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Nº 53

Notas do ICMSC - USP

ISSN 0103-2577

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São Carlos (SP)
1989

Classification of $\mathcal{A}$-simple germs from k^n to k^2

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Introduction and notation

There exist extensive classifications of map-germs from n-space ($n \geq 2$) into the plane up to contact equivalence ($\mathcal{K}$-equivalence), see for example $[D, Da, G, M, W]$. In the present paper we refine a small part of these $\mathcal{K}$-classifications by studying right-left equivalence classes ($\mathcal{A}$-classes) contained in certain $\mathcal{K}$-orbits. In particular we obtain a classification of $\mathcal{A}$-simple germs from (complex and real) $n-$space ($n \geq 2$) into the plane (the $\mathcal{A}$-simple germs of plane curves $\mathbf{C} \to \mathbf{C}^2$ have been classified in [Bru Gaf]).

Let $f : k^n, 0 \to k^p, 0$ be a smooth map-germ (where $k = \mathbf{C}$ or $\mathbf{R}$, and where smooth means analytic in the former and C^∞ or analytic in the latter case). Let $\mathcal{A} = \mathrm{Diff}(k^n, 0) \times \mathrm{Diff}(k^p, 0)$ denote the group of right-left equivalences, which acts on the space of smooth germs f as follows: $(h, k).f = h \circ f \circ k^{-1}$, where $(h, k) \in \mathcal{A}$. Replacing the action on the left, i.e. the composition with elements of $\mathrm{Diff}(k^p, 0)$, by composition with elements of $G\ell(p, k)$ with entries in C_n (where C_n = local ring of smooth function germs $k^n, 0 \to k, 0$) gives the group $\mathcal{K}$ of contact equivalences. A $\mathcal{G}$-orbit U(where $\mathcal{G} = \mathcal{A}$ or $\mathcal{K}$) is said to be adjacent to another $\mathcal{G}$-orbit V , denoted by $U \to V$, if any representative f of U can be embedded in an unfolding $F(u, \bar{f}(u, x))$, where $\bar{f}(0, x) = f(x)$, such that the set $\{u, x\}$ for which $\bar{f}(u, x) \in V$ contains $u = x = 0$ in its closure. A $\mathcal{G}$-orbit U is said to be $\mathcal{G}$-simple if it is adjacent to only a finite number of other $\mathcal{G}$-orbits.

*Research partially supported by Vitae, CNPq and British Council

Let C_n and C_p denote the local rings of function-germs in source and target whose respective maximal ideals are m_n and m_p . Let θ_f denote the C_n-module of vector fields over f , and set $\theta_n = \theta(1_{k^n})$ and $\theta_p = \theta(1_{k^p})$. One can then define the homomorphisms

$$tf : \theta_n \to \theta_f \ , \quad tf(\psi) = Df.\psi \ ,$$

and

$$wf : \theta_p \to \theta_f \ , \quad wf(\emptyset) = \emptyset \circ f \ .$$

The tangent space to the $\mathcal{A}$-orbit at f can be calculated to be $T\mathcal{A}.f = tf(m_n.\theta_n) + wf(m_p.\theta_p)$, and the $\mathcal{A}$-codimension of f is $\mathrm{cod}(\mathcal{A}, f) = \dim_k m_n.\theta_f / T\mathcal{A}.f$.

Let $J^k(n,p)$ denote the space of k^{th} order Taylor polynomials without constant terms, and write $j^k f$ for the k-jet of f . The Lie group $\mathcal{A}^k := j^k(\mathcal{A})$ acts smoothly on $J^k(n,p)$, and we shall write $T\mathcal{A}^k.f$ for the corresponding tangent space $T_{j^k f}\mathcal{A}^k.j^k f$. A germ f is said to be k-determined if, for any g , $j^k f = j^k g$ implies that $f \sim g$. The calculation of $\mathcal{A}^k$-orbits (using Mather's Lemma [Ma IV, Lemma 3.1]) and of the determinacy degree of a germ f (using an estimate of duPlessis [dP, Corollary 3.9]) are the main tools that we use in the present classification. See [Rie] for further details on notation and techniques.

1 Classification of $\mathcal{A}$-simple Σ^1-germs from $k^n, 0$ to $k^2, 0$, $n > 1$

We can assume that $f : k^n, 0 \to k^2, 0$ is of the form $f(x,y) = (x, f_2(x,y))$, and we denote by X, Y the coordinates in the target. The following splitting lemma is almost content-free but clarifies the subsequent classification,

Lemma 1.1 *Every map-germ $f : k^n, 0 \to k^2, 0$, $n > 1$, of corank 1 is $\mathcal{A}$-equivalent to a germ of the form*

$$h(x,y,z) = (x, g(x, y_1, \cdots, y_m) + \sum_{i=1}^{n-m-1} \epsilon z_i^2) \ ,$$

where $g(0, y_1, \cdots, y_m) \in m_n^3$ and $\epsilon = \pm 1$ (for $k = \mathbf{C}$, $\epsilon = 1$); and $\mathrm{cod}(\mathcal{A}, h) = \mathrm{cod}(\mathcal{A}, (x, g)) + n - m - 1$.

Proof. Any $j^k f(x,y,z) = (x, f_2(x,y,z))$ can be reduced to h by right-coordinate changes of the form $\bar{z}_i = \phi(x,y,z)$, where the bar denotes new coordinates and not complex conjugation. It is also clear that $T\mathcal{A}^k.j^kf$, $k > 1$, contains all monomials containing powers of z_i except for $z_i.\partial/\partial Y$, which implies the last statement of the Lemma . □

Hence, if we take $m = 1$ the lemma above reduces the classification of germs $f : k^n, 0 \to k^2, 0$ to the classification of germs $g : k^2, 0 \to k^2, 0$ of corank 1. The $\mathcal{A}$-simple germs of the plane of corank 1 have been classified by one of the authors in [Rie]. Next, consider the case $m = 2$.

Lemma 1.2 *For $m = 2$ (or indeed for $m \geq 2$) there are no $\mathcal{A}$-simple germs $f(x, y) = (x, g(x, y_1, \cdots, y_m))$, where $g(0, y_1, \cdots, y_m) \in m^3_{m+1}$ as in Lemma 1.1 .*

Proof. The 2-jet of f is, by the hypothesis on g , given by $j^2f = (x, ax^2 + bxy_1 + cxy_2)$, which can either be reduced to (x, xy_1) , provided that either b or c is non-zero,or else to $(x, 0)$. First, we show that the $\mathcal{A}^3$- orbits over $j^2f = (x, xy_1)$ are at least uni-modal. Note that we can reduce any 3-jet over (x, xy_1) to $h := (x, xy_1 + ay_1^3 + by_1^2y_2 + cy_1y_2^2 + dy_2^3)$. There are exactly four generators, namely $wh(X, Y) - th(x, 0, 0)$, $th(0, y_1, 0) - th(x, 0, 0) + wh(X, 0)$, $th(0, 0, y_1)$, and $th(0, 0, y_2)$, for the subspace $V := k\{(0, y_1^iy_2^j), i + j = 3\}$ of $T\mathcal{A}^3.h$, leading to the following matrix of coefficients:

$$\begin{bmatrix} a & b & c & d \\ 3a & 2b & c & \\ b & 2c & 3d & \\ & b & 2c & 3d \end{bmatrix}$$

which doesn't have maximal rank (because (row 4) $- 3 \times$ (row 1) $=$ $-$(row 2)) . It follows from Mather's lemma [Ma IV] that V is foliated by (at least) a 1-parameter family of $\mathcal{A}^3-$orbits.

The same is true for the $\mathcal{A}^3-$orbits over $(x, 0)$. Note that $\dim_k[(m_3^3/m_3^4)\partial/\partial Y] = 10$ while there are only nine generators for elements of $T\mathcal{A}^3.f$ "downstairs":

$$wf(0, X^3), wf(0, Y), tf(x, 0, 0) - wf(X, 0), tf(0, x, 0), tf(0, y_1, 0), \\ \ldots, tf(0, 0, y_2) \ ;$$

where $f := (x, h_3(x, y_1, y_2))$, h_3 being a general cubic form.

For $m > 2$ the $\mathcal{A}^2-$orbits are still those of (x, xy_1) and $(x, 0)$, and the modality of the $\mathcal{A}^3-$orbits over (x, xy_1) and $(x, 0)$ is clearly greater than or equal to one. Lemma 1.2 now follows. □

Using the results of [Rie] we get the following classification

Proposition 1.3 *An $\mathcal{A}$-simple map-germ $f : k^n, 0 \to k^2, 0$ $(n > 1)$ of corank 1 is equivalent to one of*

type	$f(x, y, z_1, \ldots, z_{n-2}) =$	$cod(\mathcal{A}, f)$
1	(x, y)	0
2	$(x, y^2 + \Sigma \varepsilon z_i^2)$	$n-1$
3	$(x, xy + y^3 + \Sigma \varepsilon z_i^2)$	n
4_k	$(x, y^3 + \varepsilon^{k-1} x^k y + \Sigma \varepsilon z_i^2),\ k > 1$	$n+k-1$
5	$(x, xy + y^4 + \Sigma \varepsilon z_i^2)$	$n+1$
6	$(x, xy + y^5 + \varepsilon y^7 + \Sigma \varepsilon z_i^2)$	$n+2$
7	$(x, xy + y^5 + \Sigma \varepsilon z_i^2)$	$n+3$
11_{2k+1}	$(x, xy^2 + y^4 + y^{2k+1} + \Sigma \varepsilon z_i^2),\ k > 1$	$n+k$
12	$(x, xy^2 + y^5 + y^6 + \Sigma \varepsilon z_i^2)$	$n+3$
13	$(x, xy^2 + y^5 + \varepsilon y^9 + \Sigma \varepsilon z_i^2)$	$n+4$
14	$(x, xy^2 + y^5 + \Sigma \varepsilon z_i^2)$	$n+5$
16	$(x, x^2 y + y^4 + \varepsilon y^5 + \Sigma \varepsilon z_i^2)$	$n+3$
17	$(x, x^2 y + y^4 + \Sigma \varepsilon z_i^2)$	$n+4$

(where $\varepsilon = \pm 1$ for $k = \mathbf{R}$, and $\varepsilon = 1$ for $k = \mathbf{C}$).

Remark 1.4 *An $\mathcal{A}$-versal (and for that matter any) deformation of one of these germs does not increase m. Hence, these $\mathcal{A}$-classes are only adjacent to other $(m = 1)$-germs and therefore are simple for all n.*

2 Classification of $\mathcal{A}$-simple $\Sigma^{2,0}$ germs from $k^n, 0$ to $k^2, 0$

The main result in this section is the following classification.

Proposition 2.1 *Any simple germ $f : k^n, 0 \to k^2, 0$ (for $n > 1$) of corank 2 is $\mathcal{A}$-equivalent to some member of the following series of germs:*

(i) $k = \mathbf{R}$: $I_{2,2}^{\ell,m} = (x^2 + y^{2\ell+1}, y^2 + x^{2m+1})$, $\ell \geq m \geq 1$, *or*
$II_{2,2}^{\ell} = (x^2 - y^2 + x^{2\ell+1}, xy)$, $\ell \geq 1$;

(ii) $k = \mathbf{C}$: $I_{2,2}^{\ell,m} = (x^2 + y^{2\ell+1}, y^2 + x^{2m+1})$, $\ell \geq m \geq 1$.

The $\mathcal{A}$-codimension of $I_{2,2}^{\ell,m}$ and $II_{2,2}^{\ell}$ are $\ell+m+2$ and $2(\ell+1)$ respectively.

To prove this statement we classify $\mathcal{A}$-orbits contained in $\mathcal{K}$-simple orbits of germs $f : k^n, 0 \to k^2, 0$ of corank 2. Such $\mathcal{K}$-simple germs have been classified in [D,M,Da] (see Theorem 2.3.1, below). In the present classification of $\mathcal{A}$-simple germs we can discard all $\mathcal{K}$-orbits adjacent to some $\mathcal{K}$-orbit that doesn't contain any $\mathcal{A}$-simple orbits.

At the end of the section, we also consider non-simple germs of type $\sum^{2,0}$. Proposition 2.4.1 gives a partial normal form for any corank 2, $\mathcal{A}$-finitely determined complex germ with non-degenerate 2-jet. An estimate for the degree of $\mathcal{A}$-determinary of such germs is also proved.

2.1 Classification of $\mathcal{A}$-orbits in $\mathcal{K}(x^2, y^2)$, for $k = \mathbf{C}$ or $\mathbf{R}$

Proposition 2.1.1 *Any germ contained in the $\mathcal{K}$-orbit of (x^2, y^2) is $\mathcal{A}$-equivalent to some member of the series*

$$I_{2,2}^{\ell,m} = (x^2 + y^{2\ell+1}, y^2 + x^{2m+1}), \quad \ell \geq m \geq 1 .$$

The $I_{2,2}^{\ell,m}$ are $(2\ell+1)$-determined, and $\mathrm{cod}(\mathcal{A}, I_{2,2}^{\ell,m}) = \ell + m + 2$.

Proof. Any k-jet $(x^2 + \sum a_{i,j}x^iy^j,\ y^2 + \sum b_{i,j}x^iy^j)$, where the x^iy^j are of degree $k > 1$, can be reduced to $(x^2 + a_{0,k}y^k,\ y^2 + b_{k,0}x^k)$ by the right-coordinate change $(\bar{x}, \bar{y}) = (x - \frac{1}{2}(a_{k,0}x^{k-1} + \ldots + a_{1,k-1}y^{k-1}), y - \frac{1}{2}(b_{k-1,1}x^{k-1} + \ldots + b_{0,k}y^{k-1}))$. Now, suppose $k = 2\ell$: the left-coordinate changes $\bar{X} = X - a_{0,2\ell}Y^\ell$ and $\bar{Y} = Y - b_{2\ell,0}X^\ell$ give (x^2, y^2), which is the single $\mathcal{A}^{2\ell}$-orbit over $j^{2\ell-1}f = (x^2, y^2)$. If $k = 2\ell+1$, we have three $\mathcal{A}^{2\ell+1}$-orbits over $j^{2\ell}f = (x^2, y^2)$: *(i)* $(x^2 + y^{2\ell+1},\ y^2 + x^{2\ell+1})$, *(ii)* $(x^2, y^2 + x^{2\ell+1})$, and *(iii)* (x^2, y^2). By a result of du Plessis [dP, Example 3.18], $(x^2 + y^{2\ell+1},\ y^2 + x^{2\ell+1})$ is $(2\ell+1)$- determined. Now, consider $\mathcal{A}^k$-orbits over $j^{2m+1}f = (x^2, y^2 + x^{2m+1})$. If $k = 2\ell > 2m+1$, we find a single $\mathcal{A}^{2\ell}$-orbit $(x^2, y^2 + x^{2m+1})$, by the same coordinate changes as above; and if $k = 2\ell+1 > 2m+1$, we can reduce to $(x^2 + a_{0,2\ell+1}y^{2\ell+1},\ y^2 + x^{2m+1})$ leading to two $\mathcal{A}^{2\ell+1}$-orbits given by $a_{0,2\ell+1} = 0, 1$. Now, $I_{2,2}^{\ell,m}$ is $(2\ell+1)$-determined, again by [dP, Example 3.18]. Finally, we check that

$$k\{(x,0), (y^{2i+1}, 0), (0, y), (0, x^{2j+1}) \ :\ \ell > i \in \mathbf{N},\ m > j \in \mathbf{N}\}$$

forms a free basis for $m_n.\theta_f/T\mathcal{A}.f$, where $f = I_{2,2}^{\ell,m}$, which proves the proposition. □

2.2 Classification of $\mathcal{A}$-orbits contained in $\mathcal{K}(x^2-y^2, xy)$ over R

Proposition 2.2.1 *Any germ contained in the $\mathcal{K}$-orbit of (x^2-y^2, xy) is $\mathcal{A}$-equivalent to some member of the series*

$$II^{\ell}_{2,2} = (x^2 - y^2 + x^{2\ell+1}, xy) \ , \quad \ell \geq 1 \ .$$

The $II^{\ell}_{2,2}$ are $(2\ell+1)$-determined, and $\operatorname{cod}(\mathcal{A}, II^{\ell}_{2,2}) = 2(\ell+1)$.

Proof. Let $f = (x^2 - y^2 + \sum a_{i,j}x^iy^j, xy + \sum b_{i,j}x^iy^j)$, where $\sum a_{i,j}x^iy^j$ and $\sum b_{i,j}x^iy^j$ are forms of degree 2ℓ , denote a 2ℓ-jet over (x^2-y^2, xy) . One easily checks that $m_n^{2\ell}/m_n^{2\ell+1}.\theta_f \subset T\mathcal{A}^{2\ell}.f$, independent of the $a_{i,j}$ and $b_{i,j}$'s, which, by Mather's Lemma [Ma IV], implies that $j^{2\ell}f = (x^2-y^2, xy)$ is the only $\mathcal{A}^{2\ell}$-orbit over $j^{2\ell-1}f = (x^2 - y^2, xy)$.

There are two $\mathcal{A}^{2\ell+1}$-orbits over $j^{2\ell}f = (x^2-y^2, xy)$. Take f as above but with monomials x^iy^j of degree $2\ell+1$. Working modulo elements of $m_n^{2\ell+2}$, we can reduce f by right-coordinate changes to $f = (x^2-y^2, xy + \mu(a_{i,j}, b_{i,j})xy^{2\ell} + \gamma(a_{i,j}, b_{i,j})y^{2\ell+1})$, where μ and γ are linear forms. (The coordinate changes $\bar{x} = x - \frac{1}{2}(a_{2\ell+1,0}x^{2\ell} + \ldots + a_{1,2\ell}y^{2\ell})$ and $\bar{y} = y + \frac{1}{2}a_{0,2\ell+1}y^{2\ell}$ remove the terms of degree $2\ell+1$ from the first component function. The pairs of successive coordinate changes

$$\bar{y} = y - b_{2\ell+1,0}x^{2\ell} \ , \quad \bar{x} = x + b_{2\ell+1,0}x^{2\ell-1}$$

$$\bar{y} = y - (b_{2\ell,1} - \tfrac{1}{2}a_{2\ell+1,0})x^{2\ell-1}y \ , \quad \bar{x} = x + (b_{2\ell,1} - \tfrac{1}{2}a_{2\ell+1,0})x^{2\ell-2}y^2$$

$$\vdots$$

$$\bar{y} = y - \phi(a_{i,j}, b_{i,j})xy^{2\ell-1} \ , \quad \bar{x} = \bar{x} + \phi(a_{i,j}, b_{i,j})y^{2\ell} \ ,$$

where ϕ is some linear form, yield the above $(2\ell+1)$-jet f.)

The linear forms μ and γ do, together, involve all the coefficients $a_{i,j}, b_{i,j}$ (this can be seen by inspecting the coordinate changes more carefully). Considering the elements of $T\mathcal{A}^{2\ell+1}.f$ that generate $m_n^{2\ell+1}/m_n^{2\ell+2}.\theta_f$ leads to the following $4(\ell+1) \times 4(\ell+1)$ matrix:

	$(x^{2\ell+1},0)$	$(x^{2\ell}y,0)$	$\cdots$	$(xy^{2\ell},0)$	$(y^{2\ell+1},0)$	$(0,x^{2\ell+1})$	$(0,x^{2\ell}y)$	$\cdots$	$(0,xy^{2\ell})$	$(0,y^{2\ell+1})$
$tf(x^{2\ell},0)$	2	0	$\cdots$	$\cdots$	0	0	1	0	$\cdots$	0
$tf(x^{2\ell-1}y,0)$	0				$\vdots$	$\vdots$				$\vdots$
$\vdots$	$\vdots$	$\ddots$			$\vdots$	$\vdots$		$\ddots$		$\vdots$
$\vdots$	$\vdots$		$\ddots$		$\vdots$	$\vdots$			$\ddots$	$\vdots$
$tf(y^{2\ell},0)$	0	$\cdots$	$\cdots$	2	0	0	$\cdots$	$\cdots$	$\cdots$	1
$tf(0,x^{2\ell})$	0	-2	0	$\cdots$	0	1	0	$\cdots$	$\cdots$	0
$\vdots$	$\vdots$		$\ddots$		$\vdots$	0	$\ddots$			$\vdots$
$\vdots$	$\vdots$			$\ddots$	$\vdots$	$\vdots$		$\ddots$		$\vdots$
$tf(0,y^{2\ell})$	0	$\cdots$	$\cdots$	$\cdots$	-2	$\vdots$			1	0
α	0	$\cdots$	$\cdots$	$\cdots$	0	0	$\cdots$	0	μ	γ
β	0	$\cdots$	$\cdots$	4μ	4γ	0	$\cdots$	$2\ell\mu$	$(2\ell+1)\gamma$	$-\mu$

where $\alpha = (2\ell-1)^{-1}(tf(x,y) - 2wf(X,Y))$ and $\beta = tf(-y,x) + wf(4Y,-X)$.

By row operations we get:

2									
	$\ddots$							*	
		2							
			-2						
				1					
					$\ddots$				
						1			
							μ		γ
		0							
							$(2\ell+3)\gamma$		$-(2\ell+3)\mu$

Hence there are two $\mathcal{A}^{2\ell+1}$-orbits, depending on whether $\mu^2+\gamma^2$ vanishes or not, whose codimensions differ by two (over $\mathbf{R}$). Inspecting the linear forms (in $a_{i,j}, b_{i,j}$) μ and γ , it turns out that $\mu^2+\gamma^2 \neq 0$ if we set one of the coefficients to one and all the others to zero. Therefore we may take $(x^2-y^2+x^{2\ell+1}, xy)$ and (x^2-y^2, xy) as representatives of the two $\mathcal{A}^{2\ell+1}$-orbits.

Next, we want to show that $f = (x^2-y^2+x^{2\ell+1}, xy)$ is $(2\ell+1)$-determined. Clearly, $m_n^2.\theta_f \subseteq T\mathcal{K}_e.f$ and we have shown above that $m_n^{2\ell}.\theta_f \subseteq T\mathcal{A}_e.f + m_n^{2\ell+2}.\theta_f$. Hence, by a result of du Plessis [dP, Cor.3.9], f is $(2\ell+2)$-determined. But there is only a single $\mathcal{A}^{2\ell+2}$-orbit over f so that the exact order of determinacy is $2\ell+1$.

Finally, one checks that

$$\mathbf{R}\{(x,0),(y,0),(0,xy^{2i}),(0,y^{2i+1}):\ \ell > i \in \mathbf{N}\}$$

forms a free basis for $m_n.\theta_f/T\mathcal{A}.f$, which implies that $\text{cod}(\mathcal{A}, II^{\ell}_{2,2}) = 2(\ell+1)$ — the proposition now follows. □

2.3 Other $\mathcal{K}$-orbits of type $\Sigma^{2,0}$ do not contain $\mathcal{A}$-simple orbits

We first recall the classifications of $\mathcal{K}$-simple germs. In the complex case, these results were obtained by Dimca and Gibson [D] $(n = p = 2)$, and by Giusti [G] and Wall [W] (complete intersections, $p = 2$, $n > p$). The real case was treated earlier by Mather [M] and Damon [Da] $(n \leq p)$.

When $n = p = 2$, the following theorem summarizes the results:

Theorem 2.3.1 ([D],[M],[Da]) *Let A denote the set of all $\mathcal{K}$-finite germs $f:(k^2,0)\to(k^2,0)$, $k = \mathbf{R}$ or $\mathbf{C}$. Then any $\mathcal{K}$-simple germ in A is $\mathcal{K}$-equivalent to exactly one of the germs in the following list:*

Boardman Symbol	Mather's Notation	Complex Notation	Normal Form	Condition	cod($\mathcal{K}$, −)
Σ^1	A_k	A_k	(x, y^{k+1})	$k \geq 0$	k
$\Sigma^{2,0}$	$I_{k,\ell}$		$(xy, x^k + y^\ell)$	$2 \leq k \leq \ell$	$k+\ell$
	$II_{k,\ell}$	$B_{k,\ell}$	$(xy, x^k - y^\ell)$	$2 \leq k \leq \ell$	$k+\ell$
	IV_k		(x^2+y^2, x^k)	$3 \leq k$	$2k$
$\Sigma^{2,1}$	Group I	C_4	(x^2, y^4)		10
		$E_{3,k}$	(x^2+y^3, y^k)	$k \geq 3$	$2k+1$
		$F_{3,k}$	(x^2+y^3, xy^{k-1})	$k \geq 4$	$2k+3$
$\Sigma^{2,1}$	Group II	$F_{k,2}$	(x^2+y^k, xy^2)	$k \geq 3$	$k+5$
			(x^2-y^k, xy^2)	$k \geq 3$	$k+5$

TABLE 1

Remark 2.3.2

(i) In the complex case $I_{k,\ell} \sim II_{k,\ell}$ and $IV_k \sim I_{k,k}$.

(ii) The germs in group II, $(x^2 \pm y^k, xy^2)$ are equivalent as complex germs.

The adjacencies between the equidimensional real map-germs of type $\sum^1$ and $\sum^{2,0}$ were described by Les Lander [L., Theorem 2.1], and we recall them in Figure 1 .

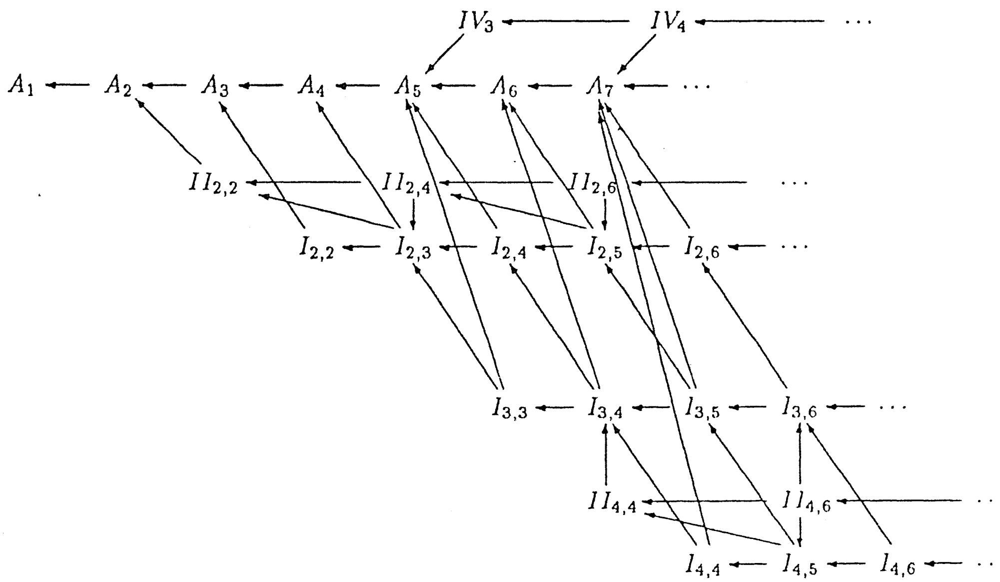


Figure 1: Adjacencies of real $\mathcal{K}$-orbits of types $\sum^1$ and $\sum^{2,0}$

By forgetting in Figure 1 the singularities of type $II_{k,\ell}$ and IV_k , and all the arrows meeting them, we obtain the diagram of adjacencies between complex germs of type $\sum^1$ and $\sum^{2,0}$. (Figure 2) (See [G]).

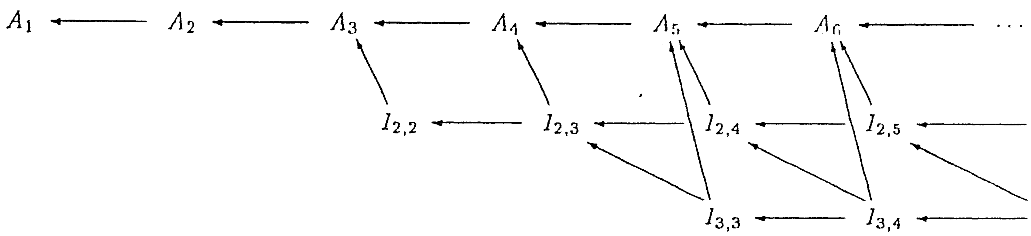


Figure 2: Adjacencies of $\mathcal{K}$-orbits of complex germs of types $\sum^1$ and $\sum^{2,0}$

To show that all $\mathcal{A}$-simple germs of type $\sum^{2,0}$ are within $I_{2,2}$, we shall need the following lemmas:

Lemma 2.3.3 *The $\mathcal{A}$-orbits within $I_{2,3}$ are all non-simple.*

Proof. We show that there is no open $\mathcal{A}$-orbit within $\mathcal{K}(x^2+y^3, xy)$, which will imply that this $\mathcal{K}$-orbit is filled up entirely with non-simple $\mathcal{A}$-orbits.

After some simple coordinate changes, we may assume that any $\mathcal{A}$-finitely determined germ in $I_{2,3}$ has the form:

$$(x^2 + y^3 + axy^2 + bxy^3 + cy^4 + \Phi(x,y) ,\ xy) , \quad \Phi \in m_2^4$$

One easily checks that $m_2^3\theta_f \subset T\mathcal{K}.f$.

Now, the relevant relations in $T\mathcal{A}^4.f$ are given by $tf(x^iy^i,0)$, $tf(0,x^iy^i)$, $i+j = 1,2,3$, $wf(X,0)$, $wf(Y,0)$, $wf(0,X)$, $wf(0,Y)$.

Thus there are only 22 generators for the vector subspace $m_2^2\theta_f/m_2^5\theta_f$, which has dimension 24. In particular, $T\mathcal{A}^4.f \not\supseteq m_2^3\theta_f/m_2^5\theta_f$, and the modality of the $\mathcal{A}$-orbit of f within $\mathcal{K}(x^2+y^3,xy)$ is greater than one. □

Lemma 2.3.4 *The $\mathcal{A}$-orbits within IV_3 are all non-simple.*

Proof. Any f in $\mathcal{K}(x^2+y^2,x^3)$ is $\mathcal{A}$-equivalent to:

$$(x^2+y^2, x^3+p(x,y)) \ , \quad p \in m_2^3\theta_f$$

We proceed as in Lemma 2.3.1 to show that the $\mathcal{A}$-orbit of f cannot be open in $\mathcal{K}(x^2+y^2,x^3)$.

We have: $m_2^3\theta_f \subset T\mathcal{K}.f$.

It is easy to see that $T\mathcal{A}^3.f \supseteq k\{(x^iy^i,0), i+j=3\}$. However $tf(x,0)$, $tf(y,0)$, $tf(0,x)$, $tf(0,y)$, $wf(X,0)$ and $wf(0,Y)$ give only two relevant relations "downstairs":

$$(0,Y) \text{ and } (0,\Delta) \quad (\text{Mod } m_2^4\theta_f) \ ,$$

where $\Delta = -6x^2y + 2xp_y - 2yp_x$.

Therefore, $T\mathcal{A}^3.f \not\supseteq m_2^3\theta_f/m_2^4\theta_f$, and $\mathcal{A}$-modality of f is greater or equal to one.□

Using Lemmas 2.3.3 and 2.3.4 and the diagrams in Figures 1 and 2, we get the following result.

Proposition 2.3.5 *Let $f:(k^2,0)\to(k^2,0)$, $(k=\mathbb{R},\mathbb{C})$ be an $\mathcal{A}$-finitely determined germ of type $\sum^{2,0}$. If f is $\mathcal{A}$-simple then the $\mathcal{K}$-orbit of f is of type $I_{2,2}$ or $II_{2,2}$.*

We consider now the case $n \geq 3$, $p=2$.

It is well known that if $n \geq 4$, the $\mathcal{K}$-modality of a pair of quadrics is greater or equal to one ([W,1]). Therefore, we only have to consider the case $n=3$, $p=2$.

In the complex case, there is only one $\mathcal{K}$-orbit of type $\sum^{2,0}$, whose normal form is (x^2+y^2,y^2+z^2) .

Proposition 2.3.6 *Any $\mathcal{A}$-orbit of a finitely determined germ within $\mathcal{K}(x^2+y^2,y^2+z^2)$ is at least 1-modal.*

Proof. Let $f : (\mathbf{C}^3,0) \to (\mathbf{C}^2,0)$ be any $\mathcal{A}$-finitely determined germ within $\mathcal{K}(x^2 + y^2, y^2 + z^2)$. Then, with simple coordinate changes, $j^3 f$ can be reduced to:

$$(x^2 + y^2 + ax^3 + cz^3 \, , \, x^2 + z^2 + by^3)$$

As before, the result follows from the information given by $T\mathcal{K}^3.f$ and $T\mathcal{A}^3.f$:

i) $T\mathcal{K}.f + m_2^4\theta_f \supset m_2^3\theta_f$.

ii) Inspecting $T\mathcal{A}^3.f$ we see that the elements of degree three are given by $tf(x,0,0)$, $tf(0,y,0)$, $tf(0,0,z)$, $\frac{1}{3}[tf(x,y,z) - 2wf(X,0) - 2wf(0,Y)]$ and by $J(f).m_2$ (where $J(f)$ is the Jacobian ideal) (Mod $m_2^4\theta_f$).

They generate the following subspace of $m_2^3/m_2^4\theta_f$:
$\mathbf{C}$ { all mixed terms of degree three, (x^3,x^3), $(y^3,0)$, $(0,z^3)$ and (ax^3+cz^3, by^3) } (Mod $m_2^4\theta_f$).
Hence $T\mathcal{A}^3.f \not\supset m_2^3\theta_f$, and comparing with i) we get the result. □

Remark 2.3.7 *$\mathcal{K}(x^2 + y^2, y^2 + z^2)$ splits into various real orbits. These real $\mathcal{K}$-orbits do not have open $\mathcal{A}$-orbit either. In fact, if the condition*

$$tf(m_2\theta_2) + f^*(m_2)\theta_f = tf(m_2\theta_2) + wf(m_2\theta_2)$$

were true for any such real germ, we should have:

$$tf_C(m_2\theta_2) + f(m_2)\theta_f = tf_C(m_2\theta_2) + wf_C(m_2\theta_2) \quad ,$$

where f_C is its complexification. This clearly contradicts the above lemma.

The following proposition summarizes the discussion:

Proposition 2.3.8 *Let $f : (k^n,0) \to (k^2,0)$, $n \geq 3$, $(k = \mathbf{R}, \mathbf{C})$ be any $\mathcal{A}$-finitely determined germ of type $\sum^{2,0}$. Then f is non simple.*

2.4 $\mathcal{A}$-determinacy of $\sum^{2,0}$-germs

We will obtain partial normal forms for corank 2, $\mathcal{A}$-finitely determined germs $f : k^n,0 \to k^2,0$, with non-degenerate 2-jet.
Following Wall, [W,2], we define

Definition 2.4.1 *For any map-germs with the same target* $f : k^s, 0 \to k^p, 0$ $g : k^t, 0 \to k^p, 0$ $(k = \mathbf{R}$ *or* $\mathbf{C})$ *let* $f \oplus g : k^{s+t}, 0 \to k^p, 0$ *be defined by*

$$(f \oplus g)(x, y) = f(x) + g(y)$$

Definition 2.4.2 *Let* $f : k, 0 \to k^2, 0$ *be defined by*

$$f(x) = (x^2 + Ax^{2\ell+1}\ ,\ ax^2 + Bx^{2m+1})\ .$$

We say that f *is* <u>*adequate*</u> *if one of the following conditions holds:*

(a) $\ell = m$ *and* $\det \begin{vmatrix} 1 & A \\ a & B \end{vmatrix} \neq 0$

(b) $\ell \neq m$ *and* $\det \begin{vmatrix} 1 & A \\ a & 0 \end{vmatrix} \neq 0$ *or* $\det \begin{vmatrix} 1 & 0 \\ a & B \end{vmatrix} \neq 0$

We define the germs

$$\begin{aligned} f_i : k, 0 &\to k^2, 0 \\ x_i &\mapsto (x_i^2 + A_i x^{2\ell_i+1}\ ,\ a_i x_i^2 + B_i x_i^{2m_i+1})\ , \end{aligned}$$

$i = 1, ..., n$, *and denote by* $f : k^n, 0 \to k^2, 0$ *the map-germ:*

$$f(x) = \bigoplus_{i=1}^{n} f_i(x_i) = (\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} A_i x_i^{2\ell_i+1}\ ,\ \sum_{i=1}^{n} a_i x_i^2 + \sum_{i=1}^{n} B_i x_i^{2m_i+1})$$

Then, we have the following proposition:

Proposition 2.4.3 *Let* $f_i : k, 0 \to k^2, 0$ *be adequate map-germs for all* $i = 1, ..., n$. *If* $a_i \neq a_j$ *for* $i \neq j$, *then the map germ* $f : k^n, 0 \to k^2, 0$, $f(x) = \bigoplus_{i=1}^{n} f_i(x_i)$ *is* $\mathcal{A}$*-finitely determined. Furthermore, any* $\mathcal{A}$*-finitely determined germ* g *of type* $\sum^{2,0}$, $\mathcal{K}$*-equivalent to* $(\sum_{i=1}^{n} x_i^2\ ,\ \sum_{i=1}^{n} a_i x_i^2)$, *is* $\mathcal{A}$*-equivalent to a sum* $g = \bigoplus_{i=1}^{n} g_i$, *where* $g_i : (k, 0) \to (k^2, 0)$ *is defined by* $g_i(x_i) = (x_i^2 + p_i(x_i)\ ,\ a_i x_i^2 + q_i(x))$, p_i *and* q_i *being polynomials of degree* ≥ 3 .

To prove this proposition we shall need the following lemmas:

Lemma 2.4.4 *Let* $j^2 f = (\sum_{i=1}^{n} x_i^2\ ,\ \sum_{i=1}^{n} a_i x_i^2)$. *Then if* $a_i \neq a_j$ *for* $i \neq j$, *the following relations hold:*

(i) $\quad tf(m_n^2\theta_n) + f^*(m_2)m_n\theta_f + m_n^4\theta_f = m_n^3\theta_f$ *(Hence f is $2-\mathcal{K}$-determined).*

(ii) $\quad J(f).m_n^{2k-2} + f^*(m_2^k)C_n + m_n^{2k+1} = m_n^{2k} \ , \ \forall k \geq n-1$.

Proof. We have: $J(f) \supset \{$ all mixed terms of degree 3 $\}$ (Mod m_n^4). Furthermore, $\forall j, f_1.x_j = x_j^3$ (Mod $m_n^4 + J(f)$) (where f_1 denotes the first component of f).

Hence $J(f).m_n + f^*(m_2)C_n + m_n^4 = m_n^3$, which implies condition *(i)* .

Now, the generators of $f^*(m_2^{n-1})C_n$ (Mod $J(f).m_n^{2n-4} + m_n^{2n-1}$) give the matrix:

	x_1^{2n-2}	x_2^{2n-2}	x_3^{2n-2}	....	x_n^{2n-2}
f_1^{n-1}	1	1	1	....	1
$f_1^{n-2}f_2$	a_1	a_2	a_3	....	a_n
$f_1^{n-3}f_2^2$	a_1^2	a_2^2	a_3^2	....	a_n^2
$\vdots$					
$\vdots$					
$\vdots$					
f_2^{n-1}	a_1^{n-1}	a_2^{n-1}	a_3^{n-1}	....	a_n^{n-1}

The condition $a_i \neq a_j$ implies that this matrix has maximal rank.

Hence, we obtain *(ii)*. □

Lemma 2.4.5 *If*

(a) $\quad f^*(m_p^k)\theta_f \subseteq tf(m_n^{2k-1}\theta_n) + wf(m_p^k\theta_p) + m_n^s\theta_f \ , \ i \leq k \leq s$

and

(b) $\quad m_n^s\theta_f \subseteq tf(m_n^{2k+1}\theta_n) + f^*(m_p^{k+1})\theta_f + m_n^{s+1}\theta_f$

Then f is $(s-1)$-$\mathcal{A}$-determined.

Proof. This lemma is a slight variant of Prop. 3.8 in [dP].

Let g be such that $j^{s-1}g(0) = j^{s-1}f(0)$.

Then g satisfies conditions *(a)* and *(b)*:

(a) $\quad g^*(m_p^k)\theta_g \subseteq tg(m_n^{2k-1}\theta_n) + wg(m_p^k\theta_p) + m_n^s\theta_g$

and

(b) $\quad m_n^s\theta_g \subseteq tg(m_n^{2k+1}\theta_n) + g^*(m_p^{k+1})\theta_g + m_n^{s+1}\theta_g$

Let $\quad E = \dfrac{tg(m_n^{2k+1}\theta_n) + g^*(m_p^{k+1})\theta_g + m_n^s\theta_g}{tg(m_n^{2k+1}\theta_n) + g^*(m_p^{k+1})\theta_g + m_n^{2s+2}\theta_g}$.

Then E is a C_n-module and using *(b)* we obtain that $m_n E = E$. Hence,

$$\begin{aligned} m_n^s\theta_g \;\subset\; & tg(m_n^{2k+1}\theta_n) + g^*(m_p^{k+1})\theta_g + m_n^{2s+2}\theta_g = \\ & tg(m_n^{2k+1}\theta_n) + g^*(m_p^k).g^*(m_p)\theta_g + m_n^{2s+2}\theta_g \quad . \end{aligned}$$

Now, substituting for $g^*(m_p^k)$, we get:

$$m_n^s\theta_g \subset tg(m_n^{2k+1}\theta_n) + wg(m_p^{k+1}\theta_p) + g^*(m_p).m_n^s\theta_g + m_n^{2s+2}\theta_g \quad .$$

Let $\quad E' = \dfrac{tg(m_n^{2k+1}\theta_n) + wg(m_p^{k+1}\theta_p) + m_n^s\theta_g}{tg(m_n^{2k+1}\theta_n) + wg(m_p^{k+1}\theta_p) + m_n^{2s+2}\theta_g}$.

E' is a finitely generated $g^*(m_p)$-module and

$$g^*(m_p).E' = E' \quad \Longrightarrow$$

$$m_n^s\theta_g \subset tg(m_n^{2k+1}\theta_n) + wg(m_p^{k+1}\theta_p) + m_n^{2s+2}\theta_g \quad .$$

Since this holds for all g with the same $s-1$-jet as f , it follows that f is $s-1$-$\mathcal{A}$-determined. □

Proof of the Proposition 2.4.1. Let

$$f(x) = \bigoplus_{i=1}^{n} f_i(x_i) = \Big(\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} A_i x_i^{2\ell_i+1} \,,\, \sum_{i=1}^{n} a_i x_i^2 + \sum_{i=1}^{n} B_i x_i^{2m_i+1}\Big)$$

then, since $a_i \neq a_j$, Lemma 2.4.4 applies, f is 2-$\mathcal{K}$-determined and

$$J(f).m_n^{2k-2} + f^*(m_2^k)C_n + m_n^{2k+1} = m_n^{2k} \,, \quad \forall k \geq n-1 \qquad (*)$$

We can obtain the following elements in $T\mathcal{A}.f$:

$$(i) \quad tf(x_1,\ldots,x_n) - 2\,wf(X_1,0) - 2\,wf(0,X_2) = \begin{pmatrix} \sum_{i=1}^{n}(2\ell_i - 1)\ A_i\ x_i^{2\ell_i+1} \\ \sum_{i=1}^{n}(2m_i - 1)\ B_i\ x_i^{2m_i+1} \end{pmatrix}$$

$$(ii)_j \quad tf(0,\ldots,x_j^s,0\ldots,0) = \begin{pmatrix} 2x_j^{s+1} \\ 2a_j x_j^{s+1} \end{pmatrix} (\mathrm{Mod}\, m_n^{s+2}\theta_f) \quad , \quad \forall j = 1,\ldots,n$$

From $(*)$ and *(i)*, we get, for all $k \geq n-1$:

$$(iii)_j \quad \begin{pmatrix} \sum_{i=1}^n (2\ell_i - 1)\ A_i\ x_i^{2\ell_i+1} \\ \sum_{i=1}^n (2m_i - 1)\ B_i\ x_i^{2m_i+1} \end{pmatrix} \cdot x_j^{2k} = \begin{pmatrix} (2\ell_j - 1)\ A_j\ x_j^{2(\ell_j+k)+1} \\ (2m_j - 1)\ B_j\ x_j^{2(m_j+k)+1} \end{pmatrix}$$

belong to $T\mathcal{A}(f)$ (Mod $m_n^{2(r_j+k)+2}\theta_f$) , where $r_j = \mathrm{Max}\{\ell_j, m_j\}$.

Since, by hypothesis, $f_j(x_j) = (x_j^2 + A_j x_j^{2\ell_j+1}\ ,\ a_j x_j^2 + B_j x_j^{2m_j+1})$ is *adequate*, we can use $(ii)_j$ and $(iii)_j$ with convenient choices of s in $(ii)_j$, and of k in $(iii)_j$ to obtain:

$$\left\{ \begin{pmatrix} x_j^{2(r_j+k)+1} \\ 0 \end{pmatrix} , \begin{pmatrix} 0 \\ x_j^{2(r_j+k)+1} \end{pmatrix} \right\} \in T\mathcal{A}.f(\mathrm{Mod}\, m_n^{2(r_j+k)+2}\theta_f) ,$$

$\forall j = 1, \ldots, n$ and $k \geq n-1$.

If $r = \max_{1 \leq j \leq n}\{r_j\}$, then we also have:

$$\left\{ \begin{pmatrix} x_j^{2(r+n-1)+1} \\ 0 \end{pmatrix} , \begin{pmatrix} 0 \\ x_j^{2(r+n-1)+1} \end{pmatrix} \right\} \in T\mathcal{A}.f(\mathrm{Mod}\, m_n^{2(r+n-1)+2}\theta_f) .$$

Hence, the conditions *(a)* and *(b)* of Lemma 2.4.5 are verified for f , and taking

$$s = 2(r+n-1)+2 = 2r+2n = 2(r+n) \text{ , and } k = r+n-1 ,$$

it follows that f is $(2(r+n)-1)-\mathcal{A}$-determined.

To prove the last part of the proposition, we consider an $\mathcal{A}$-finitely determined germ g, where $j^2 g = (\sum_{i=1}^n x_i^2\ , \sum_{i=1}^n a_i x_i^2)$.

Suppose that $j^{\ell-1}g$ has no mixed terms.

Now, since $J(f).m_n^{\ell-2} \supseteq$ { all mixed terms of degree ℓ } (Mod $m_n^{\ell+1}$) , we see that $j^\ell g$ is $\mathcal{R}_{\ell-1}^\ell$-equivalent to $(\sum_{i=1}^n x_i^2 + p_i(x_i)\, , \sum_{i=1}^n a_i x_i^2 + q_i(x_i))$ where $3 \leq$ degree $p_i \leq \ell$, $3 \leq$ degree $q_i \leq \ell$, ($\mathcal{R}_{\ell-1}^\ell$ is the subgroup of $\mathcal{R}^\ell$ consisting of diffeomorphisms $h : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ such that $j^{\ell-1}h(0) = j^{\ell-1}I_d$).

To complete the proof, we repeat the argument for all $\ell \leq$ the degree of $\mathcal{A}$-determinacy of g . □

3 Adjacencies of $\mathcal{A}$-simple $\Sigma^{2,0}$-germs $f : k^2, 0 \to k^2, 0$

The adjacencies between $\mathcal{A}$-simple $\sum^1$-germs from $\mathbf{C}^2$ to $\mathbf{C}^2$ are shown in [Rie]. For corank 2 germs we have the following

Proposition 3.1 *Figures* **A** *and* **B** *show the adjacencies of $\mathcal{A}$-simple $\sum^{2,0}$-germs $f : k^2,0 \to k^2,0$ for $k = \mathbf{C}$ and $k = \mathbf{R}$, respectively. (To denote $\mathcal{A}$-classes we use the notation of Propositions 1.3 and 2.1).*

Proof. As in [Rie] we use three invariants $m(f)$, $c(f)$, and $d(f)$, which are upper-semicontinuous under deformation, to rule out certain adjacencies. Let $\sum$ and Δ denote the critical set and the discriminant of f, which are both germs of plane curves, and let $\delta(C)$ denote the well-known δ-invariant of a germ of a plane curve C (see [Mi]). The three invariants of f can then be calculated as follows: $m(f) = \dim_k C_n/f^*m_p$, $c(f) = \dim_k C_n/I$ (where I = ideal defined by the vanishing of 2×2 minors of $\begin{bmatrix} Df \\ \nabla|Df| \end{bmatrix}$), and $d(f) = \delta(\Delta) - \delta(\sum) - c(f)$. (For germs $f : \mathbf{C}^2,0 \to \mathbf{C}^2,0$ these invariants have the following geometrical meaning: $m(f)$ is the number of preimages of a target point off the discriminant Δ of f; and $c(f)$ and $d(f)$ are the numbers of cusps and transverse fold crossings of a generic deformation of f).

For $\sum^1$-germs these invariants have been calculated in [Rie] and for the $\mathcal{A}$-simple $\sum^{2,0}$-germs of the classification in §2 we have the following

Lemma 3.2 *The invariants m, c, and d associated with the members of the series of germs $I_{2,2}^{\ell,m}$ and $II_{2,2}^{\ell}$ have the values:*

$$m(I_{2,2}^{\ell,m}) = 4 \ , \ c(I_{2,2}^{\ell,m}) = 3 \ , \ d(I_{2,2}^{\ell,m}) = \ell + m \ ;$$

and

$$m(II_{2,2}^{\ell}) = 4 \ , \ c(II_{2,2}^{\ell}) = 3 \ , \ d(II_{2,2}^{\ell}) = 2\ell \ .$$

These expressions also make sense for the "stems" of these series $I_{2,2}^{\infty,\infty} = (x^2, y^2)$ and $II_{2,2}^{\infty} = (x^2 - y^2, xy)$.

Proof. These are just a trivial calculation. Note that the critical sets and the discriminants of the germs $I_{2,2}^{\ell,m}$ consist of two branches, so that $\delta(\sum)$ and $\delta(\Delta)$ are sums of δ-invariants of each branch and the (local) intersection numbers of the branch pairs. Also note that, as complex-analytic germs, $II_{2,2}^{\ell} \sim I_{2,2}^{\ell,\ell}$ (and that the dimensions of the relevant local algebras are not altered by complexifying).

Also notice that the Milnor number μ of the critical sets of the germs $I_{2,2}^{\ell,m}$ and $II_{2,2}^{\ell}$ is equal to one. The upper semicontinuity of these invariants and the adjacencies of $\mathcal{K}$-classes described in §2.3, together with the following lemma, conclude the proof of Proposition 3.1.

Lemma 3.3

(i) $k = \mathbf{C}$ *: the degenerate mono-germs in a versal deformation of* $I^{1,1}_{2,2}$ *are of type* 4_2 *; and*

(ii) $k = \mathbf{R}$ *: the degenerate mono-germs in a versal deformation of* $I^{1,1}_{2,2}$ *are of type* 4_2^- *, and there are no degenerate mono-germs in a deformation of* $II^1_{2,2}$ *.*

Proof. We consider $\mathcal{A}$-versal unfoldings $F : k^d \times k^2, 0 \to k^d \times k^2, 0$, given by $F(u, x, y) = (u, f(u, x, y))$, where $f(0, x, y) = I^{2,2}_{2,2}$ or $II^2_{2,2}$. The set $B := \{u \in (k^d, 0) : c(f(u, 0, 0)) \geq 2\}$ gives all degenerate mono-germs in a deformation of $I^{1,1}_{2,2}$ or $II^1_{2,2}$, because $c \geq 2$ for any degenerate mono-germ of the plane and the origins in source and target are preserved under $\mathcal{A}$. Let $m_1(x, y)$, $m_2(x, y)$, and $m_3(x, y)$ denote the determinants of the 2×2 minors of $\begin{bmatrix} Df(u,x,y) \\ \nabla \mid Df(u,x,y) \mid \end{bmatrix}$, where D is the differential of f with respect to x and y .

Now $c(f(u, 0, 0)) = \dim_k C_n/(m_1, m_2, m_3) \geq 2$ if and only if

$$m_1(0,0) = m_2(0,0) = m_3(0,0) = 0$$

and the 2×2 minors of

$$\begin{bmatrix} \partial m_1(0,0)/\partial x & \partial m_1(0,0)/\partial y \\ \partial m_2(0,0)/\partial x & \partial m_2(0,0)/\partial y \\ \partial m_3(0,0)/\partial x & \partial m_3(0,0)/\partial y \end{bmatrix}$$

vanish. The six equations define an ideal I in $k[u_1, \ldots, u_d]$.

First, consider the $\mathcal{A}$-versal deformation $f(u, x, y) = (u, x^2 + y^3 + u_1 x + u_2 y\ ,\ y^2 + x^3 + u_3 x + u_4 y)$ of $I^{1,1}_{2,2}$. One calculates that $I = (u_1 u_3 - u_4^2\ ,\ u_1 u_4 - u_2 u_3\ ,\ u_1^2 - u_2 u_4\ ,\ u_4(3u_1 u_3 + 2u_2) + u_1(4u_1 + 3u_2^2)\ ,\ u_4(3u_3^2 + 4u_4) + u_1(2u_3 + 3u_2 u_4)\ ,\ -(4u_1 + 3u_2^2)(3u_3^2 + 4u_4) + (3u_1 u_3 + 2u_2)(2u_3 + 3u_2 u_4))$, and, calculating a standard basis for I with respect to some lexicographical ordering of the variables in $k[u_1, \ldots, u_4]$, one finds the following set of degenerate mono-germs: $B = \{\, u \in (k^4, 0) : u_1 = u_4 = u_2 u_3 = 0 \,\}$. Now, by direct coordinate changes, $f(0, u_2, 0, 0, x, y) \sim 4_2^-$ for $u_2 \in \mathbf{R} - \{0\}$ and $f(0, 0, u_3, 0, x, y) \sim 4_2^-$ for $u_3 \in \mathbf{R} - \{0\}$ (in the case $k = \mathbf{R}$), and $f(0, u_2, 0, 0, x, y) \sim f(0, 0, u_3, 0, x, y) \sim 4_2$ for $u_2, u_3 \in \mathbf{C} - \{0\}$ (for $k = \mathbf{C}$).

Finally, we consider the $\mathcal{A}$-versal deformation $f(u, x, y) = (x^2 - y^2 + x^3 + u_1 x + u_2 y\ ,\ xy + u_3 x + u_4 y)$ of the real germ $II^1_{2,2}$. Repeating the calculations above, one finds that $B = \{ u \in (\mathbf{R}^4, 0) : u_1 = u_2 = u_3 = u_4 = 0 \}$. Hence there are no degenerate mono-germs in a versal deformation of $II^1_{2,2}$, and the lemma follows.

Proof of Proposition 3.1; conclusion. Lemma 3.3 says that $I^{1,1}_{2,2}$ and $II^1_{2,2}$ are not adjacent to the $\sum^1$-germ $(x, xy + y^4)$, which is the open $\mathcal{A}$-orbit in the $\mathcal{K}$-class A_3 . From the adjacencies in **[Rie]** of $\sum^1$-germs it follows that none of the germs $I^{\ell,m}_{2,2}$ and $II^{\ell}_{2,2}$ is adjacent to some $\mathcal{A}$-orbit in A_3 . Finally, one checks that $II^1_{2,2} \to 3$. □

(1) $\underline{k = \mathbf{C}}$

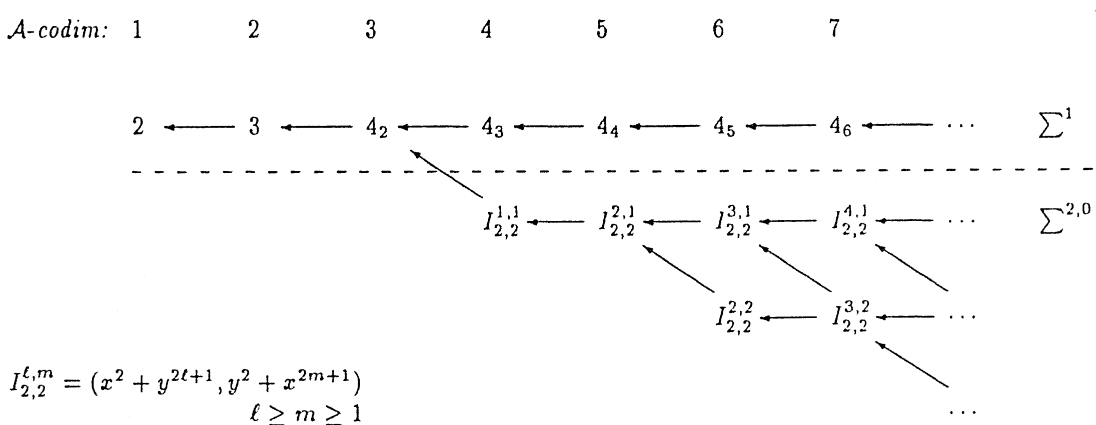
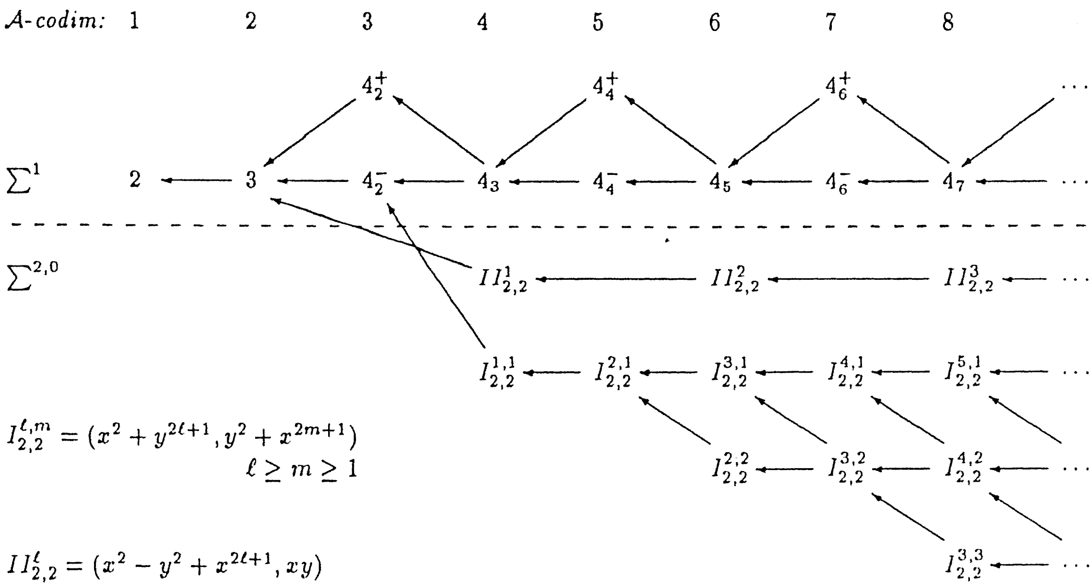


Fig. A

(2) $\underline{k = \mathbf{R}}$



$I_{2,2}^{\ell,m} = (x^2 + y^{2\ell+1}, y^2 + x^{2m+1})$
$\ell \geq m \geq 1$

$II_{2,2}^{\ell} = (x^2 - y^2 + x^{2\ell+1}, xy)$
$\ell \geq 1$

Fig. B

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