UNIVERSIDADE DE SÃO PAULO
CAMPUS DE SÃO CARLOS
INSTITUTO DE CIÊNCIAS MATEMÁTICAS DE SÃO CARLOS

I.C.M.S.C.

On harmonic and subharmonic solutions of nonlinear second order equations: symmetry and bifurcation.

FÜRKOTTER, M. & RODRIGUES, H.M.

nº 52

Notas do ICMSC - USP

ISSN 0103-2577

On harmonic and subharmonic solutions of nonlinear second order equations: symmetry and bifurcation.

FÜRKOTTER, M. & RODRIGUES, H.M.

nº 52

São Carlos (SP)
1989

ON HARMONIC AND SUBHARMONIC SOLUTIONS OF NONLINEAR SECOND ORDER EQUATIONS: SYMMETRY AND BIFURCATION

MONICA FÜRKOTTER
Faculdade de Ciências e Tecnologia, UNESP, Presidente Prudente, SP, Brazil.

HILDEBRANDO MUNHOZ RODRIGUES
Instituto de Ciências Matemáticas de São Carlos, USP, São Carlos, SP, Brazil.

AMS(MOS): 34A34, 34C15, 34C25

Abstract Consider the equation $\ddot{u} + u = g(u,p) + \mu f(t)$, where p, μ are small parameters, f is an even continuous π/m-odd-harmonic function (i.e., $f(t+\pi/m) = -f(t)$, for every t in $\mathbb{R}$), $m \geq 2$ and g is an odd function of u. Under certain conditions on f and g it is proved that the small 2π-periodic solutions of the above equation maintain some symmetry properties of the forcing $f(t)$, when $\mu \neq 0$. Other interesting results describe the changes of the number of such solutions, as p and μ vary in a small neighborhood of the origin. As another contribution of this paper, it was proved that a central assumption which was required in the main results, is generic. The main tool used in this work is the Liapunov-Schmidt Method.

Key Words: Periodic solutions, symmetry, bifurcation, nonlinear equations, small solutions, odd-harmonic.

1. INTRODUCTION

We consider the equation

$$\ddot{u} + u = g(u,p) + \mu f(t) \tag{1.1}$$

where p,μ are small parameters, f is an even π/m-odd-harmonic function (that is, $f(t+\pi/m) = -f(t)$, for every t in $\mathbb{R}$), g is an odd function of u, sufficiently smooth and $m \geq 2$ is an integer.

Our main results are that, under certain conditions on g anf f, the small 2π-periodic solutions of (1.1) maintain some symmetry properties of the forcing $f(t)$, when $\mu \neq 0$. We also find the bifurcation curves and describe the changes of the number of such solutions, as (p,μ) varies in a small neighhorhood of the origin. A conjecture which was stated in Fürkotter-Rodrigues [2] is proved. In order to guarantee that the 2π-periodic solutions of (1.1) maintain some symmetry of $f(t)$, we require here, and it was required in earlier works, that a certain coefficient, which depends on $g(\cdot,0)$ and f, is nonzero. As a contribution of this work we prove that the cited coefficient is generically nonzero.

Hale-Rodrigues [5] studying Duffing's Equation $\ddot{u}+u = pu - u^3 + \mu\cos t$ showed that the only small 2π-periodic solutions are even functions of t if $\mu \neq 0$.

Rodrigues-Vanderbauwhede [6] generalized this result for equations like (1.1), with the assumption that $\rho = \rho(f) = \int_0^{2\pi} f(s)\cos s\,ds \neq 0$. They also presented an abstract version for equations in Banach spaces. Vanderbauwhede [7] also considers problems related to the above ones in an abstract form.

Fürkotter-Rodrigues [2] considered the case where f is $\pi/2$-odd-harmonic, that is, the case $m=2$. They also require that a certain coefficient $\eta = \eta(f,g) \neq 0$.

Fürkotter-Rodrigues [3] consider the case where f is $2\pi/m$-periodic. They also require that a certain coefficient $\rho = \rho(f,g) \neq 0$. We point out that if f is π/m-odd-harmonic, m even, the conditions of the former work are not satisfied.

A conclusion of this work is that, when m is even, as a consequence of the fact that f has more symmetries, the problem becomes technically harder and more degenerate than the case m odd. We also need the assumption that a certain coefficient $\rho = \rho(f,g) \neq 0$.

In § 2, using Liapunov-Schmidt Method, we show that symmetries in (1.1) imply symmetries of the solutions of the auxiliary equation. The main result of this chapter is Theorem 2.1, but the most difficult part is handled in Lemma 2.5.

In § 3, we analyze the bifurcation equations. Among the main results of this paper we indicate Theorem 3.2 and Theorem 3.3.

In § 4 we prove that the condition $\rho = \rho(f,g) \neq 0$, which is a central assumption of this paper, is generic. Theorem 4.1 gives one of our main results, but Lemma 4.1 is also interesting and was very helpfull in the proof of the main theorem.

An application which can be reduced to equation (1.1) is the equation $\ddot{v} + \omega_o^2 v + G(v) = \sigma f(\omega t)$ where $G(v) = O(v^2)$, as v tend to 0, f is an even π/m-odd-harmonic function and we look for $2\pi/\omega$-periodic solutions for ω close to ω_o. This includes the pendulum equation and many other mechanical and electrical oscillators. If we let $u(t) \stackrel{def}{=} v(t/\omega)$, $\omega_o^2/\omega^2 = 1 - p$ and $\mu = \sigma/\omega^2$ we obtain one equation like (1.1).

The results of this paper indicates that if one truncates the Taylor series of the nonlinearity in the wrong place, the information given by our theorems may be lost. For example, if one wants to use our results in the pendulum equation $\ddot{v} + \frac{g}{L} \operatorname{sen} v = \mu \cos 50\omega t$, for ω close to $\omega_o = \sqrt{g/L}$, one should know all the terms of the Taylor expansion of $\operatorname{sen} v$, arond $v = 0$, up to the order v^{101}.

2. THE AUXILIARY EQUATION

Let p and μ be small parameters, that is (p,μ) varies in a small neighborhood of the origin. Let $m \geq 2$ be an integer.

Consider the equation

$$\ddot{u} + u = g(u,p) + \mu f(t) \tag{1.1}$$

and the hypotheses:

(H_1) g is defined in a neighborhood of $(u,p) = (0,0)$. It is an odd function of u, $g \in C^{m+2}$ if m is odd, $g \in C^{2m+3}$ if m is even, $g(u,p) = \sum_{k=1}^{\frac{m-1}{2}} \alpha_{2k+1} u^{2k+1} + pu + O(|u|^{m+2} + |p||u|^3)$ if m is odd, $g(u,p) = \sum_{k=1}^{m} \alpha_{2k+1} u^{2k+1} + pu + o(|u|^{2m+3} + |p||u|^3)$ if m is even.

(H_2) f is a real, even and continuous $2\pi/m$-periodic function.

(OH_2) f is a real, even and continuous π/m-odd-harmonic function, that is, $f(t+\pi/m) = -f(t)$, for every t in $\mathbb{R}$.

Let P be the space of all 2π-periodic continuous functions, with the norm $\|w\| = \max\{|w(t)|,\ 0 \le t \le 2\pi\}$.

On this spaces, we consider the projection

$$(Pw)(t) \overset{\text{def}}{=} \frac{\cos t}{\pi}\int_0^{2\pi} w(s)\cos s\,ds + \frac{\sin t}{\pi}\int_0^{2\pi} w(s)\sin s\,ds. \tag{2.1}$$

The Fredholm Alternative implies that the equation $\ddot{u} + u = h(t)$, with h in $\mathcal{P}$, has a solution in $\mathcal{P}$ if and only if $Ph = 0$. Moreover, if $Ph = 0$ then there exists a unique solution $u(t)$ in $\mathcal{P}$, such that $Pu = 0$. We indicate this solution by $\mathcal{K}h$. From the variation of constants formula, we obtain

$$(\mathcal{K}h)(t) = (I-P)[\cos(\cdot)\int_0^{(\cdot)} h(s)\sin s\,ds + \sin(\cdot)\int_0^{(\cdot)} h(s)\cos s\,ds]. \tag{2.2}$$

Following the usual procedure of Liapunov-Schmidt Method, see Hale [4], the problem of finding a 2π-periodic solution $u(t)$ of (1.1) is reduced to that of finding a solution w in $(I-P)\mathcal{P}$ of the following system of equations:

$$\begin{aligned}
&(a)\ \ w = \mathcal{K}(I-P)[g(r\cos(\cdot-\phi) + w, p) + \mu f(\cdot)] \\
&(b)\ \ P[g(r\cos(\cdot-\phi) + w, p) + \mu f(\cdot)] = 0
\end{aligned} \tag{2.3}$$

where $u(t) = r\cos(t-\phi) + w(t)$, $r \in \mathbb{R}$ and $\phi \in (-\pi/2, \pi/2]$.

The equations (a) and (b) are called the auxiliary and bifurcation equation, respectively.

It follows from the implicit function theorem that the equation (2.3.a) has a unique small solution in $(I-P)\mathcal{P}$, for (p,μ) in a sufficiently small neighborhood of the origin. We denote this solution by $w^*(r,\phi,p,\mu)(t)$. If we substitute it into (2.3.b) we obtain the following equivalent system of equations:

$$\begin{aligned}
&(a)\ \ F(r,\phi,p,\mu) \overset{\text{def}}{=} \frac{1}{\pi}\int_0^{2\pi} g(r\cos s + w^*(r,\phi,p,\mu)(s+\phi), p)\cos s\,ds = 0 \\
&(b)\ \ G(r,\phi,p,\mu) \overset{\text{def}}{=} \frac{1}{\pi}\int_0^{2\pi} g(r\cos s + w^*(r,\phi,p,\mu)(s+\phi), p)\sin s\,ds = 0\ .
\end{aligned} \tag{2.4}$$

The following lemmas give informations about some symmetries and estimates of $w^*(r,\phi,p,\mu)(t)$.

Lemma 2.1: If hypothesis (H_1) and (H_2) are satisfied, then the solution w^* of (2.3.a) has the following properties:

$w^*(0,\phi,p,\mu)(t)$ is an even $2\pi/m$-periodic function of t and is independent of ϕ (2.5)

$w^*(r,k\pi/m,p,\mu)(t+k\pi/m)$ is even in t for $-m/2 < k \le m/2$. (2.6)

$w^*(r,\phi,p,0)(t+\phi)$ is even in t. (2.7)

$$w^*(0,\phi,p,\mu) = \mu Kf + 0(|p\mu| + |\mu^3|), \quad \text{as} \quad (p,\mu) \to (0,0). \tag{2.8}$$

$$w^*(r,\phi,p,\mu) = w^*(0,\phi,p,\mu) + rS(r,\phi,p,\mu)$$

where $S(r,\phi,p,\mu) = 0(r^2 + |\mu|)$, as $(r,p,\mu) \to (0,0,0)$. (2.9)

If m is even then the following properties hold:

$$w^*(r,\phi,p,\mu)(t) = w^*(-r,\phi,p,\mu)(t-\pi). \tag{2.10}$$

$$w^*(r,\phi,p,\mu)(t) = -w^*(r,\phi,p,-\mu)(t-\pi). \tag{2.11}$$

If $m = 3$, then the following properties hold:

$$w^*(r,-\pi/3,p,\mu)(t-\pi/3) = w^*(r,\pi/3,p,\mu)(t+\pi/3). \tag{2.12}$$

$$w^*(-r,\pi/3,p,\mu)(t-2\pi/3) = w^*(r,0,p,\mu)(t). \tag{2.13}$$

If hypothesis (H_1) and (OH_2) are satisfied then $w^*(0,\phi,p,\mu)(t)$ is a π/m-odd-harmonic function of t. Moreover,

$w^*(r,\frac{k\pi}{2m},p,\mu)(t+\frac{k\pi}{2m})$ is even in t if k is even and

$w^*(r,\frac{k\pi}{2m},p,\mu)(t+\frac{k\pi}{2m}-\frac{\pi}{2})$ is odd in t if k is odd and m is even.

Proof: Properties (2.8) and (2.9) can be proved in a natural way. All the remaining properties follow essentially from the fact that the auxiliary equation is invariant under certain transformations. ❐

Let $P_{2\pi\times 2\pi} \stackrel{def}{=} \{f: R\times R \to R : f(t+2\pi,\phi) = f(t,\phi) = f(t,\phi+2\pi)$ for every $(t,\phi) \in R\times R$, f continuous$\}$

with the sup norm.

Let m and n be positive integers, with $m \ge 2$, such that $n \le m-1$ if m is odd and $n \le 2m-1$ if m is even.

If n is even we define $\mathcal{O}_n$ as the space of all functions

$y \in P_{2\pi\times 2\pi}$, such that $y(t,\phi)$ can be written into the form:

$$\sum_{j=0}^{n/2}[a_{n,2j}(t)\cos 2j(t-\phi) + b_{n,2j}(t)\sin 2j(t-\phi)], \tag{2.15}$$

where $a_{n,2j}$ is an even π/m-odd-harmonic function and $b_{n,2j}$ is an odd π/m-odd-harmonic function, for $j = 0,1,\ldots,n/2$.

If n is an odd integer, we define $\mathcal{O}_n$ as the space of all functions $y \in P_{2\pi\times 2\pi}$, such that $y(t,\phi)$ can be written into the form:

$$\sum_{j=0}^{(n-1)/2}[a_{n,2j+1}(t)\cos(2j+1)(t-\phi) + b_{n,2j+1}(t)\sin(2j+1)(t-\phi)] \tag{2.16}$$

where $a_{n,2j+1}$ is an even π/m-periodic function and $b_{n,2j+1}$ is an odd π/m-periodic function, for $j = 0,1,\ldots,(n-1)/2$.

__Remark 2.1__: In Lemma 2.2, 2.3, 2.5 and Theorem 2.1 we allow ϕ to vary in R. To avoid picking up twice the same solution, in § 3 we restrict ϕ to $(-\frac{\pi}{2},\frac{\pi}{2}]$.

__Lemma 2.2__: Let m and n positive integers such that $n \le m-1$ if m is odd and $n \le 2m-1$ if m is even, then:

a) If n is even and $n \ne m-1$ then $(I-P)\mathcal{O}_n = \mathcal{O}_n$.

b) If n is odd and $n < 2m-1$ then a function belongs to $(I-P)\mathcal{O}_n$ if and only if it has the form (2.16), it satisfies the conditions that define $\mathcal{O}_n$ and $Ma_{n,1} = 0$, where M indicates the mean value.

c) If $n = m-1$, n even then the elements of $(I-P)\mathcal{O}_n$ have the same properties that define $\mathcal{O}_n$ and the additional condition:

$$\int_0^{2\pi}[a_{n,m-1}(t)\cos mt + b_{n,m-1}(t)\sin nmt]dt = 0. \tag{2.17}$$

d) If $n = 2m-1$, m is even then the elements of $(I-P)\mathcal{O}_n$ satisfy the conditions that define $\mathcal{O}_n$, $Ma_{n,1} = 0$ and

$$\int_0^{2\pi}[a_{n,2m-1}(s)\cos 2ms + b_{n,2m-1}(s)\sin 2ms]ds = 0\,. \tag{2.18}$$

Proof: Let us assume first that n is even and that $f(t,\phi)$ is an element of $\mathcal{O}_n$ given by (2.15). Some calculations show that $Pf(\cdot,\phi)(t)$ is given by:

$$\frac{1}{2\pi}\sum_{j=0}^{n/2}\Big\{\int_0^{2\pi}[a_{n,2j}(s)\cos(2j+1)s+b_{n,2j}(s)\sin(2j+1)s]ds. \tag{2.19}$$

$$.[\cos(2j+1)t\cos 2j(t-\phi)+\sin(2j+1)t\sin 2j(t-\phi)]+$$

$$+\int_0^{2\pi}[a_{n,2j}(s)\cos(2j-1)s+b_{n,2j}(s)\sin(2j-1)s]ds.$$

$$.[\cos(2j-1)t\cos 2j(t-\phi)-\sin(2j-1)t\sin 2j(t-\phi)]\Big\}.$$

Consider first statement (a). Since $a_{n,2j}$, $b_{n,2j}$ are π/m-odd-harmonic, when m is even they are π-periodic and so all the integrals of (2.19) vanish.

If m is odd, since $j\le n/2<(m-1)/2$ we have that $2\pi/(2j+1)>2\pi/m$ and thus all the integrals of (2.19) vanish because $a_{n,2j}$, $b_{n,2j}$ are $2\pi/m$-periodic.

The above remarks imply that $Py(\cdot,\phi)=0$ if $n\ne m-1$.

In order to prove (b) we first point out that if $f(t,\phi)$ is an element of $\mathcal{O}_n$ given by (2.16), then $Pf(\cdot,\phi)(t)$ is given by:

$$\frac{1}{2\pi}\sum_{j=0}^{(n-1)/2}\Big\{\int_0^{2\pi}[a_{n,2j+1}(s)\cos 2(j+1)s+b_{n,2j+1}(s)\sin 2(j+1)s]ds. \tag{2.20}$$

$$.[\cos 2(j+1)t\cos(2j+1)(t-\phi)+\sin 2(j+1)t\sin(2j+1)(t-\phi)]+$$

$$+\int_0^{2\pi}[a_{n,2j+1}(s)\cos 2js+b_{n,2j+1}(s)\sin 2js]ds.$$

$$.[\cos 2jt\cos(2j+1)(t-\phi)+\sin 2jt\sin(2j+1)(t-\phi)]\Big\}.$$

If $0\le j\le(n-1)/2$ and $n<2m-1$ then $2\pi/2(j+1)>(2\pi)/m$. Therefore the first integral of (2.20) vanishes and the second integral of (2.20) vanishes for $0<j\le(n-1)/2$. Thus in this case we have

$$Pf(\cdot,\phi)(t)=\frac{1}{2\pi}\int_0^{2\pi}a_{n,1}(s)ds\cos(t-\phi)$$

and (b) follows easily.

To analyze (c) we use again (2.19) and show that in this case $Pf(\cdot,\phi)(t)$ is given by

$$\frac{1}{2\pi}\int_0^{2\pi}[a_{n,m-1}(s)\cos ms + b_{n,m-1}(s)\sin ms]ds \, .$$

$$.[\cos mt\cos(m-1)(t-\phi) + \sin mt\sin(m-1)(t-\phi)],$$

since all the other integrals of (2.19) vanish.

Therefore an element f of $\mathcal{O}_n$, given by (2.16) belongs to $(I-P)\mathcal{O}_n$ if and only if (2.17) is satisfied.

To prove (d) we use (2.20) for $n=2m-1$ to show that if $f\in\mathcal{O}_n$ is given by (2.16) then $Pf(\cdot,\phi)(t)$ is given by

$$\frac{1}{2\pi}\int_0^{2\pi}[a_{n,2m-1}(s)\cos 2ms + b_{n,2m-1}(s)\sin 2ms]ds \, .$$

$$.[\cos 2mt\cos(2m-1)(t-\phi) + \sin 2mt\sin(2m-1)(t-\phi)]+$$

$$+ \frac{1}{2\pi}\int_0^{2\pi}a_{n,1}(s)ds\cos(t-\phi).$$

The statement (d) follows easily. □

Lemma 2.3: If n_1, n_2, m and $n=n_1+n_2$ are positive integers, such that $n\le m-1$ if m is odd and $n\le 2m-1$ if m is even, then

a) $\mathcal{O}_n$ is closed in $P_{2\pi\times 2\pi}$.

b) If $y_i\in\mathcal{O}_{n_i}$ then $\prod_{i=1}^{2}y_i$ has the form (2.15) or (2.16) depending on whether n is even or odd, where $a_{n,k}$ is even, $b_{n,k}$ is odd, either π/m-odd-harmonic or π/m-periodic functions.

The proof of the above lemma follows the ideas of Lemma 2.2 [3] Fürkotter-Rodrigues.

Corollary 2.1: If n_i,β,m are positive integers, q_i is a nonnegative integer, $y_i\in\mathcal{O}_{n_i}$ for $i=1,\dots,\beta$, $n\overset{\text{def}}{=}\sum_{i=1}^{\beta}q_in_i$, $1\le n\le m-1$ if m is odd, $n\le 2m-1$ if m is even, then $\prod_{i=1}^{\beta}y_i^{q_i}$ has the form (2.15) or

(2.16) depending on whether n is even or odd, where $a_{n,k}$ is even, $b_{n,k}$ is odd, either π/m-odd-harmonic or π/m-periodic functions.

Let OH_T the space of the continuous $f : R \to R^n$, such that $f(t+T) = -f(t)$ (i.e., f is T-odd-harmonic), with the usual sup norm.

Using the ideas of Hale [4, p.276, Lemma 2.1] one can prove the following similar result.

Lemma 2.4: Let $B(t)$ be a T-periodic $n\times n$ continuous real matrix. Let $\bar{P}$, $\bar{Q}$ be the projections given below. Then the equation $\dot{x} = B(t)x + f(t)$, for f in OH_T, has a solution in OH_T if and only if $\bar{Q}f = 0$. If $\bar{Q}f = 0$ then exists a unique solution $\bar{K}f$ of that equation, in OH_T, such that $\bar{P}\bar{K}f = 0$. Moreover, the operator $\bar{K}(I-\bar{Q}) : OH_T \to OH_T$ is linear and continuous.

Now we will define $\bar{P}$ and $\bar{Q}$.

Let $\Phi(t)$ (resp. $\Psi(t)$) a matriz $n \times p$ (resp. $p \times n$) whose columns (resp. rows) form a basis of the space of the T-odd-harmonic solutions of $\dot{x} = B(t)x$ (resp. $\dot{y} = -yB(t)$).

If $f \in OH_T$ then we define

$$(\bar{P}f)(t) \overset{def}{=} \Phi(t)\left[\int_0^T \Phi'(s)\Phi(s)ds\right]^{-1}\int_0^T \Phi'(s)f(s)ds$$

and

$$(\bar{Q}f)(t) \overset{def}{=} \Psi'(t)\left[\int_0^T \Psi(s)\Psi'(s)ds\right]^{-1}\int_0^T \Psi(s)f(s)ds$$

where (') indicates the transpose.

The next lemma plays an important role in this work.

Lemma 2.5: Let m and n be positive integers, such that $n \leq m-1$ if m is odd and $n \leq 2m-1$ if m is even. If $f \in (I-P)O_n$ then Kf, that is the function $(t,\phi) \to Kf(\cdot,\phi)(t)$, belongs to $(I-P)O_n$.

Proof: We will use the ideas of the method which was introduced in [3] Fürkotter-Rodrigues and was modified conveniently to be used in the odd-harmonic case.

Let us consider first the case n odd and let $f \in (I-P)O_n$ be given by

$$\sum_{j=0}^{(n-1)/2} [A_{2j+1}(t)\cos(2j+1)(t-\phi) + B_{2j+1}(t)\sin(2j+1)(t-\phi)].$$

We will prove that there exists a solution $x(t)$ of $\ddot{x}+x = f(t,\phi)$ of the form

$$\sum_{j=0}^{(n-1)/2} [a_{2j+1}(t)\cos(2j+1)(t-\phi) + b_{2j+1}(t)\sin(2j+1)(t-\phi)],$$

such that $x \in (I-P)\mathcal{O}_n$. Then the result of our theorem will follow from the Fredholm Alternative.

If we substitute $x(t)$ into the equation $\ddot{x} + x = f(t,\phi)$, we obtain the system:

$$\ddot{a}_{2j+1} + [1-(2j+1)^2]a_{2j+1} + 2(2j+1)\dot{b}_{2j+1} = A_{2j+1}(t)$$

$$\ddot{b}_{2j+1} + [1-(2j+1)^2]b_{2j+1} - 2(2j+1)\dot{a}_{2j+1} = B_{2j+1}(t).$$

If we let $y_1 = a_{2j+1}$, $\dot{y}_j = y_2$, $y_3 = b_{2j+1}$, $\dot{y}_3 = y_4$, $y = \mathrm{col}(y_1,y_2,y_3,y_4)$ we obtain the system

$$\dot{y} = C_j y + F_j(t),$$

where $F = \mathrm{col}(0,A_{2j+1},0,B_{2j+1})$ and

$$C_j = \begin{bmatrix} 0 & 1 & 0 & 0 \\ (2j+1)^2-1 & 0 & 0 & -2(2j+1) \\ 0 & 0 & 0 & 1 \\ 0 & 2(2j+1) & (2j+1)^2-1 & 0 \end{bmatrix}$$

The eigenvalues of C_j are $\pm 2ji$ and $\pm 2(j+1)i$. Following Hale [4], we have that the above system is critical with respect to $P\pi/m$ if and only if there exists an integer k, such that either $j = mk$ or $j = mk-1$. Since $0 \le j \le (n-1)/2$ and $n \le 2m-1$, for m even we conclude that the system is critical, for $n < m-1$, only if $j = 0$. Also it is critical, for $m-1 \le n \le 2m-1$, only if $j = 0$ or $j = m-1$. When m is odd, since $0 \le j \le m-1$, the only critical case is $j = 0$.

For the noncritical cases, with respect to $P\pi/m$, the equation $\dot{y} = C_j y + F_j(t)$ has a unique π/m-periodic solution $y^j(t) \overset{\text{def}}{=} \text{col}(y_1^j(t), y_2^j(t), y_3^j(t), y_4^j(t))$. Since $z^j(t) \overset{\text{def}}{=} \text{col}(y_1^j(-t), -y_2^j(-t), -y_3^j(-t), y_4^j(-t))$ is also a π/m-periodic solution, it follows that $y_1^j(t)$ must be even and $y_3^j(t)$ must be odd.

Let us consider now the critical case $j = 0$. Following Hale [4, p.275] we have that

$$\Phi(t) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}$$

is a matrix whose columns form a basis of the space of π/m-periodic solutions of $\dot{y} = C_o y$ and

$$\Psi(t) = \begin{bmatrix} 1 & 0 & 0 & -1/2 \\ 0 & -1/2 & -1 & 0 \end{bmatrix}$$

is a matrix whose rows form a basis of the space of π/m-periodic solutions of the adjoint equation $\dot{z} = -zC_o$.

If we define projections $\overline{P}$, $\overline{Q}$ as in Hale [4, p.276, (2.5)], we obtain for the π/m-periodic function $G(t) = \text{col}(G_1(t), G_2(t), G_3(t), G_4(t))$:

$$\overline{Q}_o G = \frac{4m}{5\pi} \int_o^{\pi/m} \begin{bmatrix} G_1(s) - \dfrac{G_4(s)}{2} \\[2ex] \dfrac{G_2(s)}{4} + \dfrac{G_3(s)}{2} \\[2ex] \dfrac{G_2(s)}{2} + G_3(s) \\[2ex] -\dfrac{G_1(s)}{2} + \dfrac{G_4(s)}{4} \end{bmatrix} ds$$

$$\overline{P}_o G = \begin{bmatrix} MG_1 \\ 0 \\ MG_3 \\ 0 \end{bmatrix}$$

$Pf(\cdot,\phi) = 0$ implies $\overline{Q}_o F_o = 0$. Then the equation $\dot{y} = C_o y + F_o(t)$ has a unique π/m-periodic solution $y^o(t)$ such that $P_o y^o = 0$.

If $z^o(t) \overset{def}{=} col(y_1^o(-t), -y_2^o(-t), -y_3^o(-t), y_4^o(-t))$ then $z^o(t)$ is also a solution of the same equation and $\overline{P}_o z^o = 0$. This implies that $y_1^o(t)$ is even and $y_3^o(t)$ is odd.

The same procedure can be used for $j = m-1$, where m is even. In this case $\Phi(t)$ (resp. $\Psi(t)$) is a matrix whose columns (resp. rows) form a basis of the space of the π/m-periodic solutions of $\dot{y} = C_{m-1}y$ (resp. $\dot{z} = -zC_{m-1}$), where,

$$\Phi(t) = \begin{bmatrix} \cos 2mt & \sin 2mt \\ -2m\sin nmt & 2m\cos 2mt \\ \sin 2mt & -\cos 2mt \\ 2m\cos 2mt & 2m\sin 2mt \end{bmatrix}$$

$$\Psi(t) = \begin{bmatrix} -2(m-1)\sin 2mt & \cos 2mt & 2(m-1)\cos 2mt & \sin 2mt \\ 2(m-1)\cos 2mt & \sin 2mt & 2(m-1)\sin 2mt & -\cos 2mt \end{bmatrix}$$

Following Hale [4, p.276, (2.5)], we define the following projections:

$$(\overline{P}_{m-1}G)(t) \overset{def}{=} \begin{bmatrix} A\cos 2mt + B\sin 2mt \\ -2mA\sin 2mt + 2mB\cos 2mt \\ A\sin 2mt - B\cos 2mt \\ 2mA\cos 2mt + 2mB\sin 2mt \end{bmatrix}$$

where,

$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{m}{\pi(1+4m^2)} \int_0^{\pi/m} \begin{bmatrix} h_1(s) \\ h_2(s) \end{bmatrix} ds$$

and

$$h_1(s) \overset{\text{def}}{=} G_1(s)\cos 2ms - 2mG_2(s)\sin 2ms + G_3(s)\sin 2ms + 2mG_4(s)\cos 2ms$$

$$h_2(s) \overset{\text{def}}{=} G_1(s)\sin 2ms + 2mG_2(s)\cos 2ms - G_3(s)\cos 2ms + 2mG_4(s)\sin 2ms$$

and

$$(\bar{Q}_{m-1}G)(t) \overset{\text{def}}{=} \begin{bmatrix} -2(m-1)\bar{A}\sin 2mt + 2(m-1)\bar{B}\cos 2mt \\ \bar{A}\cos 2mt + \bar{B}\sin 2mt \\ 2(m-1)\bar{A}\cos 2mt + 2(m-1)\bar{B}\sin 2mt \\ \bar{A}\sin 2mt - \bar{B}\cos 2mt \end{bmatrix}$$

where

$$\begin{bmatrix} \bar{A} \\ \bar{B} \end{bmatrix} = \frac{m}{\pi[4(m-1)^2+1]} \int_0^{\pi/m} \begin{bmatrix} d_1(s) \\ d_2(s) \end{bmatrix} ds$$

and

$$d_1(s) \overset{\text{def}}{=} -2(m-1)G_1(s)\sin 2ms + G_2(s)\cos 2ms + 2(m-1)G_3(s)\cos 2ms + G_4(s)\sin 2ms$$

$$d_2(s) \overset{\text{def}}{=} 2(m-1)G_1(s)\cos 2ms + G_2(s)\sin 2ms + 2(m-1)G_3(s)\sin 2ms - G_4(s)\cos 2ms$$

and $\mathrm{col}(G_1(s), G_2(s), G_3(s), G_4(s))$ is a continuous π/m-periodic function.

Now we follow the same procedure used for $j = 0$.

Since $Pf(\cdot,\phi) = 0$, after some calculations using Lemma 2.2, we obtain that $\bar{Q}_{m-1}F_{m-1} = 0$, where $F_{m-1} = \mathrm{col}(0, A_{2m-1}, 0, B_{2m-1})$.

Thus there exists a unique π/m-periodic solution $y^{m-1}(t)$ of $\dot{y} = C_j y + F_j(t)$, such that $P_{m-1}y^{m-1} = 0$.

If we let $a_{2j+1} = y_1^j$, $b_{2j+1} = y_3^j$ and

$$x(t) = \sum_{j=0}^{(n-1)/2} [a_{2j+1}(t)\cos(2j+1)(t-\phi) + b_{2j+1}(t)\sin(2j+1)(t-\phi)],$$

after some calculations and using Lemma 2.2, we can prove that $Px = 0$.

Let us consider now the case n even.

Let $f \in (I-P)\mathcal{O}_n$ be given by

$$f(t,\phi) = \sum_{j=0}^{n/2} [A_{2j}(t)\cos 2j(t-\phi) + B_{2j}(t)\sin 2j(t-\phi)]. \tag{2.21}$$

We will prove that there exists a solution x in $(I-P)\mathcal{O}_n$, of the equation $\ddot{x} + x = f(t,\phi)$, given by

$$x(t) = \sum_{j=0}^{n/2} [a_{2j}(t)\cos 2j(t-\phi) + b_{2j}(t)\sin 2j(t-\phi)] \tag{2.22}$$

and the conclusion of our lemma follows easily.

If we substitute $x(t)$ given by (2.22) into $\ddot{x} + x = f(t,\phi)$, where $f(t,\phi)$ is given by (2.21), we obtain the system

$$\ddot{a}_{2j} - (4j^2-1)a_{2j} + 4j\dot{b}_{2j} = A_{2j}(t)$$

$$\ddot{b}_{2j} - (4j^2-1)b_{2j} - 4j\dot{a}_{2j} = B_{2j}(t)$$

for $j = 0, \dots, n/2$.

If we let $y_1 = a_{2j}$, $\dot{y}_1 = y_2$, $y_3 = b_{2j}$, $\dot{y}_3 = y_4$ and $y = \mathrm{col}(y_1, y_2, y_3, y_4)$, we obtain the equation $\dot{y} = C_j y + F_j(t)$, where $F_j = \mathrm{col}(0, A_{2j}, 0, B_{2j})$ and

$$C_j = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 4j^2-1 & 0 & 0 & -4j \\ 0 & 0 & 0 & 1 \\ 0 & 4j & 4j^2-1 & 0 \end{bmatrix}$$

for $j = 0,\dots,n/2$.

If m is even the system $\dot{y} = C_j y$ is noncritical with respect to space $OH_{\pi/m}$ that is, the only π/m-odd-harmonic solution of that system is the null function. If m is odd, since $n \le m-1$, the only critical case is obtained for $j = (m-1)/2$.

For the noncritical cases, we can use the same approach used in the case n odd.

Let us consider now the case $j = (m-1)/2$, where m is odd.

In order to use Lemma 2.4, we consider the matrix

$$\Phi(t) = \begin{bmatrix} \cos mt & \sin mt \\ -m\sin mt & m\cos mt \\ \sin mt & -\cos mt \\ m\cos mt & m\sin mt \end{bmatrix}$$

whose columns form a basis of the space of the π/m-odd-harmonic solutions of $\dot{y} = C_{\frac{m-1}{2}} y$, and the matrix

$$\Psi(t) = \begin{bmatrix} -(m-2)\sin mt & \cos mt & (m-2)\cos mt & \sin mt \\ (m-2)\cos mt & \sin mt & (m-2)\sin mt & -\cos mt \end{bmatrix}$$

whose rows form a basis of the sapece of the π/m-odd-harmonic solutions of $\dot{z} = -zC_{\frac{m-1}{2}}$.

In this case, the projections of Lemma 2.4 are given by:

$$(\bar{P}G)(t) \overset{\text{def}}{=} \begin{bmatrix} A\cos mt + B\sin mt \\ -mA\sin mt + mB\cos mt \\ A\sin mt - B\cos mt \\ mA\cos mt + mB\sin mt \end{bmatrix}$$

where

$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{m}{\pi(1+m^2)} \int_o^{\pi/m} \begin{bmatrix} h_1(s) \\ h_2(s) \end{bmatrix} ds$$

and

$$h_1(s) \overset{\text{def}}{=} G_1(s)\cos ms - mG_2(s)\sin ms + G_3(s)\sin ms + mG_4(s)\cos ms$$

$$h_2(s) \overset{\text{def}}{=} G_1(s)\sin ms + mG_2(s)\cos ms - G_3(s)\cos ms + mG_4(s)\sin ms \;.$$

$$(\bar{Q}G)(t) \overset{\text{def}}{=} \begin{bmatrix} (m-2)\bar{A}\sin mt - (m-2)\bar{B}\cos mt \\ \bar{A}\cos mt + \bar{B}\sin mt \\ (m-2)\bar{A}\cos mt + (m-2)\bar{B}\sin mt \\ -\bar{A}\sin mt + \bar{B}\cos mt \end{bmatrix} ,$$

where

$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{m}{\pi[(m-2)^2+1]} \int_o^{\pi/m} \begin{bmatrix} d_1(s) \\ d_2(s) \end{bmatrix}$$

and

$$d_1(s) \overset{\text{def}}{=} -(m-2)G_1(s)\sin ms + G_2(s)\cos ms + (m-2)G_3(s)\cos ms + G_4(s)\sin ms$$

$$d_2(s) \overset{\text{def}}{=} (m-2)G_1(s)\cos ms + G_2(s)\sin ms + (m-2)G_3(s)\sin ms - G_4(s)\cos ms.$$

Since $Pf(\cdot,\phi)$ implies $\overline{Q}F_{\frac{m-1}{2}} = 0$, then the equation $\dot{y} = C_{\frac{m-1}{2}}y + F_{\frac{m-1}{2}}(t)$ has a unique π/m-odd-harmonic solution $y^{\frac{m-1}{2}}(t)$ such that $\overline{P}y^{\frac{m-1}{2}} = 0$ and the remainder goes as in the case n odd.

The solution $x(t)$ given by (2.22) satisfies $Px = 0$ and this completes the proof. $\square$

Lemma 2.6: Let X be a Banach space and $I \subset R$ an interval. Let $\xi : I \to X$ and $g : X \to X$ be functions with continuous derivatives up to the order n. Let $H = g\circ\xi$. Then for $n \geq 1$, $\frac{\partial^n H}{\partial r^n}(r)$ can be written as a sum of terms of the form,

$$\gamma_i \frac{\partial^i g}{\partial u^i}(\xi(r))(d^{\alpha_1^i}\xi/dr^{\alpha_1^i})^{\beta_1^i} \dots (d^{\alpha_{k_i}^i}\xi/dr^{\alpha_{k_i}^i})^{\beta_{k_i}^i}$$

where $\alpha_1^i\beta_1^i + \dots + \alpha_{k_i}^i\beta_{k_i}^i = n$, $\beta_1^i + \dots + \beta_{k_i}^i = i$ and γ_i are constants, for $i = 1,\dots,n$. Moreover, if $i > 1$ then $\alpha_j^i < n$ and $\frac{\partial g}{\partial u}(\xi(r))\frac{d^n\xi}{dr^n}$ is the only term containing $\frac{d^n\xi}{dr^n}$.

Our main results depend heavely on the next theorem.

Theorem 2.1: Suppose hypothesis (H_1) and (OH_2) satisfied and that m and n are positive integers, such that $n \leq m-1$ if m is odd and $n \leq 2m-1$ if n is even. Then $\frac{\partial^n w^*}{\partial r^n}(0,\cdot,p,\mu)(\cdot)$ belongs to $(I-P)\mathcal{O}_n$, moreover it has the form (2.15) or (2.16) if n is even and odd, respectively, where $a_{ij}(t) = a_{ij}(p,\mu)(t)$ and $b_{ij}(t) = b_{ij}(p,\mu)(t)$.

Proof: We will do the proof by induction.

If $n = 1$ then $\frac{\partial w^*}{\partial r}(0,\cdot,p,\mu)(\cdot)$ is the unique solution of $y = \mathcal{H}y$, where

$$(\mathcal{H}y)(t,\phi) \overset{def}{=} K(I-P)[\frac{\partial g}{\partial u}(w^*(0,\phi,p,\mu)(\cdot),p)(y+\cos(\cdot.\phi))](t).$$

H is a uniform contraction with respect to p,μ for (p,μ) in a small neighhorhood of $(0,0)$.

From Lemma 2.1 and Lemma 2.4, after some calculations, we prove that $(I-P)\mathcal{O}_1$ is invariant under H. Since by Lemma 2.2, $(I-P)\mathcal{O}_1$ is a Banach space it follows that H has a fixed point and the fixed point is in $(I-P)\mathcal{O}_1$.

Now let us assume that the result is true up to the order $n-1$. We will prove that it is true for n.

$y = \frac{\partial^n w^*}{\partial r^n}(0,\cdot,p,\mu)(\cdot)$ is the unique solution of $y = Hy$ where

$$(Hy)(t,\phi) \overset{def}{=} K(I-P)[\frac{\partial g}{\partial u}(w^*(0,\phi,p,\mu),p)y + T(\phi,p,\mu)](t)$$

and $T(\phi,p,\mu)$, by Lemma 2,6, can be written as a sum of terms of the form:

$$\gamma_i \frac{\partial^i g}{\partial u^i}(w^*,p)(\cos(\cdot-\phi) + \frac{\partial w^*}{\partial r})^{\beta_1^i}(\partial^{\alpha_2^i} w^*/\partial r^{\alpha_2^i})^{\beta_2^i} \ldots (\partial^{\alpha_{k_i}^i} w^*/\partial r^{\alpha_{k_i}^i})^{\beta_{k_i}^i}$$

where $i > 1$ and $\frac{\partial^\ell w^*}{\partial r^\ell}$ means $\frac{\partial^\ell w^*}{\partial r^\ell}(0,\phi,p,\mu)$ for $\ell = 0,1,\ldots,\alpha_{k_i}^i$.

Moreover $\alpha_j^i < n$, $\alpha_1^i\beta_1^i + \ldots + \alpha_{k_i}^i\beta_{k_i}^i = n$ and $\beta_1^i + \ldots + \beta_{k_i}^i = i$, for $j = 1,\ldots,k_i$, $i = 1,\ldots,n$.

As before, H is a uniform contraction in $\mathcal{O}_n$ for (p,μ) in a small neighborhood of $(0,0)$.

From Lemma 2.1, 2.2, 2.3 and 2.4, after some claculations we prove that $(I-P)\mathcal{O}_n$ is invariant under H. It follows from Lemma 2.3 that $(I-P)\mathcal{O}_n$ is closed in $\mathcal{P}_{2\pi\times 2\pi}$. Then the fixed point of H belongs to $(I-P)\mathcal{O}_n$. □

3. THE BIFURCATION EQUATIONS

Now we return to equations (2.4).

Lemma 3.1: Under hypotheses (H_1) and (H_2) the following hold

(i) $G(r,\phi,p,0) \equiv 0$

(ii) If m is even then F and G are odd functions of r and even functions of μ.

Proof: The first part follows from Lemma 2.1, (2.7) and the second part follows from Lemma 2.1 (2.10) and (2.11). □

Theorem 3.1: Suppose (H_1) and (OH_2) are satisfied. Then for (r,p,μ) in a small neighborhood of the origin, $G(r,\phi,p,\mu) = r^{m-1}\mu\sin m\phi(\rho + \ldots)$, if m is odd and $G(r,\phi,p,\mu) = r^{2m-1}\mu^2\sin 2m\phi(\rho + \ldots)$, where ρ is a constant, that depends on $g(\cdot,0)$ and f and $(\ldots)$ indicates a function, which is zero for $(r,p,\mu) = (0,0,0)$.

Proof: Let us consider first the case m even. We will prove that

$$\frac{\partial^\ell G}{\partial r^\ell}(0,\phi,p,\mu) = 0, \quad \text{for } \ell = 1,2,\ldots,2m-2.$$

If we let $H(r,s) \overset{\text{def}}{=} g(r\cos(s-\phi) + w^*(r,\phi,p,\mu)(s),p)$, then:

$$\frac{\partial^\ell G}{\partial r^\ell}(0,\phi,p,\mu) = \frac{1}{\pi}\int_0^{2\pi}\frac{\partial^\ell H}{\partial r^\ell}(0,s)\sin(s-\phi)ds.$$

From Lemma 2.6 it follows that $\frac{\partial^\ell H}{\partial r^\ell}(0,s)$ is a sum of terms of the form:

$$\gamma_i\frac{\partial^i g}{\partial u^i}(w^*,p)(\cos(\cdot-\phi) + \frac{\partial w^*}{\partial r})^{\beta_1^i}(\partial^{\alpha_2^i}w^*/\partial r^{\alpha_2^i})^{\beta_2^i} \ldots (\partial^{\alpha_{k_i}^i}w^*/\partial r^{\alpha_{k_i}^i})^{\beta_{k_i}^i}$$

where $\frac{\partial^q w^*}{\partial r^q} = \frac{\partial^q w^*}{\partial r^q}(0,\phi,p,\mu)(s)$, $q = 0,\alpha_1^i,\ldots,\alpha_{k_i}^i$,

$\alpha_1^i\beta_1^i + \ldots + \alpha_{k_i}^i\beta_{k_i}^i = \ell$ and $\beta_1^i + \ldots \beta_{k_i}^i = i$.

If ℓ is odd, from Theorem 2.1 and Corollary 2.1, it follows that $\frac{\partial^\ell G}{\partial r^\ell}(0,\phi,p,\mu)$ is a sum of integrals of the form:

$$\int_0^{2\pi}[a(s)\cos(2j+1)(s-\phi) + b(s)\sin(2j+1)(s-\phi)]\sin(s-\phi)ds$$

where $0 \le j \le (\ell-1)/2 < m-1$ and $a(s)$ and $b(s)$ are even and odd, respectively, π/m-periodic functions.

That integral can be written as

$$-\frac{1}{2}\{\sin(2j+2)\phi\int_0^{2\pi}[a(s)\cos(2j+2)s + b(s)\sin(2j+2)s]ds$$

$$-\sin 2j\phi\int_0^{2\pi}[a(s)\cos 2js + b(s)\sin 2js]ds. \tag{3.1}$$

Since $j \le (\ell-1)/2 < m-1$ implies that $\frac{2\pi}{2j+2} > \frac{\pi}{m}$ and since $a(s)$ and $b(s)$ are π/m-periodic it follows that the above integrals vanish.

Let us consider now the case ℓ even. From Lemma 2.6, Theorem 2.1 and Corollary 2.1, it follows that $\frac{\partial^\ell G}{\partial r^\ell}(0,\phi,p,\mu)$ can be written as a sum of integrals of the form:

$$\int_0^{2\pi}[a(s)\cos 2j(s-\phi) + b(s)\sin 2j(s-\phi)]\sin(s-\phi)ds,$$

where $a(s)$ and $b(s)$ are, respectively, even and odd π/m-odd-harmonic functions.

A simple calculation shows that the above integral can be written as,

$$\frac{1}{2}[-\sin(2j+1)\phi\int_0^{2\pi}[a(s)\cos(2j+1)s + b(s)\sin(2j+1)s]ds +$$

$$+ \sin(2j-1)\phi\int_0^{2\pi}[a(s)\cos(2j-1)s + b(s)\sin(2j-1)s]ds\}. \tag{3.2}$$

Since the fact that $a(s)$, $b(s)$ are π/m-odd-harmonic implies that they are π-periodic functions and since $\cos(2j\pm1)s$ and

$\sin(2j\pm1)s$ are π-odd-harmonic we have that the integrals of (3.2) vanish for $0 \le j \le \ell/2$ and $1 \le \ell \le 2m-2$.

A similar procedure shows that $\frac{\partial^{2m-1}G}{\partial r^{2m-1}}(0,\phi,p,\mu)$ is a sum of integrals of the form

$$\sin 2m\phi \int_0^{2\pi} [a(s)\cos 2ms + b(s)\sin 2ms]ds,$$

where $a(s)$, $b(s)$ are, respectively, even and odd π/m-periodic functions of s. They are of course functions of p,r and μ, but they do not depend on ϕ.

From the above remarks and from Lemma 3.1 it follows that there exists a constant ρ, independent of ϕ,r,p,μ, such that $G(r,\phi,p,\mu) = r^{2m-1}\mu^2\sin 2m\phi(\rho + \dots)$, where ... indicates terms that vanish for $(r,p,\mu) = (0,0,0)$.

Let us consider now the case m odd. In this case the problem of proving that $\frac{\partial^{\ell}G}{\partial r^{\ell}}(0,\phi,p,\mu) \equiv 0$ if $1 \le \ell < m-1$ is also reduced to expressions like (3.1) or (3.2). Since $0 \le j < (m-1)/2$ and as $a(s)$ and $b(s)$ are $2\pi/m$-periodic we have that the integrals of (3.1) and (3.2) vanish. Therefore, $\frac{\partial^{\ell}G}{\partial r^{\ell}}(0,\phi,p,\mu) \equiv 0$ if $1 \le \ell < m-1$.

A similar procedure shows that $\frac{\partial^{m-1}G}{\partial r^{m-1}}(0,\phi,p,\mu)$ is a sum of integrals of the form

$$\sin m\phi \int_0^{2\pi} [a(s)\cos ms + b(s)\sin ms]ds$$

where $a(s)$, $b(s)$ are $2\pi/m$-periodic functions of s, depending also on r,p,μ but do not depend on ϕ.

The conclusion now follows as in the former case. □

Remark 3.1: To evaluate ρ, it suffices to compute

$[\frac{\partial^{m}G}{\partial r^{m-1}\partial\mu}(0,\phi,0,0)]/\sin m\phi$, if m is odd and $[\frac{\partial^{2m+1}G}{\partial r^{2m-1}\partial\mu^2}(0,\phi,0,0)]/\sin 2m\phi$,

if m is even.

The next theorem is one of our main results.

Theorem 3.2: Suppose (H_1), (OH_2) are satisfied and $\rho \neq 0$. If m is odd, $\mu \neq 0$ and $u(t)$ is a 2π-periodic solution of (1.1), sufficiently small, then there exists an integer k, $-m/2 < k \leq m/2$ such that $u(t+k\pi/m)$ is even in t. If m is even, $\mu \neq 0$ and $u(t)$ is a 2π-periodic solution of (1.1), sufficiently small, then there exists an integer k, $-m < k \leq m$ such that $u(t+k\pi/2m)$ is an even function of t if k is even and $u(t+k\pi/2m - \pi/2)$ is an odd function of t if k is odd.

Proof: Let us consider first the case m odd.

From Theorem 3.1 it follows that if $G = 0$ then $r^{m-1}\mu \sin m\phi = 0$. Since $u(t) = r\cos(t-\phi) + w^*(r,\phi,p,\mu)(t)$, from Lemma 2.1 it follows that $u(t)$ is even in t, if $r = 0$.

If $\sin m\phi = 0$ then there exists k, $-m/2 < k \leq m/2$ such that $\phi = k\pi/m$. Also from Lemma 2.1 it follows that $u(t+k\pi/m) = r\cos t + w^*(r,k\pi/m,p,\mu)(t+k\pi/m)$ is even in t.

Now we consider the case m even. Like in the former case, using Theorem 3.1, it remains to consider $\sin 2m\phi = 0$. In this case there exists an integer k, $-m < k \leq m$, such that $\phi = k\pi/2m$. If k is even from Lemma 2.1 it follows that $u(t+k\pi/2m) =$
$= r\cos t + w^*(r,k\pi/2m,p,\mu)(t+k\pi/2m)$ is even in t.

If k is odd then $u(t+\frac{k\pi}{2m}-\frac{\pi}{2}) =$
$= r\sin t + w^*(r,k\pi/2m,p,\mu)(t+\frac{k\pi}{2m}-\frac{\pi}{2})$ is an odd function of t.

In what follows we assume $\rho \neq 0$.

Now let us analyze the first bifurcation equation (2.4a).

If we suppose (H_1) and (OH_2) satisfied, after some calculations, we obtain for $m \geq 3$,

$$F(r,\phi,p,\mu) = \frac{1}{\pi}\int_0^{2\pi} g(r\cos s + w^*(r,\phi,p,\mu)(s+\phi),p)\cos s\,ds =$$

$$= r(p + \frac{3}{4}\alpha_3 r^2 + 3\alpha_3 \eta r\mu + 3\alpha_3\lambda\mu^2 + \ldots) = 0$$

where ... indicates higher order terms and

$$\lambda \overset{\text{def}}{=} \frac{1}{2\pi}\int_0^{2\pi}[(Kf)(t)]^2 dt, \qquad \eta \overset{\text{def}}{=} \frac{\cos 3\phi}{4\pi}\int_0^{2\pi}(Kf)(t)\cos 3t dt.$$

It can be proved that $\eta=0$, if $m>3$.
If $J \overset{\text{def}}{=} r^{-1}F$ then,

$$J(r,\phi,p,\mu) = p + \frac{3}{4}\alpha_3 r^2 + 3\alpha_3\eta r\mu + 3\alpha_3\lambda\mu^2 + \dots = 0.$$

$$J_r(r,\phi,p,\mu) = \frac{3}{2}\alpha_3 r + 3\alpha_3\eta\mu + \dots = 0$$

and

$$\det\frac{\partial(J,J_r)}{\partial(p,r)} = \frac{3}{2}\alpha_3 \quad \text{for} \quad r = p = \mu = 0.$$

If we suppose $\alpha_3 \neq 0$, from the implicit functions theorem, it follows that p and r can be found as functions of μ in a small neighborhood of the origin, for each fixed ϕ.

From Theorem 2.1 it follows that there exist π/m-periodic functions $a(s)$, $b(s)$ and π/m-odd-harmonic functions $A_o(s)$, $A(s)$ and $B(s)$ such that:

$$\frac{\partial w^*}{\partial r}(0,\phi,p,\mu)(s) = a(s)\cos(s-\phi) + b(s)\sin(s-\phi)$$

$$\frac{\partial^2 w^*}{\partial r^2}(0,\phi,p,\mu)(s) = A_o(s) + A(s)\cos 2(s-\phi) + B(s)\sin(s-\phi)$$

where a, A_o, A are even and b, B are odd.

After some calculations one shows that for $m > 3$,

$$J_r(0,\phi,p,\mu) = \frac{1}{2}\frac{\partial^2 F}{\partial r^2}(0,\phi,p,\mu) =$$

$$= \frac{1}{2}\{\int_0^{2\pi}\frac{\partial^2 g}{\partial u^2}(w^*(0,\phi,p,\mu)(s),p)[1+a(s)]^2\frac{1}{4}[\cos 3(s-\phi) + 3\cos(s-\phi)]ds +$$

$$+ 2\int_0^{2\pi}\frac{\partial^2 g}{\partial u^2}(w^*(0,\phi,p,\mu)(s),p)[1+a(s)]\frac{b(s)}{4}[\sin 3(s-\phi) + \sin(s-\phi)]ds +$$

$$+ \int_0^{2\pi}\frac{\partial^2 g}{\partial u^2}(w^*(0,\phi,p,\mu)(s),p)\frac{(b(s))^2}{4}[\cos(s-\phi) - \cos 3(s-\phi)]ds +$$

$$+ \int_0^{2\pi}\frac{\partial g}{\partial u}(w^*(0,\phi,p,\mu)(s),p)\frac{A(s)}{2}[\cos 3(s-\phi) + \cos(s-\phi)]ds +$$

$$+ \int_0^{2\pi}\frac{\partial g}{\partial u}(w^*(0,\phi,p,\mu)(s),p)\frac{B(s)}{2}[\sin 3(s-\phi) + \sin(s-\phi)]ds +$$

$$+ \int_0^{2\pi}\frac{\partial g}{\partial u}(w^*(0,\phi,p,\mu)(s),p)A_o(s)\cos(s-\phi)ds\} = 0.$$

The above integrals vanish because $w^*(0,\phi,p,\mu)$, a,b,A_o,A,B are $2\pi/m$-periodic functions, with $m>3$.

A similar calculation shows that

$$J(0,\phi,p,\mu) = F_r(0,\phi,p,\mu) = \frac{1}{2\pi}\int_0^{2\pi}\frac{\partial g}{\partial u}(w^*(0,\phi,p,\mu)(t),p)[1+a(t)]dt.$$

Since $J(0,\phi,p,\mu)$ does not depend on ϕ and $J_r(0,\phi,p,\mu) \equiv 0$, it follows that the unique solution $p = p(\mu)$ which we obtain solving $J(0,\phi,p,\mu) = 0$ and $r=0$, provides the unique solution $(p(\mu),0)$ of the system $J=0$, $J_r=0$, which was obtained earlier using the implicit function theorem.

Therefore for $m>3$ we have a unique bifurcation curve, given by $p = -3\alpha_3\lambda\mu^2 + O(\mu^3)$.

The case $m=2$ is analyzed in Fürkotter-Rodrigues [2].

If $m=3$ the admissible values of ϕ are $\phi=0$, $\pm\pi/3$. From Lemma 2.1 (2.12) and (2.13) it is possible to prove that there also exists a unique bifurcation curve which is given by $p = 3\alpha_3(\eta^2-\lambda)\mu^2 + O(\mu^3)$.

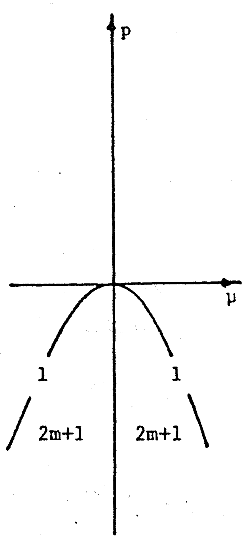
Using the fact that $J(r,\phi,p,\mu)$ starts with quadratic terms in r, using the admissible values of ϕ and the above remarks, one can prove the following theorem.

<u>**Theorem 3.3**</u>: Suppose (H_1), (OH_2) are satisfied, $m \geq 3$, $\rho \neq 0$ (given in Remark 3.1) and $\alpha_3 \neq 0$. Then there exists a bifurcation curve Γm, which is given by $p = 3\alpha_3(\eta^2 - \lambda)\mu^2 + O(|\mu|^3)$, where

$$\lambda = \frac{1}{2\pi}\int_0^{2\pi} [(Kf)(s)]^2 ds, \qquad \eta = \frac{\cos 3\phi}{4\pi}\int_0^{2\pi} (Kf)(s)\cos 3s\,ds \quad \text{and} \quad Kf \text{ is}$$

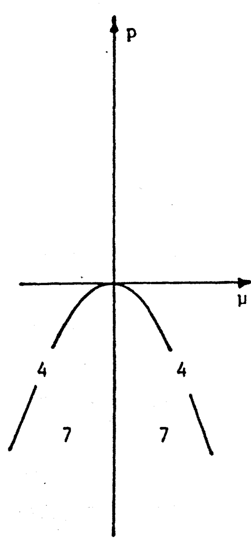
the unique π/m-odd-harmonic solution of $\ddot{u} + u = f(t)$.

The curve Γ_m divides a neighborhood of the origin into regions as it is shown in Fig. 3.1, Fig. 3.2 and Fig. 3.3. The number of 2π-periodic solutions of (1.1) is indicated in the pictures.



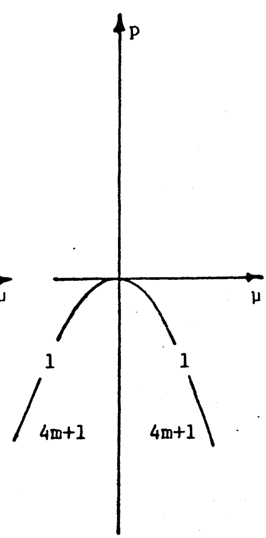
$m > 3$, odd

Fig. 3.1



$m = 3$

Fig. 3.2



$m > 3$, even

Fig. 3.3

Proof: The proof was essentially done before the statement of this theorem. We just point out how to find the number of 2π-periodic solutions. For m even, $m>3$ the admissible phases are $\phi = k\pi/2m$, $-m < k \leq m$. We know that $r=0$ solves $F=0$ and gives rise to the solution $u(t) = w^*(0,\phi,p,\mu)(t)$ which is π/m-odd-harmonic and does not depend on ϕ.

The other solutions are obtained by solving $J=0$, for each fixed admissible phases ϕ. If we find the number of r's that solve $J=0$ and combine with the admissible phases, we obtain the number of solution given by $u(t) = r\cos(t-\phi) + w^*(r,\phi,p,\mu)(t)$.

The same procedure can be used to analyze the case $m=3$ and $m>3$, when m is odd. $\square$

Remark 3.2: The bifurcation is not so degenerate as it appears. In fact the above proof shows that, besides the $2\pi/m$-periodic solution, for each admissible phase we have a quadratic bifurcation in r.

4. THE GENERICITY OF THE CONDITION $\rho \neq 0$

Since to find ρ we must consider $p=0$, throught this chapter we will use $g(u)$ in place of $g(u,0)$.

In this chapter we will prove that $\rho = \rho(f,g) \neq 0$, generically.

In order to prove that we show first that the solution of the auxiliary equation has a very interesting representation.

Lemma 4.1: Suppose (H_1), (OH_2) are satisfied, $m \geq 2$ is even (resp. $m>3$ is odd) and k is an integer, $1 \leq k \leq m$ (resp. $2 \leq k \leq (m-1)/2$). Then there exist continuous functions $W_1 = W_1(\alpha_3, \ldots, \alpha_{2k-1}, r, \phi, \mu)(t)$ and $W_2 = W_2(r,\phi,\mu)(t)$, such that W_1 does not depend on the coefficients α_i for $i > 2k-1$, $W_1 = O((|r|+|\mu|)^3)$, $W_2 = O((|r|+|\mu|)^{2k+1})$, such that:

$$w^*(r,\phi,0,\mu)(t) = \mu(Kf)(t) + W_1 + \alpha_{2k+1}W_2 + O((|r|+|\mu|)^{2k+3}).$$

Proof: The proof can be done by induction on k. Let us consider first the case $k=1$.

Since $w^*(r,\phi,p,\mu) = O(|r|+|\mu|)$, if we let $w^* \overset{\text{def}}{=} w^*(r,\phi,0,\mu)$ we have that, for $p=0$,

$$w^* = \mu Kf + \alpha_3 K(I-P)[(r\cos(\cdot-\phi) + w^*)^3] + O(|r|+|\mu|)^5) =$$

$$= \mu Kf + \alpha_3 K(I-P)[(r\cos(\cdot-\phi) + \mu Kf)^3] + O((|r|+|\mu|)^5).$$

If we let $W_2(r,\phi,\mu) \overset{def}{=} K(I-P)[(r\cos(\cdot-\phi) + \mu Kf)^3]$, we have that $w^*(r,\phi,0,\mu)(t) = \mu Kf(t) + \alpha_3 W_2 + O((|r|+|\mu|)^5)$. In this case $W_1 = 0$.

Let us assume now that the result is true for $k-1$. Then

$$g(u) = \sum_{i=1}^{k-1}\alpha_{2i+1}u^{2i+1} + O(|u|^{2k+1})$$

and $w^* = \mu Kf + W_1 + \alpha_{2k-1}W_2 + O((|r|+|\mu|)^{2k+1})$, where W_1 depends only on $\alpha_3,\ldots,\alpha_{2k-3}$, but not on α_i, for $i \geq 2k-1$, $W_1 = O((|r|+|\mu|)^3)$, and $W_2 = O((|r|+|\mu|)^{2k-1})$.

Then we have

$$w^* = K(I-P)[\sum_{i=1}^{k}\alpha_{2i+1}(r\cos(\cdot-\phi) + w^*)^{2i+1} + O((|r|+|\mu|)^{2k+3}) + \mu f] =$$

$$= \mu Kf + K(I-P)[\sum_{i=1}^{k}\alpha_{2i+1}(r\cos(\cdot-\phi) + \mu Kf + W_1 + \alpha_{2k-1}W_2)^{2i+1}] + \ldots =$$

$$= \mu Kf + \alpha_{2k+1}K(I-P)(r\cos(\cdot-\phi) + \mu Kf + W_1 + \alpha_{2k-1}W_2)^{2k+1} +$$

$$+ K(I-P)\sum_{i=1}^{k-1}\alpha_{2i+1}(r\cos(\cdot-\phi) + \mu Kf + W_1 + \alpha_{2k-1}W_2)^{2i+1} + \ldots =$$

$$= \mu Kf + \alpha_{2k+1}K(I-P)(r\cos(\cdot-\phi)+\mu Kf)^{2k+1} + K(I-P)\sum_{i=1}^{k-1}\alpha_{2i+1}(r\cos(\cdot-\phi) +$$

$$+ \mu Kf + W_1 + \alpha_{2k-1}W_2)^{2i+1} + \ldots .$$

If we let $\overline{W}_2 \overset{def}{=} K(I-P)(r\cos(\cdot-\phi) + \mu Kf)^{2k+1}$,

$$\overline{W}_1 \overset{def}{=} K(I-P)\sum_{i=1}^{k-1}\alpha_{2i+1}(r\cos(\cdot-\phi) + \mu Kf + W_1 + \alpha_{2k-1}W_2)^{2i+1},$$

we have

$$w^* = \mu Kf + \alpha_{2k+1}\overline{W}_2 + \overline{W}_1 + \ldots$$

where $\ldots$ indicates $O((|r|+|\mu|)^{2k+3})$. □

__Lemma 4.2__: Suppose (H_1), (OH_2) are satisfied and $m \geq 2$ is even. Then $\rho = \rho(f,g)$ can be written in the form

$$\rho = -\frac{m(2m+1)}{2^{2m-1}\pi}\alpha_{2m+1}\int_0^{2\pi}[(Kf)(s)]^2\cos 2ms\,ds + K(\alpha_3,\dots,\alpha_{2m-1},f),$$

where K is continuous.

<u>**Proof**</u>: From Lemma 4.1, it follows that,

$w^* = \mu Kf + W_1 + \alpha_{2m+1}W_2 + \dots$, where $\dots \overset{def}{=} O((|r|+|\mu|)^{2m+3})$.

Then we have,

$$G(r,\phi,0,\mu) = \frac{1}{\pi}\int_0^{2\pi}\sum_{\ell=1}^{m}\alpha_{2\ell+1}[r\cos s + w^*(s+\phi)]^{2\ell+1}\sin s\,ds + \dots =$$

$$= \frac{1}{\pi}\int_0^{2\pi}\alpha_{2m+1}(r\cos + w^*(s+\phi))^{2m+1}\sin s\,ds +$$

$$+ \frac{1}{\pi}\int_0^{2\pi}\sum_{\ell=1}^{m-1}\alpha_{2\ell+1}[r\cos s + w^*(s+\phi)]^{2\ell+1}\sin s\,ds + \dots =$$

$$= \frac{1}{\pi}\int_0^{2\pi}\alpha_{2m+1}(r\cos s + \mu(Kf)(s+\phi) + W_1 + \alpha_{2m+1}W_2)^{2m+1}\sin s\,ds +$$

$$+ \frac{1}{\pi}\int_0^{2\pi}\sum_{\ell=1}^{m-1}\alpha_{2\ell+1}[r\cos s + \mu(Kf)(s+\phi) + W_1 + \alpha_{2m+1}W_2]^{2\ell+1}\sin s\,ds + \dots =$$

$$= \frac{1}{\pi}\int_0^{2\pi}\alpha_{2m+1}[r\cos s + \mu(Kf)(s+\phi)]^{2m+1}\sin s\,ds +$$

$$+ \frac{1}{\pi}\int_0^{2\pi}\sum_{\ell=1}^{m-1}\alpha_{2\ell+1}[r\cos s + \mu(Kf)(s+\phi) + W_1]^{2\ell+1} + \dots .$$

Since from Theorem 3.1, we have that $G = r^{2m-1}\mu^2\sin 2m\phi[\rho + O(|r|+|u|)]$, then to find ρ we should only consider the part of G that contains terms envolving $r^{2m-1}\mu^2$, for $p = 0$.

Let us consider first the term

$$\frac{\alpha_{2m+1}}{\pi}\int_0^{2\pi}[r\cos s + \mu(Kf)(s+\phi)]^{2m+1}\sin s\,ds.$$

The part of this term which envolves $r^{2m-1}\mu^2$ is given by:

$$\frac{\alpha_{2m+1}}{\pi} m(2m+1)\int_0^{2\pi} r^{2m-1}\mu^2\cos^{2m-1}s[(Kf)(s+\phi)]^2 \sin s ds =$$

$$= \frac{\alpha_{2m+1}}{\pi} m(2m+1) r^{2m-1}\mu^2\int_0^{2\pi} \cos^{2m-1}s[(Kf)(s+\phi)]^2 \sin s ds.$$

By induction one can prove that

$$\cos^{2m-1}s \sin s = \frac{1}{2^{2m-1}}[\sin 2ms - \sin(2m-2)s] +$$

$$+ \sum_{k=1}^{m-1}\beta_k\{\sin 2(m-k)s - \sin 2[m-(k+1)]s\}$$

where β_k are constants.

Since $[(Kf)(s)]^2$ is π/m-periodic we have that

$\int_0^{2\pi}[(Kf)(s+\phi)]^2 \sin js ds = 0$ if $0 \leq j < 2m$. Therefore

$$\frac{\alpha_{2m+1}}{\pi} m(2m+1) r^{2m-1}\mu^2\int_0^{2\pi}[(Kf)(s+\phi)]^2\cos^{2m-1}s \sin s ds =$$

$$= \frac{\alpha_{2m+1}}{\pi} m(2m+1) r^{2m-1}\mu^2\int_0^{2\pi}\frac{1}{2^{2m-1}}[(Kf)(s+\phi)]^2 \sin 2ms ds =$$

$$= -\frac{\alpha_{2m+1}}{2^{2m-1}\pi} m(2m+1) r^{2m-1}\mu^2 \sin 2m\phi\int_0^{2\pi}[(Kf)(\tau)]^2\cos 2m\tau d\tau.$$

If we let $K(\alpha_3,\ldots,\alpha_{2m-1},f)$ be the coefficient of the term $r^{2m-1}\mu^2\sin 2m\phi$, obtained from

$$\frac{1}{\pi}\int_0^{2\pi}\sum_{\ell=1}^{m-1}\alpha_{2\ell+1}[r\cos + \mu(Kf)(s+\phi) + W_1]^{2\ell+1}\sin s ds$$

we can conclude that ρ has the form stated in our lemma. □

Theorem 4.1: Suppose (H_1), (OH_2) are satisfied and $m \geq 2$ is even.

Then $\rho \neq 0$, generically.

Proof: From Lemma 4.2 it follows that

$$\rho = \alpha_{2m+1} c_m \int_0^{2\pi} [(Kf)(s)]^2 \cos 2ms\,ds + K(\alpha_3, \ldots, \alpha_{2m-1}, f)$$

where $c_m \neq 0$ is a constant.

Let us suppose first that $\int_0^{2\pi} [(Kf)(s)]^2 \cos 2ms\,ds \neq 0$ and that $\rho = 0$.

If we let $\bar{g}(u) \overset{def}{=} g(u) + \varepsilon u^{2m+1}$ for ε sufficiently small, we have that $\rho(f,\bar{g}) = \varepsilon c_m \int_0^{2\pi} [(Kf)(s)]^2 \cos 2ms\,ds \neq 0$ and $\bar{g}$ is close to g in the C^{2m+3} - topology.

It is not hard to prove that if $\int_0^{2\pi} [(Kf)(s)]^2 \cos 2ms\,ds = 0$ then $\int_0^{2\pi} [(K\bar{f})(s)]^2 \cos 2ms\,ds \neq 0$, where $\bar{f}(s) = f(s) + \tau \cos ms$ for a convenient sufficiently small τ.

Since $\rho = \rho(f,g)$ depends continuously on (f,g) we can conclude easily that set $\{(f,g) \in OH_{\pi/m} \times C^{2m+3} : \rho(f,g) \neq 0\}$ is open. □

Let us consider now the case m odd.

Following the above ideas and Remark 4.1, one can prove the following lemma.

Lemma 4.3: Suppose (H_1), (OH_2) are satisfied and $m > 3$ is odd. Then $\rho = \rho(f,g)$ can be written in the form:

$$\rho = -\frac{m}{2^{m-1}\pi} \alpha_m \int_0^{2\pi} (Kf)(s) \cos ms\,ds + K(\alpha_3, \ldots, \alpha_{m-2}, f),$$

where K is continuous and does not depend on α_i, $i \geq m$.

Using Lemma 4.3 and the ideas of Theorem 4.1, on can prove the following theorem.

Theorem 4.2: Suppose (H_1), (OH_2) are satisfied and $m \geq 3$ is odd. Then $\rho = \rho(f,g) \neq 0$, generically.

Remark 4.2: Fürkotter [3] considered the case where f is $2\pi/m$-periodic. Using the above ideas with the assumption (H_1), (H_2) one can also prove that the condition $\rho \neq 0$ is generic also in that case.

In fact, one can observe that the cases where f is $2\pi/m$-periodic considered in Fürkotter [3], for any $m \geq 2$, and the case where f is π/m-odd-harmonic and m is odd are less degenerate than the case where f is π/m-odd-harmonic and m is even.

ACKNOWLEDGEMENTS: This research was supported in part by CNPq (Conselho Nacional de Desenvolvimento Científico e Tecnológico, Brazil) under processo 301994/85-MA and FAPESP (Fundação de Amparo à Pesquisa de Estado de São Paulo).

REFERENCES

1. S. N. Chow and J. K. Hale: Methods of Bifurcation Theory. Springer Verlag, London Heidelberg-Berlin-Tokio, (1982).
2. M. Fürkotter and H. M. Rodrigues: Periodic solutions of forced nonlinear second order equations: symmetry and bifurcations. SIAM J. Math. Anal., Vol. 17, nº 6, November, 1986, pp. 1319-1331.
3. M. Fürkotter: Sobre Bifurcação e Simetria de Soluções Periódicas de Equações Diferenciais Não Lineares. Doctoral Thesis. Adviser: H. M. Rodrigues. ICMSC-USP (1989).
4. J. K. Hale: Ordinary Differential Equations. Krieger, New York, (1980).
5. J. K. Hale and H. M. Rodrigues: Bifurcation in the Duffing Equation with independent parameters II. Proc. Royal Soc. Edinburgh A, 79 (1977), pp. 317-326.
6. H. M. Rodrigues and A. Vanderbauwhede: Symmetric perturbations of nonlinear equations: symmetry of small solutions. Nonlinear Anal. TMA, 2(1978), pp. 27-46.
7. A. Vanderbauwhede: Local Bifurcation and Symmetry. Pitman Advanced Publishing Program, Boston-London-Melbourne, (1982).