$, $\frac{\partial^n H}{\partial r^n}(r)$ can be written as a sum of terms of the form*

$$\gamma_i \frac{\partial^i g}{\partial u^i}(\xi(r))(d^{\alpha_1^i}\xi/dr^{\alpha_1^i})^{\beta_1^i} \cdots (d^{\alpha_{k_i}^i}\xi/dr^{\alpha_{k_i}^i})^{\beta_{k_i}^i}$$

where $\alpha_1^i\beta_1^i + \cdots + \alpha_{k_i}^i\beta_{k_i}^i = n$, $\beta_1^i + \cdots + \beta_{k_i}^i = i$ and γ_i are constants, for $i = 1, \ldots, n$. Moreover, if $i > 1$ then $\alpha_j^i < n$ and $\frac{\partial g}{\partial u}(\xi(r))\frac{d^n\xi}{dr^n}$ is the only term containing $\frac{d^n\xi}{dr^n}$.

The next theorem will be very important in the proof of our main results.

Theorem 2.1 *Suppose hypotheses (A) and (B) are satisfied. If $1 \leq n \leq m-1$ then $\frac{\partial^n w^*}{\partial r^n}(0, \cdot, p, \mu)(\cdot)$ belongs to $(I-P)\mathcal{F}_n$. Moreover it has the form (2.14) or (2.15), for m even or odd, respectively, with the coefficients $a_{ij}(t) = a_{ij}(p, \mu)(t)$ and $b_{ij}(t) = b_{ij}(p, \mu)(t)$.*

Proof. We will do the proof by induction.

If $n = 1$ then $\frac{\partial w^*}{\partial r}(0, \cdot, p, \mu)(\cdot)$ is the unique solution of $\mathcal{H}y = y$, where $(\mathcal{H}y)(t, \phi) \stackrel{\text{def}}{=} \mathcal{K}(I-P)[\frac{\partial g}{\partial u}(w^*(0, \phi, p, \mu)(\cdot), p)(y + \cos(\cdot - \phi))](t)$, is a uniform contraction with respect to p, μ for (p, μ) in a small neighborhood of $(0, 0)$.

From Lemma 2.1 and Lemma 2.4, after some calculations we prove that $(I-P)\mathcal{F}_1$ is invariant under $\mathcal{H}$. Since, by Lemma 2.2, $(I-P)\mathcal{F}_1$ is closed in $\mathcal{P}_{2\pi \times 2\pi}$ it follows that the fixed point of $\mathcal{H}$ is in $(I-P)\mathcal{F}_1$.

Now let us assume that the result is true up to order $n-1$. We will prove that it is true for n.

$y = \frac{\partial^n w^*}{\partial r^n}(0, \cdot, p, \mu)$ is the unique solution of $y = \mathcal{H}y$ where

$$(\mathcal{H}y)(t, \phi) \stackrel{\text{def}}{=} \mathcal{K}(I-P)[\frac{\partial g}{\partial u}(w^*(0, \phi, p, \mu), p)y + T(\phi, p, \mu)](t),$$

and $T(\phi, p, \mu)$, by Lemma 2.5, can be written as a sum of terms of the form

$$\gamma_i \frac{\partial^i g}{\partial u^i}(w^*, p)(\cos(\cdot - \phi) + \frac{\partial w^*}{\partial r})^{\beta_1^i}(\partial^{\alpha_2^i} w^*/\partial r^{\alpha_2^i})^{\beta_2^i} \cdots (\partial^{\alpha_{k_i}^i} w^*/\partial r^{\alpha_{k_i}^i})^{\beta_{k_i}^i}$$

where $i > 1$ and $\frac{\partial^\ell w^*}{\partial r^\ell}$, above, means $\frac{\partial^\ell w^*}{\partial r^\ell}(0, \phi, p, \mu)$ for $\ell = 0, 1, \cdots, \alpha_{k_i}^i$. Moreover $\alpha_j^i < n$, $\alpha_1^i\beta_1^i + \cdots + \alpha_{k_i}^i\beta_{k_i}^i = n$ and $\beta_1^i + \cdots + \beta_{k_i}^i = i$, for $j = 1, \ldots, k_i$, $i = 1, \ldots, n$.

As before $\mathcal{H}$ is an uniform contraction in $\mathcal{P}_{2\pi\times2\pi}$ for (p,μ) in a small neighborhood of $(0,0)$.

From Lemma 2.1, Lemma 2.3, Lemma 2.4, after some calculations it follows that $(I-P)\mathcal{F}_n$ is closed in $\mathcal{P}_{2\pi\times2\pi}$. Then the fixed point of $\mathcal{H}$ belongs to $(I-P)\mathcal{F}_n$. □

3 The Bifurcation Equations

Since in Chapter 2 we obtained many informations about the solution of the auxiliary equation, now we are in conditions to analyze the bifurcation equations given in (2.4).

Lemma 3.1 *Under hypotheses* (A), (B) *the following hold:*

(i) $G(r,\phi,p,0)\equiv 0$.

(ii) If m is even then F and G are odd functions of r and even functions of μ.

Proof. The first part follows from Lemma 2.1, (2.7) and the second part follows from Lemma 2.1, (2.10) and (2.11). □

Theorem 3.1 *Suppose hypotheses* (A), (B) *are satisfied. Then for (r,p,μ) in a small neighborhood of the origin, $G(r,\phi,p,\mu)=r^{m-1}\mu\sin m\phi(\rho+\cdots)$, if m is odd and $G(r,\phi,p,\mu)=r^{m-1}\mu^2\sin m\phi(\rho+\cdots)$, if m is even, where ρ is independent of ϕ,p,μ and $(\cdots)$ indicates terms of order $0(|p|+|\mu|+|r|)$ uniformly on ϕ, as (r,p,μ) goes to $(0,0,0)$.*

Proof. We will prove first that

$$\frac{\partial^\ell G}{\partial r^\ell}(0,\phi,p,\mu)=0,\ \ \ell=1,2,\ldots,m-2\ .$$

If we let $H(r,s)\stackrel{\text{def}}{=}g(r\cos(s-\phi)+w^*(r,\phi,p,\mu)(s),p)$, then:

$$\frac{\partial^\ell G}{\partial r^\ell}(0,\phi,p,\mu)=\frac{1}{\pi}\int_0^{2\pi}\frac{\partial^\ell H}{\partial r^\ell}(0,s)\sin(s-\phi)ds\ .$$

From Lemma 2.5 it follows that $\frac{\partial^\ell H}{\partial r^\ell}(0,s)$ is a sum of terms of the form:

$$\gamma_i \frac{\partial^i g}{\partial u^i}(w^*,p)(\cos(\cdot-\phi)+\frac{\partial w^*}{\partial r})^{\beta_1^i}(\partial^{\alpha_2^i}w^*/\partial r^{\alpha_2^i})^{\beta_2^i}\cdots(\partial^{\alpha^i_{\beta_{k_i}}}w^*/\partial r)^{\alpha^i_{\beta_{k_i}}}$$

where $\frac{\partial^q w^*}{\partial r^q}=\frac{\partial^q w^*}{\partial r^q}(0,\phi,p,\mu)(s)$, $q=0,\ \alpha_1^i,\ldots,\alpha_{k_i}^i,\ \alpha_1^i\beta_1^i+\cdots+\alpha_{k_i}^i\beta_{k_i}^i=\ell$ and $\beta_1^i+\cdots\beta_{k_i}^i=i$.

Let us assume first that ℓ is even. From Theorem 2.1 and Corollary 2.1 it follows that $\frac{\partial^\ell G}{\partial r^\ell}(0,\phi,p,\mu)$ is a sum of integrals of the form:

$$\int_0^{2\pi}[a(s)\cos 2j(s-\phi)+b(s)\sin 2j(s-\phi)]\sin(s-\phi)ds$$

where, $0\le j\le \ell/2<(m-1)/2$ and $a(s),\ b(s)$ are $2\pi/m$-periodic functions.

That integral can be written as

$$\frac{1}{2}\{-\sin(2j+1)\phi\int_0^{2\pi}[a(s)\cos(2j+1)s+b(s)\sin(2j+1)s]ds+$$

$$+\sin(2j-1)\phi\int_0^{2\pi}[a(s)\cos(2j-1)s+b(s)\sin(2j-1)s]ds\}\ .$$

Since $j<(m-1)/2$ implies that $(2j+1)<m$ and since $a(s),\ b(s)$ are $2\pi/m$-periodic it follows that the above integrals vanish.

The case ℓ odd, $\ell<m-1$ is similar.

The same idea shows that $\frac{\partial^{m-1}}{\partial r^{m-1}}G(0,\phi,p,\mu)$ is a sum of integrals of the form

$$-\frac{\sin m\phi}{2}\int_0^{2\pi}[a(s)\cos ms+b(s)\sin ms]ds$$

where $a(s)=a(p,\mu)(s)$, $b(s)=b(p,\mu)(s)$ are $2\pi/m$- periodic functions of s.

From Lemma 2.1, (2.6) it follows that $G(r,k\pi/m,p,\mu)\equiv 0$.

The proof can be completed by using Lemma 3.1 and the above results. □

Remark 3.1 *To evaluate ρ it is sufficient to compute*

$$(\frac{\partial^{m+1}G}{\partial r^{m-1}\partial\mu^2}(0,\phi,0,0))/\sin m\phi,\quad\text{or}\quad(\frac{\partial^{m}G}{\partial r^{m-1}\partial\mu}(0,\phi,0,0))/\sin m\phi)\ ,$$

for m even or odd, respectively.

The case $m=2$ is considered in Fürkotter-Rodrigues [2]. The cases $m=3,4,5$ are considered with details in §3 in the section of examples.

Theorem 3.2 *Suppose (A), (B) are satisfied and $\rho \neq 0$. Then the only small 2π-periodic solutions of (1.1) are such that $u(t+k\pi/m)$ is even in t, for some k, $-m/2 < k \leq m/2$, for (p,μ) small and $\mu \neq 0$.*

Proof. Since $G(r,\phi,p,\mu) = r^{m-1}\mu \sin m\phi(\rho + \cdots)$ if m is odd, $G = 0$, $\mu \neq 0$ implies $r = 0$ or $\sin m\phi = 0$. Since $u(t) = r\cos(t-\phi) + w^*(r,\phi,p,\mu)(t)$, from Lemma 2.1 it follows that $u(t)$ is even if $r = 0$. If $\sin m\phi = 0$ then $\phi = k\pi/m$, $-m/2 < k \leq m/2$. Still from Lemma 2.1 it follows that $u(t + k\pi/m) = r\cos t + w^*(r, k\pi/m, p,\mu)(t + k\pi/m)$ is even in t. □

In what follows we will assume that $\rho \neq 0$.

Now let us analyze the first bifurcation equation (2.4.a),

$$F(r,\phi,p,\mu) = \frac{1}{\pi}\int_0^{2\pi} g(r\cos s + w^*(r,\phi,p,\mu)(s+\phi), p)\cos s\, ds = 0\ .$$

If we let $g(u,p) = pu + \alpha u^3 + 0(|pu^3| + |u^5|)$ and since $F(0,\phi,p,\mu) \equiv 0$, after some calculations we obtain for $m > 3$,

$$F(r,\phi,p,\mu) = r(p + \frac{3}{4}\alpha r^2 + 3\alpha\eta r\mu + 3\alpha\lambda\mu^2 + \cdots) = 0$$

where

$$\lambda \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_0^{2\pi}[(\mathcal{K}f)(t)]^2 dt,\ \eta \stackrel{\text{def}}{=} \frac{\cos 3\phi}{4\pi}\int_0^{2\pi}(\mathcal{K}f)(t)\cos 3t\, dt$$

and $\cdots$ indicates higher order terms.

We know that $r = 0$ solves $F = 0$ and $u(t) = w^*(0,\phi,p,\mu)(t)$ is $2\pi/m$-periodic in t.

In order to find other solutions, we consider $J \stackrel{\text{def}}{=} F/r$. To find multiple roots of $J = 0$ we consider the system:

$$J(r,\phi,p,\mu) = p + \frac{3}{4}\alpha r^2 + 3\alpha\eta r\mu + 3\alpha\lambda\mu^2 + \cdots = 0$$

$$J_r(r,\phi,p,\mu) = \frac{3}{2}\alpha r + 3\alpha\eta\mu + \cdots = 0\ .$$

Since $\det\frac{\partial(J,J_r)}{\partial(p,r)} = \frac{3}{2}\alpha$, for $r = p = \mu = 0$, if $\alpha \neq 0$, from the implicit function theorem it follows that p and r can be found as functions of μ in a small neighborhood of the origin, for each fixed ϕ.

In what follows, the case $m = 3$ requires a different treatment from the case $m > 3$.

If $m = 3$ the admissible values of ϕ are $\phi = 0$ and $\phi = \pm\pi/3$. From Lemma 2.1 (2.12) it follows that $F(r, -\pi/3, p, \mu) = F(r, \pi/3, p, \mu)$, which implies that the bifurcation equations are the same for $\phi = -\pi/3$ and $\phi = \pi/3$.

Also from Lemma 2.1 (2.13) it follows that $F(-r, \pi/3, p, \mu) = -F(r, 0, p, \mu)$ which implies $J(-r, \pi/3, p, \mu) = J(r, 0, p, \mu)$. Therefore the bifurcation curve, $p = p(\mu)$ for $\phi = 0$ is the same as for $\phi = \pi/3$, while $r = r(\mu)$ changes sign.

The bifurcation curve for $\phi = 0$ is given by $p = 3\alpha(\eta^2 - \lambda)\mu^2 + 0(|\mu|^3)$. The value of r where the bifurcation occurs is given by $r = -2\eta\mu + 0(\mu^2)$.

If $m > 3$ we assure that $J_r(0, \phi, p, \mu) \equiv 0$. This follows from Theorem 2.1 and Lemma 2.1.

It also follows from Theorem 2.1 that $J(0, \phi, p, \mu)$ is independent of ϕ.

Since $J(0, \phi, p, \mu)$ does not depend on ϕ and $J_r(0, \phi, p, \mu) \equiv 0$, it follows that the solution $p = p(\mu)$ which we obtain solving $J(0, \phi, p, \mu) = 0$ and $r = 0$ is the unique solution of $J(0, \phi, p, \mu) = 0$, $J_r(0, \phi, p, \mu) = 0$ of r and p as functions of μ. Therefore for $m > 3$ we have a unique bifurcation curve, which is given by

$$p = -3\alpha\lambda\mu^2 + 0(|\mu|^3) .$$

The next theorem is very interesting and it describes the changes of the number of the small $2\pi/m$-periodic solutions of (1.1) as (p, μ) crosses the bifurcation curve.

Theorem 3.3 *Suppose* (A), (B) *are satisfied,* $m \geq 3$, $\alpha \neq 0$ *and* ρ, *given in Remark 3.1, is nonzero. Then there exists a unique bifurcation curve* Γ, *which is given by* $p = -3\alpha\lambda\mu^2 + 0(\mu^3)$ *where* $\lambda = \frac{1}{2\pi}\int_0^{2\pi}[(\mathcal{K}f)(s)]^2 ds$ *and* $\mathcal{K}f$ *is the* $2\pi/m$*-periodic solution of* $\ddot{u} + u = f(t)$, *if* $m > 3$ *and by* $p = 3\alpha(\eta^2 - \lambda)\mu^2 + 0(\mu^3)$ *if* $m = 3$, *where* $\eta^2 = (\frac{\cos 3\phi}{4\pi}\int_0^{2\pi}(\mathcal{K}f)(t)\cos 3t\, dt)^2$.

The curve Γ *divides a neighborhood of the origin into regions as it is shown in Fig. 3.1 and Fig. 3.2 for* $m > 3$ *and* $m = 3$ *respectively, for* $\alpha, \lambda > 0$, $\mu \neq 0$. *The number of* 2π*-periodic solutions of (1.1) is indicated in the pictures.*

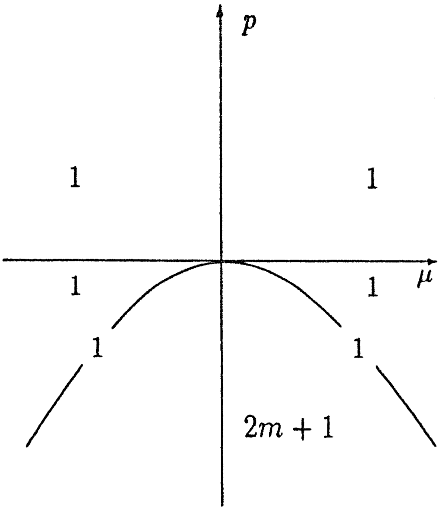

Fig. 3.1

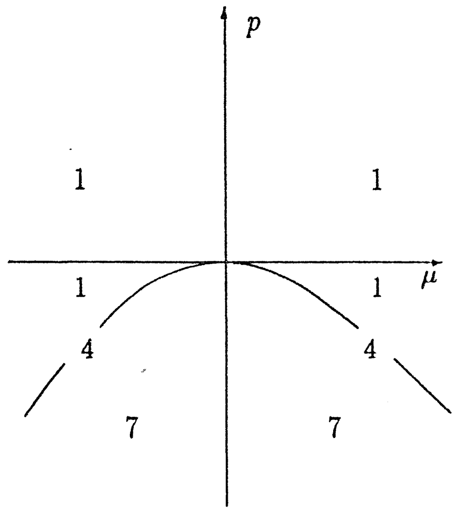

Fig. 3.2

Examples

In what follows we will analyze the cases $m = 3, 4, 5$. We recall that the case $m = 2$ was studied in Fürkotter-Rodrigues [2] where was considered the example $f(t) = 1 + \cos 2t$. From the calculation that will be presented below, if $g(u,0) = \alpha u^3 + \beta u^5 + 0(|u^7|)$, to compute the value of ρ for $m = 3$, we only use α, while for $m = 4, 5$ both α and β have contribution on the value of ρ.

For $m = 3$, we have $\rho = -\frac{3}{4\pi}\alpha \int_0^{2\pi} (\mathcal{K}f)(t) \cos 3t \, dt$.

For $m = 4$,

$$\begin{aligned}
\rho \;=\; & -\frac{9}{4\pi}\alpha^2 \int_0^{2\pi} \mathcal{K}(I-P)[\cos(\cdot)(\mathcal{K}f)^2](s) \cos 3s \, ds \\
& -\frac{9}{4\pi}\alpha\Big(\int_0^{2\pi} (\mathcal{K}f)(s)\mathcal{K}[\cos 2(\cdot)\mathcal{K}f](s) \cos 2s \, ds \\
& -\int_0^{2\pi} (\mathcal{K}f)(s)\mathcal{K}[\sin 2(\cdot)\mathcal{K}f](s) \sin 2s \, ds\Big) \\
& +\frac{3}{64\pi}\alpha^2 \int_0^{2\pi} [(\mathcal{K}f)(s)]^2 \cos 4s \, ds - \frac{5\beta}{4\pi}\int_0^{2\pi} [(\mathcal{K}f)(s)]^2 \cos 4s \, ds
\end{aligned}$$

For $m = 5$,

$$\rho = \frac{45}{64\pi}\alpha^2 \int_0^{\pi/5} (\mathcal{K}f)(s)\cos 5s\, ds + \frac{3}{64\pi}\alpha^2 \int_0^{2\pi} (\mathcal{K}f)(s)\cos 5s\, ds - \frac{5}{16\pi}\beta \int_0^{2\pi} (\mathcal{K}f)(s)\cos 5s\, ds\ .$$

Let us consider now the equation $\ddot{u} + u = g(u,p) + \mu f(t)$, where $g(u,p) = pu + \alpha u^3 + \beta u^5 + \cdots$, for some specific examples of $f(t)$:

(1) If $f(t) = \cos 3t$ then $\rho = \frac{3\alpha}{32}$, $\lambda = \frac{1}{128}$ and $\eta = -\frac{1}{32}\cos 3\phi$.

(2) If $f(t) = 1 + \cos 4t$ then $\rho = -\frac{1}{8}\alpha^2 + \frac{1}{6}\beta$, $\lambda = \frac{1}{450}$ and $\eta = 0$.

(3) If $f(t) = \cos 5t$ then $\rho = -\frac{5}{1024}\alpha^2 + \frac{5}{384}\beta$, $\lambda = \frac{1}{1152}$ and $\eta = 0$.

4 The Genericity of the Condition $\rho \neq 0$

In this chapter we will suppose that $g \in C^{m+2}$

$$g(u,0) = \sum_{i=1}^{(m-1)/2} \alpha_{2i+1} u^{2i+1} + O(|u|^{m+2}),$$

if m is odd and $g \in C^{m+3}$,

$$g(u,0) = \sum_{i=1}^{m/2} \alpha_{2i+1} u^{2i+1} + O(|u|^{m+3}),$$

if m is even. In both cases we suppose taht $g(\cdot,p)$ is odd.

Lemma 4.1 *Under the above assumptions and* (A), *if $k \geq 3$ is an integer, $k \leq \frac{m-1}{2}$ if m is odd $k \leq \frac{m}{2}$ if m is even, then there exist continuous functions, $W_1 = W_1(\alpha_3, \ldots, \alpha_{2k-1}, r, \phi, \mu, f) = O((|r| + |\mu|)^3)$, $W_2 = W_2(r, \phi, \mu, f) = O((|r| + |\mu|)^{2k+1})$ such that:*

$$w^*(r,\phi,0,\mu) = \mu(\mathcal{K}f) + W_1 + \alpha_{2k+1} W_2 + O((|r| + |\mu|)^{2k+3}).$$

The proof of the above lemma can be done by induction on k.

Lemma 4.2 *Under the assumptions of Lemma 4.1, the following hold:*

1) *If $m \geq 4$ is even, then there exists a continuous function* $K(\alpha_3, \ldots, \alpha_{m-1}, f)$, *such that,*

$$\rho = -\frac{m(m+1)}{2^m \pi} \alpha_{m+1} \int_0^{2\pi} [(\mathcal{K}f)(s)]^2 \cos ms \, ds + K(\alpha_3, \ldots, \alpha_{m-1}, f).$$

2) *If $m \geq 5$ is odd,then there exists a continuous function $K(\alpha_3, \ldots, \alpha_{m-2}, f)$, such that,*

$$\rho = -\frac{m}{2^{m-1}\pi} \alpha_m \int_0^{2\pi} (\mathcal{K}f)(s) \cos ms \, ds + K(\alpha_3, \ldots, \alpha_{m-2}, f).$$

Proof. If m is even, from Lemma 4.1 it follows that

$$w^* = \mu \mathcal{K} f + W_1 + \alpha_{m+1} W_2 + \cdots$$

where, $\cdots \stackrel{\text{def}}{=} O((|r| + |\mu|)^{m+3})$.

Therefore,

$$\begin{aligned}
G(r, \phi, 0, \mu) &= \frac{1}{\pi} \int_0^{2\pi} \sum_{\ell=1}^{m/2} \alpha_{2\ell+1}(r \cos s + w^*(s+\phi))^{2\ell+1} \sin s \, ds + \cdots = \\
&= \frac{1}{\pi} \int_0^{2\pi} \alpha_{m+1}(r \cos s + w^*(s+\phi))^{m+1} \sin s \, ds + \\
&+ \frac{1}{\pi} \int_0^{2\pi} \sum_{\ell=1}^{\frac{m-2}{2}} \alpha_{2\ell+1}(r \cos s + w^*(s+\phi))^{2\ell+1} \sin s \, ds + \cdots = \\
&= \frac{1}{\pi} \int_0^{2\pi} \alpha_{m+1}[r \cos s + \mu(\mathcal{K}f)(s+\phi)]^{m+1} \sin s \, ds + \\
&+ \frac{1}{\pi} \int_0^{2\pi} \sum_{\ell=1}^{\frac{m-2}{2}} \alpha_{2\ell+1}[r \cos s + \mu(\mathcal{K}f)(s+\phi) + W_1]^{2\ell+1} \sin s \, ds + \cdots
\end{aligned}$$

From Theorem 3.1, it follows that

$$G(r, \phi, p, \mu) = r^{m-1} \mu^2 \sin m\phi [\rho + O(|p| + |\mu| + |r|)] .$$

Therefore, to determine ρ it suffices to consider the part of G, which contains terms envolving $r^{m-1}\mu^2$, for $p = 0$.

Let us consider first the term

$$\frac{\alpha_{m+1}}{\pi}\int_0^{2\pi}[r\cos s+\mu(\mathcal{K}f)(s+\phi)]^{m+1}\sin s\,ds.$$

The part of this term that envolves $r^{m-1}\mu^2$ is given by:

$$\frac{\alpha_{m+1}}{\pi}\frac{m(m+1)}{2}r^{m-1}\mu^2\int_0^{2\pi}\cos^{m-1}s[(\mathcal{K}f)(s+\phi)]^2\sin s\,ds.$$

By induction one can prove that

$$\begin{aligned}\cos^{m-1}s\sin s &= \frac{1}{2^{m-1}}\{[\sin ms-\sin(m-2)s]+\\ &+\sum_{j=1}^{\frac{m-2}{2}}\beta_j[\sin(m-2j)s-\sin(m-2)(j+1))s]\}\end{aligned}$$

where β_j are constants.

Since $\mathcal{K}f$ is $2\pi/m$-periodic, we have that,

$$\int_0^{2\pi}[(\mathcal{K}f)(s+\phi)]^2\sin ks\,ds=0 \quad\text{if}\quad 0\le k<m\ .$$

Therefore,

$$\begin{aligned}&\tfrac{\alpha_{m+1}}{\pi}\tfrac{m(m+1)}{2}r^{m-1}\mu^2\textstyle\int_0^{2\pi}[(\mathcal{K}f)(s+\phi)]^2\cos^{m-1}s\sin s\,ds=\\ =\ &-\tfrac{\alpha_{m+1}}{2^m\pi}m(m+1)r^{m-1}\mu^2\sin m\phi\textstyle\int_0^{2\pi}[(\mathcal{K}f)(s)]^2\cos ms\,ds\end{aligned}$$

If we define $K(\alpha_3,\ldots,\alpha_{m-1},f)$ as the coefficient of the term $r^{m-1}\mu^2\sin m\phi$ obtained from

$$\frac{1}{\pi}\int_0^{2\pi}\sum_{\ell=1}^{\frac{m-2}{2}}\alpha_{2\ell+1}[r\cos+\mu(\mathcal{K}f)(s+\phi)+W_1]^{2\ell+1}\sin s\,ds$$

we conclude that ρ has the stated form.

The second part of our lemma is similar. □

Theorem 4.1 *Under the assumptions on g, in the beginning of this chapter, the condition $\rho\neq 0$ is generic.*

Proof. Let us consider the case $m \geq 2$, even. The case $m = 2$ is simple. For $m \geq 4$ we have, from Lemma 4.2 that

$$\rho = \alpha_{m+1} c_m \int_0^{2\pi} [(\mathcal{K}f)(s)]^2 \cos ms \, ds + K(\alpha_3, \ldots, \alpha_{m-1}, f)$$

where $c_m \neq 0$.

Without loss of generality, we can assume that

$$\int_0^{2\pi} [(\mathcal{K}f)(s)]^2 \cos ms \, ds \neq 0.$$

If $\rho(f,g) = 0$ then we consider $\bar{g}(u) = g(u,0) + \varepsilon u^{m+1}$. Then $\rho(f,\bar{g}) = \varepsilon c_m \int_0^{2\pi} [(\mathcal{K}f)(s)]^2 \cos ms \, ds \neq 0$ and $\bar{g}$ is close to g in the c^{m+3}-topology, if ε is small.

The remaining part of the proof is easy.

The case m odd is similar. □

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