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ON A RECURRENCE FORMULA ASSOCIATED WITH STRONG DISTRIBUTIONS*

A. SRI RANGA**

Abstract. Polynomials satisfying a certain three term recurrence relation are studied. The properties of these polynomials and an associated strong distribution are looked at under various conditions on the coefficients of the recurrence relation. Some examples are given to illustrate these results.

Key words. Three term recurrence relation (or formula), J-fractions, strong distribution functions, Stieltjes functions.

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1. Introduction. The recurrence formula or relation in our study takes the form

$$
(1.1)\qquad
\begin{aligned}
Q_{2n-1}(z) &= \{(1+\alpha_{2n-1})z-\beta_{2n-1}\}Q_{2n-2}(z)-\alpha_{2n-1}z^2Q_{2n-3}(z)\ , \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad n\geq 1, \\
Q_{2n}(z) &= (z-\beta_{2n})Q_{2n-1}(z)-\alpha_{2n}Q_{2n-2}(z)\ ,
\end{aligned}
$$

where $\qquad Q_{-1}(z)=0,\quad Q_0(z)=1,$

$$
(1.2)\qquad \alpha_1=0,\quad \alpha_{n+1}>0 \quad\text{and}\quad \beta_n\in\mathbb{R},\quad n\geq 1.
$$

It can easily be verified that for any $n\geq 0$ $Q_n(z)$ is a monic polynomial of degree n, satisfying in particular

$$
Q_n(z) = z^n+\ldots+q_1^{(n)}z+q_0^{(n)}\ ,
$$

where

$$
(1.3)\qquad
\begin{aligned}
& q_0^{(0)}=1,\quad q_0^{(1)}=-\beta_1,\quad q_0^{(2n)}=\prod_{r=1}^{n}(\beta_{2r-1}\beta_{2r}-\alpha_{2r})\ , \\
& q_0^{(2n+1)}=-\beta_{2n+1}q_0^{(2n)}\ , \\
& q_1^{(2n+1)}=(1+\alpha_{2n+1})q_0^{(2n)}-\beta_{2n+1}q_1^{(2n)}\ ,
\end{aligned}
$$

for all $n\geq 1$. From the theory of continued fractions one also has that $Q_n(z)$ is the denominator of the n-th convergent of the fraction

$$
(1.4)\qquad \frac{a_1}{z-\beta_1}-\frac{\alpha_2}{z-\beta_2}-\frac{\alpha_3z^2}{(1+\alpha_3)z-\beta_3}-\frac{\alpha_4}{z-\beta_4}-\frac{\alpha_5z^2}{(1+\alpha_5)z-\beta_5}-\ \ldots
$$

which is called a J-fraction.

So far, in all the attempts at resolving the so called strong Hamburger moment problem using regular continued fractions, the two regular J-fractions one of which is (1.4) and the other is of the form

$$\frac{a_1^*}{z-\beta_1^*} - \frac{\alpha_2^* z^2}{(1+\alpha_2^*)z-\beta_2^*} - \frac{\alpha_3^*}{z-\beta_3^*} - \frac{\alpha_4^* z^2}{(1+\alpha_4^*)z-\beta_4^*} - \frac{\alpha_5^*}{z-\beta_5^*} - \cdots ,$$

are knowm to provide the best partial solution to this problem (Sri Ranga [10]). Here, the term regular implies that there is a repetitive pattern in the functional forms taken by the partial coefficients.

In [11] the author examines the convergence properties of a class of J-fractions and gives some criteria for the convergence of a regular real J-fraction. This regular J-fraction, though it appears to be different from those given above, is equivalent to the fraction (1.4). Even though these two equivalent continued fractions have the same convergence behaviour, it will be seen (theorem 3.1 of this article) that some of the convergence criteria given in [11] are much simpler if they are written in terms of the coefficients α_n and β_n of (1.4). In view of this and from the fact that the denominators of (1.4) are monic polynomials, it may be considred that the fraction (1.4) is of a more natural form than its equivalent form given in [11].

A well known result on three term relations is that of the orthogonal polynomials. Associated with any positive distribution $d\psi(t)$ on $(-\infty,\infty)$, there exists a sequence of orthogonal (monic) polynomials $\{B_n(z)\}$ satisfying a recurrence formula of the form

$$(1.5) \qquad B_n(z) = (z-m_n)B_{n-1}(z)-\ell_n B_{n-2}(z) , \qquad n\geq 1 ,$$

where $B_{-1}(z)=0$, $B_0(z)=1$, $m_n\in\mathbb{R}$ and $\ell_n>0$. A not so familiar result regarding this three term relation is one which is attributed to Favard [5]. This result can be stated as: "given the relation (1.5), associated with it, there always exists a positive distribution $d\psi(t)$ such that the polynomials generated from the relation form an orthogonal sequence with respect to this distribution". However, the relation (1.5) does not in general uniquely characterize its distribution $d\psi(t)$. For the uniqueness of $d\psi(t)$, the coefficients ℓ_n and m_n must also satisfy further conditions. The simplest of such additional conditions for the uniqueness of $d\psi(t)$ is the boundedness of these coefficients.

In recent years considerable work has been done on extracting the properties of the associated distribution $d\psi(t)$ from three term formulas of the form (1.5). Many interesting results have been obtained for example by Blumenthal [1], Chihara [2,4], Nevai [8,9] and Van Assche [12]. In this article we consider an analogous study on the three term recurrence formula (1.1).

2. The strong distribution. From the studies of the J-fractions and of the strong moment problems [10], it has become evident that given a strong bounded positive distribution $d\psi(t)$ on $(-\infty,\infty)$, with the existence of the moments $c_m = \int_{-\infty}^{\infty} t^m d\psi(t)$ for all values of m, including negative, there exists a unique J-fraction of the form (1.4) corresponding to the Stiltjes function $\int_{-\infty}^{\infty} d\psi(t)/(z-t)$, provided that the moments also satisfy

$$H_{2n}^{(-2n)}>0, \quad H_{2n+1}^{(-2n)}>0 \quad \text{and} \quad H_{2n}^{(-2n+1)}\neq 0, \quad n\geq 1.$$

Here, $H_n^{(k)}$ are the Hankel determinants associated with c_m. Correspondence of this J-fraction is such that the convergents are two-point (0 & ∞) Padē approximants of the Stieltjes function.

The determinantal conditions $H_{2n}^{(-2n+1)}\neq 0$, $n\geq 1$, ensure that the denominator $Q_n(z)$ of the n-th convergent of this fraction satisfies $Q_n(0)\neq 0$, when n is even, Thus looking at (1.3) one realises that in the three term formula (1.1) satisfied by these polynomials the coefficients must also satisfy.

$$(2.1) \qquad \beta_{2n-1}\beta_{2n} - \alpha_{2n} \neq 0 \qquad n\geq 1.$$

We now start from the three term relation (1.1) and proceed to establish the existence of a distribution associated with it. We use in conjunction another sequence of polynomials $\{P_n(z)\}$ generated by

$$(2.2) \qquad \begin{aligned} P_{2n}(z) &= (z-\beta_{2n})P_{2n-1}(z) - \alpha_{2n}P_{2n-2}(z) , \\ P_{2n+1}(z) &= \{(1+\alpha_{2n+1})z - \beta_{2n+1}\}P_{2n}(z) - \alpha_{2n+1}z^2P_{2n-1}(z) , \end{aligned} \qquad n\geq 1 ,$$

where $P_0(z)=0$, $P_1(z)=a_1$, and α_n, β_n are as in (1.1).

This relation differs from (1.1) only in the initial conditions of the polynomials. For any value of $a_1 \neq 0$ the relation (2.2) generates a non-trivial sequence of polynomials $\{P_n(z)\}$. In this case we can write

$$P_n(z) = a_1 z^{n-1} + \text{lower order terms}.$$

These polynomials are also easily verified to be the numerators of the convergence of the J-fractions (1.4).

From (1.1) and (2.2) we have that the functions $T_n(z)$ and $U_n(z)$ defined by

$$T_n(z) = \{Q_n'(z)Q_{n-1}(z) - Q_{n-1}'(z)Q_n(z)\}, \qquad n \geq 1,$$

$$U_n(z) = \{P_n(z)Q_{n-1}(z) - P_{n-1}(z)Q_n(z)\}, \qquad n \geq 1,$$

Satisfy the relations

$$T_{2n}(z) = \{Q_{2n-1}(z)\}^2 + \alpha_{2n}T_{2n-1}(z),$$

$$T_{2n+1}(z) = \{Q_{2n}(z)\}^2 + \alpha_{2n+1}\{Q_{2n}(z) - zQ_{2n-1}(z)\}^2 + \alpha_{2n+1}\alpha_{2n}z^2T_{2n-1}(z),$$

$$U_{2n}(z) = \alpha_{2n}U_{2n-1}(z),$$

$$U_{2n+1}(z) = \alpha_{2n+1}z^2U_{2n}(z),$$

for $n \geq 1$, with $T_1(z)=1$ and $U_1(z)=a_1$. Under (1.2) the functions $T_n(z)$, $n \geq 1$, are hence strictly positive for all real values of z other than zero. If we also assume that the coefficients of (1.1) satisfy (2.1), (i.e, $Q_{2n}(0) \neq 0$, $n \geq 1$), then $T_n(z)$ are also positive for z=0. This enables one to establish that the roots of $Q_n(z)$ are all real, distinct and different from those of $Q_{n-1}(z)$.

We take that $a_1>0$. Then the functions $U_n(z)$, $n\geq 1$, are also positive for all real values of z other then zero. For $n\geq 3$, $U_n(z)$ takes the value zero when $z=0$. Hence one has, that all the roots of $Q_{2n}(z)$ are different from those of $P_{2n}(z)$ and that all the non zero roots of $Q_{2n+1}(z)$ are different from those of $P_{2n+1}(z)$. It is possible that at most one of the roots of $Q_{2n+1}(z)$ is zero. For any $n\geq 1$, when zero is a root of $Q_{2n+1}(z)$, it is also a root of $P_{2n+1}(z)$.

From these results it immediately follows that the quotient $P_n(z)/Q_n(z)$ has a partial decomposition of the form

$$\text{(2.3)} \qquad \frac{P_n(z)}{Q_n(z)} = \sum_{r=1}^{n} \frac{\ell_r^{(n)}}{z-z_r^{(n)}} \qquad n\geq 1 ,$$

where $z_r^{(n)}$ are the roots of $Q_n(z)$ and

$$\ell_r^{(n)} = P_n(z_r^{(n)})/Q_n'(z_r^{(n)}) , \qquad r=1,2,\ldots,n.$$

By noting $\ell_r^{(n)}$ is also equal to $U_n(z_r^{(n)})/T_n(z_r^{(n)})$, we find that it is a positive number except in the case when n is an odd number greater than 2 and that $z_r^{(n)}$ is equal to zero. In this case $\ell_r^{(n)}$ is also equal to zero.

Taking the limit of $\{zP_n(z)/Q_n(z)\}$ as $z\to\infty$ in (2.3), one also obtains

$$\sum_{r=1}^{n} \ell_r^{(n)} = a_1 .$$

Therefore, if we define a step function $\psi_n(t)$ by

$$\psi_n(t) = \begin{cases} 0, & -\infty<t\le z_1^{(n)} \\ \sum_{s=1}^{r} \ell_s^{(n)}, & z_r^{(n)} < t \le z_{r+1}^{(n)}, \quad r=1,2,\dots,n-1, \\ a_1, & z_n^{(n)} < t < \infty \end{cases}$$

then from the definition of the Stieltjes integral we have

$$(2.4) \qquad \frac{P_n(z)}{Q_n(z)} = \int_{-\infty}^{\infty} \frac{1}{z-t}\, d\psi_n(t), \qquad n\ge 1 .$$

We now state a result (see [10], p.271) on the correspondence behaviour of the quotiens $P_n(z)/Q_n(z)$, $n\ge 1$.

THEOREM 2.1. *With the conditions* (1.2) *and* (2.1) $L_n(z)=P_n(z)/Q_n(z)$, $n\ge 1$, *which are the convergents of the J-fraction* (1.4), *correspond to two formal power series expansions*

$$f(z) = \frac{c_0}{z} + \frac{c_1}{z^2} + \frac{c_2}{z^3} + \frac{c_3}{z^4} + \cdots,$$

$$g(z) = -c_1 - c_{-2}z - c_{-3}z^2 - c_{-4}z^3 - \cdots,$$

such that

$$\left.\begin{aligned} f(z) - L_{2n}(z) &= \gamma_{2n} z^{-2n-1} + 0(1/z)^{2n+2}, \\ f(z) - L_{2n+1}(z) &= \gamma_{2n+1} z^{-2n-3} + 0(1/z)^{2n+4}, \end{aligned}\right\} \quad n\ge 0,$$

where

$$\gamma_0 = a_1 = c_0\,, \qquad \gamma_{2n} = \alpha_{2n+1}\alpha_{2n}\cdots\alpha_2 a_1\,,$$

$$\gamma_{2n+1} = (1+\alpha_{2n+3})\alpha_{2n+2}\alpha_{2n+1}\cdots\alpha_2 a_1\,,$$

and

$$Q_{2n}(z)g(z)-P_{2n}(z) = \delta_{2n}z^{2n}+O(z)^{2n+1}\,,$$

$$Q_{2n+1}(z)g(z)-P_{2n+1}(z) = \delta_{2n+1}z^{2n}+O(z)^{2n+1}\,, \qquad n\geq 0,$$

where

$$\delta_{2n} = -\beta_{2n+2}\gamma_{2n}/Q_{2n+2}(0)\,, \qquad \delta_{2n+1} = \alpha_{2n+2}\gamma_{2n}/Q_{2n+2}(0)\,.$$

With the requirement $Q_{2n}(0)\neq 0$, the correspondence behaviour of $L_{2n}(z)$, $n\geq 1$, to the formal series expansions $g(z)$ is very clear. Since, we have not considered the behaviour of $Q_{2n+1}(0)$, the correspondence of $L_{2n+1}(z)$ to the series expansion $g(z)$ is not entirely evident. However, from (1.3) we have that if $Q_{2n+1}(0)=-\beta_{2n+1}Q_{2n}(0)=0$, then $Q'_{2n+1}(0)=(1+\alpha_{2n+1})Q_{2n}(0)\neq 0$. This result ensures that $L_{2n+1}(z)$ corresponds to at least (2n-1) terms of the series expansion $g(z)$, for $n\geq 1$.

Now by using the results of the above theorem with (2.4), we are able to give:

THEOREM 2.2. *When conditions* (1.2) *and* (2.1) *hold, there always exists a function* $\psi(t)$, *bounded, non-decreasing in* $(-\infty,\infty)$, *with infinitely many points of increase, such that*

$$\lim_{r\to\infty} \psi_{n(r)}(t) = \psi(t) ,$$

$$\lim_{r\to\infty} L_{n(r)}(z) = \int_{-\infty}^{\infty} d\psi(t)/(z-t) ,$$

and

$$c_m = \int_{-\infty}^{\infty} t^m d\psi(t) , \qquad m=\ldots,-2,-1,0,1,2,\ldots .$$

Here, $\{n(r)\}$ is an increasing subsequence of the sequence of positive integers.

This theorem follows from arguments analogous to those used for J-fraction and positive T-fraction by respectively, Wall [13] and Jones, Thron & Waadeland [7].

3. The $\psi(t)$ and the polynomials $Q_n(z)$ and $P_n(z)$.
Theorem 2.2 states that there exists a distribution function $\psi(t)$ such that its Stieltjes function

$$S(\psi(t);z) = \int_{-\infty}^{\infty} d\psi(t)/(z-t)$$

is a limit of a subsequence of $\{L_n(z)\}$. As a consequence, $S(\psi(t);z)$ has formal expansions $f(z)$ and $g(z)$. There may also be other distribution functions whose Stieltjes functions are limits of other subsequences of $\{L_n(z)\}$ (or even limits of other similar sequences) and hence having the same formal expansions. That is, the associated distribution function may not be unique. (We recall that all distribution functions which differ from each other only by constant values are non-distinct). At the end of this section we provide conditions on the coefficients α_n and β_n that ensure the uniqueness of $\psi(t)$.

For now, let $\psi(t)$ be any distribution function such that $S(\psi(t);z)$ has formal expansions $f(z)$ and $g(z)$. Then we have.

THEOREM 3.1. *The polynomials* $Q_n(z)$ *and* $P_n(z)$ *can be given in terms of* $\psi(t)$ *as*

$$\int_{-\infty}^{\infty} t^{-2n+s} Q_{2n}(t)d\psi(t) = \begin{cases} 0, & 0 \le s \le 2n-1 \\ \gamma_{2n}, & s = 2n \end{cases}, \qquad n \ge 1,$$

$$\int_{-\infty}^{\infty} t^{-2n+s} Q_{2n+1}(t)d\psi(t) = \begin{cases} 0, & 0 \le s \le 2n, \\ \gamma_{2n+1}, & s = 2n+1 \end{cases} \qquad n \ge 0,$$

and

$$P_n(z) = \int_{-\infty}^{\infty} \frac{Q_n(z) - Q_n(t)}{z-t}\, d\psi(t), \qquad n \ge 1,$$

where the constants γ_n *are defined in theorem* 2.1.

Proof: One needs to look at the odd and even indexed polynomials separately. However, as the proofs of both cases are very similar, only for those of odd index is given here. From the results of the theroem 2.1 and 2.2 it follows

$$(3.1) \qquad \int_{-\infty}^{\infty} \frac{1}{z-t}\, d\psi(t) - L_{2n+1}(z) = \gamma_{2n+1} z^{-2n-3} + O(1/z)^{2n+4}, \qquad n \ge 0.$$

Hence, from (2.4)

$$\int_{-\infty}^{\infty} t^{2n+2} d\psi_{2n+1}(t) = c_{2n+2} - \gamma_{2n+1}, \qquad n\geq 0.$$

Writing (3.1) in the form

$$\int_{-\infty}^{\infty} \frac{1}{z-t}\, d\psi(t) - L_{2n+1}(z) = z^{-2n-3} G_{2n+1}(z), \qquad n\geq 0,$$

one has

$$G_{2n+1}(z) = \int_{-\infty}^{\infty} \frac{zt^{2n+2}}{z-t}\, d(\psi(t) - \psi_{2n+1}(t)).$$

Consequently, for values of $z \in V \equiv \{z: z=iy,\ y>M>0\}$,

$$|G_{2n+1}(z)| < \int_{-\infty}^{\infty} t^{2n+2} d(\psi(t) + \psi_{2n+1}(t)) = 2c_{2n+2} - \gamma_{2n+1}.$$

This indicates that the function $G_{2n+1}(z)$ is bounded at least for values of $z \in V$. Hence, in the following idently

$$\int_{-\infty}^{\infty} \frac{Q_{2n+1}(z) - Q_{2n+1}(t)}{z-t}\, d\psi(t) - P_{2n+1}(z)$$

$$= z^{-2n-3} Q_{2n+1}(z) G_{2n+1}(z) - \int_{-\infty}^{\infty} \frac{Q_{2n+1}(t)}{z-t}\, d\psi(t), \quad n\geq 0,$$

the right hand side of it is a bounded function for $z \in V$ and it tends to zero as $z\to\infty$ in V. But the left hand-side is a polynomial of degree less than or equal to 2n. Therefore we must have, both sides of this identily equal to zero for all values of z to give

$$P_{2n+1}(z) = \int_{-\infty}^{\infty} \frac{Q_{2n+1}(z) - Q_{2n+1}(t)}{z-t}\, d\psi(t), \qquad n\geq 0,$$

and

$$\int_{-\infty}^{\infty} \frac{Q_{2n+1}(t)}{z-t}\, d\psi(t) = z^{-2n-3}Q_{2n+1}(z)G_{2n+1}(z) , \qquad n\geq 0.$$

The first of these two equations gives the definition of $P_n(z)$ when n is odd. Expanding the other in terms of powers of 1/z yields

(3.2)
$$\begin{aligned} &\int_{-\infty}^{\infty} tQ_{2n+1}(t)d\psi(t) = \gamma_{2n+1} , \\ &\int_{-\infty}^{\infty} Q_{2n+1}d\psi(t) = 0 , \end{aligned} \qquad n\geq 0.$$

Now, from theorem 2.1 we also have

$$Q_{2n+1}(z) \int_{-\infty}^{\infty} \frac{1}{z-t}\, d\psi(t) - P_{2n+1}(z) = \delta_{2n+1}z^{2n} + O(z)^{2n+1} , \quad n\geq 0.$$

With the definition of $P_{2n+1}(z)$, we obtain from this

$$\int_{-\infty}^{\infty} \frac{Q_{2n+1}(t)}{z-t}\, d\psi(t) = \delta_{2n+1}z^{2n} + O(z)^{2n+1} , \qquad n\geq 0.$$

Here, taking the power series expansion about the origin one obtains

$$\int_{-\infty}^{\infty} t^{-2n+s}Q_{2n+1}(t)d\psi(t) = 0, \qquad 0\leq s\leq 2n-1 , \quad n\geq 1.$$

From this and from (3.2), the definition for the odd indexed polynomials $Q_{2n+1}(z)$, $n\geq 0$, immediately follows. This completes the proof. □

To examine the conditions for the uniqueness of $\psi(t)$, we define the sequence of polynomials $\{Q_n(z,\tau)\}$ and

$\{P_n(z,\tau)\}$ by

$$Q_{2n+1}(z,\tau) = Q_{2n+1}(z) + \tau Q_{2n}(z)$$

(3.3) $\qquad n\geq 1.$

$$P_{2n+1}(z,\tau) = P_{2n+1}(z) + \tau P_{2n}(z)$$

It is easily seen from theorem 3.1 that these polynomials satisfy

$$(3.4) \qquad \int_{-\infty}^{\infty} t^{-2n+s} Q_{2n+1}(t,\tau)\, d\psi(t) = 0, \qquad 0\leq s\leq 2n-1,$$

$$(3.5) \qquad P_{2n+1}(z,\tau) = \int_{-\infty}^{\infty} \frac{Q_{2n+1}(z,\tau) - Q_{2n+1}(t,\tau)}{z-t}\, d\psi(t),$$

for $n\geq 1$, where $\psi(t)$ is any distribution with $S(\psi(t);z)$ having formal expansions $f(z)$ and $g(z)$.

It can be easily proved that equation (3.3) for $Q_{2n+1}(z,\tau)$, with τ taking all real values, defines all real monic polynomials in z of degree $2n+1$ that satisfy (3.4). We may view (3.4) as a skewed quasi orthogonality relation.

From (3.4), we find that when τ is real, all the zeros $z_r^{(2n+1)}(\tau)$, $r=1,2,\ldots,2n+1$, of $Q_{2n+1}(z,\tau)$ are real and distinct. Further, if $h(t)$ is any polynomial of degree less than $4n+1$, the following quadrature formula is satisfied

$$\int_{-\infty}^{\infty} t^{-2n} h(t)\, d\psi(t) = \sum_{r=1}^{2n+1} \lambda_r^{(2n+1)}(\tau)\, h(z_r^{(2n+1)}(\tau)) \qquad n\geq 1,$$

where

$$(3.6) \qquad \lambda_r^{(2n+1)}(\tau) = \int_{-\infty}^{\infty} t^{-2n} \left\{ \frac{Q_{2n+1}(t,\tau)}{Q'_{2n+1}(z_r^{(2n+1)}(\tau),\tau)(t - z_r^{(2n+1)}(\tau))} \right\}^m d\psi(t)$$

for $r=1,2,...,2n+1$. Here, m can take values both 1 and 2. Taking $m=2$ provides the result that all $\lambda_r^{(2n+1)}(\tau)$ are positive numbers.

Consider the rational function

$$L_{2n+1}(z,\tau) = P_{2n+1}(z,\tau)/Q_{2n+1}(z,\tau) ,$$

and, using (3.5), expand it about infinity. We obtain

$$L_{2n+1}(z,\tau) = \frac{c_0}{z} + \frac{c_1}{z^2} + \dots + \frac{c_{2n}}{z^{2n+1}} + O(1/z)^{2n+2} .$$

The function corresponds to the formal expansion f(z). Similarly, using (3.4) and (3.5), expanding $L_{2n+1}(z,\tau)$ about the origin, we obtain

$$L_{2n+1}(z,\tau) = -c_{-1}-c_{-2}z-\dots-c_{-2n}z^{2n-1} + O(z)^{2n} ,$$

provided $Q_{2n+1}(0,\tau)$ is non zero. However, if $Q_{2n+1}(0,\tau)$ is zero then from (2.1) we see that $Q'_{2n+1}(0,\tau)$ is non zero and hence this order of correspondence to the formal expansion g(z) reduces only by one.

Since the zero of $Q_{2n+1}(z,\tau)$ are distinct, can write

$$L_{2n+1}(z,\tau) = \sum_{r=1}^{2n+1} \frac{\rho_r^{(2n+1)}(\tau)}{z-z_r^{(2n+1)}(\tau)} ;$$

where $\rho_r^{(2n+1)}(\tau) = P_{2n+1}(z_r^{(2n+1)}(\tau),\tau)/Q'_{2n+1}(z_r^{(2n+1)}(\tau),\tau)$.

From relations (3.4), (3.5) and (3.6), we can also express $\rho_r^{(2n+1)}(\tau)$ as

$$\rho_r^{(2n+1)}(\tau) = (z_r^{(2n+1)}(\tau))^{2n}\lambda_r^{(2n+1)}(\tau) .$$

This indicates $\rho_r^{(2n+1)}(\tau)$ is positive unless $z_r^{(2n+1)}$ is zero, in which case $\rho_r^{(2n+1)}(\tau)$ is also zero.

Taking the limit of $z\,L_{2n+1}(z,\tau)$ as $z\to\infty$, we obtain

$$\sum_{r=1}^{2n+1} \rho_r^{(2n+1)}(\tau) = c_0$$

Hence, we can write

$$L_{2n+1}(z,\tau) = \int_{-\infty}^{\infty} \frac{1}{z-t}\, d\psi_n(t,\tau) \, ,$$

where $\psi_n(t,\tau)$ is the step function

$$\psi_n(t,\tau) = \begin{cases} 0 & -\alpha<t\leq z_1^{(2n+1)}(\tau) \\ \sum_{s=1}^{r} \rho_s^{(2n+1)}(\tau) \, , & z_r^{(2n+1)}(\tau)<t\leq z_{r+1}^{(2n+1)}(\tau) \, , \\ & r=1,2,\ldots,2n \, , \\ c_0 & z_{2n+1}^{(2n+1)}(\tau)<t<\infty \end{cases}$$

These results leads to a similar situation as in (2.4), but with rational functions of odd orders only. As the denominator of $L_{2n+1}(z,\tau)$ with τ real represents any polynomial in z of degree $2n+1$ satisfying the skewed quasi orthogonal relation (3.4) for any distribution whose Stieltjes function has formal expansions $f(z)$ and $g(z)$, we can conclude from above:

The associated distribution is unique if, and only if $\{L_{2n+1}(z,\tau)\}$ converges to the same limit for all real values of τ.

As the regular real J-fraction studied in [11] is equivalent to the fraction (1.4), we obtain, after some simple manipulation,

$$L_{2n+1}(z,\tau) = T_{2n+1}(z,w)$$

where $\tau=-\alpha_{2n+2}/w$. These functions $T_n(z,w)$ are defined on page 336 of [11].

Hence, from the limit point case of $T_n(z,w)$ considered in [11], we obtain

THEOREM 3.2. *If the coefficients* α_n *and* β_n *satisfy, in addition to* (1.2) *and* (2.1), *one or more of the following conditions*

$$\textit{I} \qquad \sum_{r=1}^{\infty} 1/(\alpha_{2r}\alpha_{2r+1})^{1/2} = \infty$$

$$\textit{II} \qquad \sum_{r=1}^{\infty} |\beta_{2r}|/(\alpha_{2r}\alpha_{2r+1})^{1/2} = \infty$$

$$\textit{III} \qquad \beta_{2n}=0, \quad n\geq 1 \quad \text{and} \quad \sum_{r=1}^{\infty} (\alpha_{2r+2}/\alpha_{2r+1})^{1/2} = \infty ,$$

then $\{L_{2n+1}(z,\tau)\}$ *converges to the same limit for all real values of* τ *(in fact, for all* τ *such that* $\mathrm{Im}(\tau)\geq 0$*). Hence the associated distribution function* $\psi(t)$ *is unique.*

4. Special cases. First we look at a case when the distribution function $\psi(t)$, may not be unique, but does have all its points of increase in the positive half of the real axis.

To prove that the zeros of $Q_n(z)$ are real, distinct, we used the fact that $T_n(z)>0$ for $n\geq 1$. By including the condition $(-1)^n Q_n(0)>0$, $n\geq 1$, with this, one can also easily show that the smallest zero of $Q_n(z)$ is greater than zero. Since the points of increase of $\psi_n(t)$ are the zeros of $Q_n(z)$ we have that $\psi(t)$, which is a limit of $\psi_n(t)$, also has its points of increase in the positive half of the real axis.

From (1.3) it follows that the condition $(-1)^n Q_n(0)>0$, $n\geq 1$, is equivalent to

$$(4.1) \qquad \beta_n>0 \quad \text{and} \quad \beta_{2n-1}\beta_{2n}-\alpha_{2n}>0\,, \qquad n\geq 1.$$

Hence, we have the following theorem

THEOREM 4.1. *If the coefficients* α_n *and* β_n *also satisfy the condition* (4.1), *then associated with* (1.1) *there exists a distribution function which has all its points of increase in* $(0,\infty)$.

We now consider the case in which all the coefficients α_n and β_n are bounded. If they also satisfy (1.2) and (2.1) then by theorem 3.2 the associated distribution function is unique. We can say more about this distribution function by considering the convergence behaviour of $\{L_n(z)\}_{n=0}^{\infty}$ which are the convergents of the J-fraction (1.4). We write this continued fraction in the equivalent form

(4.2) $$\frac{A_1(z)}{1} + \frac{A_2(z)}{1} + \frac{A_3(z)}{1} + \frac{A_4(z)}{1} + \cdots ,$$

where

$$A_1(z) = a_1/(z-\beta_1)$$

$$A_{2n}(z) = -\alpha_{2n}/[\{(1+\alpha_{2n-1})z - \beta_{2n-1}\}(z-\beta_{2n})] , \qquad n\geq 1,$$

$$A_{2n+1}(z) = -\alpha_{2n+1}z^2/[(z-\beta_{2n})\{(1+\alpha_{2n+1})z-\beta_{2n+1}\}] , \quad n\geq 1,$$

Hence, if there exist constants M, ε, ε_1, ε_2 of real numbers satisfying

(4.3a) $$0<M<\infty, \qquad 0\leq\varepsilon_1,\varepsilon_2<1, \qquad 0<\varepsilon<1,$$

and such that

$$\alpha_{2n-1}/(1+\alpha_{2n-1}) \leq (1-\varepsilon_1)(1-\varepsilon_2)(1-\varepsilon)^2 ,$$

(4.3b) $$\alpha_{2n}/(1+\alpha_{2n-1}) \leq (1-\varepsilon_1)(1-\varepsilon_2)\varepsilon^2 M^2 ,$$

$$|\beta_{2n-1}|/(1+\alpha_{2n-1}) \leq \varepsilon_1 M, \qquad |\beta_{2n}| \leq \varepsilon_2 M ,$$

for $n\geq 1$, then for all values of z such that $|z|>M$, $A_1(z)$ is finite, $|A_{2n}(z)|\leq\varepsilon^2$ and $|A_{2n+1}(z)|\leq(1-\varepsilon)^2$. Here, using a well known result (see for example [6], corollary 4.36) one obtains that for $|z|>M$ all the convergents of the fraction (4.2), (i.e., the fraction (1.4)) are finite and that they converge also to a finite limit. We note that in the case when $\varepsilon=1/2$, the fraction (4.2) satisfies the Worpitzky criteria, (p.42 [13]).

This result, and the fact that the zeros of $Q_n(z)$ are real and different from those of $P_n(z)$ indicate that all the zeros of $Q_n(z)$ lie inside the interval $[-M,M]$. It follows that the points of increase of $(\psi_n(t))$ and hence those of its unique limit $\psi(t)$, lie inside the same interval.

Further, by using (2.3), that $L_n(z)$ are uniformly bounded for all values of z which lie at a positive distance from the interval $[-M,M]$. Thus, from the Stieltjes-Vitali theorem, we obtain.

THEOREM 4.2. *If the coefficients* α_n *and* β_n *satisfy the conditions* (1.2), (2.1) *and* (4.3) *then the associated distribution function* $\psi(t)$, *which is unique, has all its points of increase inside* $[-M,M]$. *Further, the convergents* $L_n(z)$ *of the J-fraction* (1.4) *converge uniformly to the Stieltjes function* $\int_{-M}^{M} d\psi(t)/(z-1)$ *over every finite closed domain whose distance from the interval* $[-M,M]$ *is positive.*

For the condition (4.3b) to hold, it is required that α_n and β_n are bounded. On the other hand, we also can easily realise that if the coefficients α_n and β_n are bounded then constants M, ε, c_1 & c_2 can be found such that (4.3) holds. Thus,

COROLLARY 4.1. *If the coefficients α_n and β_n are bounded and satisfy the conditions (1.2) and (2.1), then there exists a positive number M and a unique distribution function $\psi(t)$, with all its points of increase inside $[-M,M]$, such that $\{L_n(z)\}$ converges to the Stieltjes function $\int_{-M}^{M} d\psi(t)/(z-t)$, uniformly over every finite closed region whose distance from the interval $[-M,M]$, is positive.*

We now give some examples to illustrate these results.

Example I: $$Q_{2n}(z) = zQ_{2n-1}(z) - Q_{2n-2}(z)\ , \qquad n\geq 1,$$

$$Q_{2n+1}(z) = (n+1)zQ_{2n}(z) - nz^2Q_{2n-1}(z)\ , \qquad n\geq 0.$$

Here $\beta_n=0$, $\alpha_{2n}=1$ and $\alpha_{2n+1}=n$ for $n\geq 1$. Hence,

$$\beta_{2n-1}\beta_{2n} - \alpha_{2n} = -1\ , \qquad n\geq 1,$$

and for example,

$$\beta_{2n}=0, \qquad n\geq 1,$$

and

$$\sum_{r=1}^{\infty} (\alpha_{2r+2}/\alpha_{2r+1})^{1/2} = \sum_{r=1}^{\infty} (1/r)^{1/2} = \infty$$

The coefficients satisfy the conditions required by theorem 3.2. This implise that there exists a distribution function and it is uniques.

The distribution associated with this recurrence relation is infact $d\psi(t)=(e/\sqrt{2\pi})e^{-(t^2+1/t^2)/2}dt$, in

$(-\infty,\infty)$. The constant $e/\sqrt{2\pi}$ is a normalising factor so that the moment $c_0=1$. All the moments of this distribution can be generated by the relations

$$c_0 = c_{-2} = 1, \qquad c_{2n-1} = 0, \qquad n\geq 0,$$

$$c_{2n+2} = (2n+1)c_{2n} + c_{2n-2}\ , \qquad n\geq 0,$$

and

$$c_{-n-2} = c_n, \qquad n\geq 0.$$

The moments c_{2n} can also be explicitly given as

$$c_{2n} = 2^{-3n} \sum_{r=0}^{n} \left[\binom{2n+1}{2r+1} \sum_{s=0}^{r} \left[\binom{r}{s}\ 2^{3s}\ \frac{(2n-2s)!}{(n-s)!}\right]\right], \qquad n\geq 0,$$

where $\binom{p}{q}$ are the binomial coefficients.

Example II: $$Q_{2n}(z) = (z-1)Q_{2n-1}(z) - (2n-1)Q_{2n-2}(z)\ , \qquad n\geq 1,$$

$$Q_{2n+1}(z) = \{(2n+1)z - (2n+2)\}Q_{2n}(z) - 2nz^2Q_{2n-1}(z)\ , \qquad n\geq 0.$$

We have $\beta_{2n-1}=2n$, $\beta_{2n}=1$, $\alpha_n=n-1$, for $n\geq 1$. Hence,

$$\beta_n>0, \qquad \beta_{2n-1}\beta_{2n}-\alpha_{2n}=1>0,$$

and also for example

$$\sum_{r=1}^{\infty} 1/\{\alpha_{2r}\alpha_{2r+1}\}^{1/2} = \sum_{r=1}^{\infty} 1/\{2r(2r-1)\}^{1/2} = \infty$$

The coefficients satisfy the conditions required by theorems 3.2 and 4.1. Hence, associated with this recurrence relation, there exists a unique distribution with all its points of increase in $(0,\infty)$.

We in fact have in this case $d\psi(t)=(e/\sqrt{2\pi})t^{-1/2}e^{-(t+1/t)/2}dt$ in $(0,\infty)$. The moments c_m associated with this distribution are same as the even moments of the first example.

Example III: Here we take $\beta_n=0$, $\alpha_{2n}=1$ and

$$\alpha_{2n+1} = \lambda^2 n^2/(4n^2-1), \quad \text{for } n\geq 1, \text{ where } 0<\lambda<\infty.$$

Therefore, $\beta_{2n-1}\beta_{2n}-\alpha_{2n}=-1$, for $n\geq 1$. Furthermore, if we let

$$b = \frac{\lambda+\sqrt{\lambda^2+4}}{2} \quad ,$$

then

$$b>1, \qquad \lambda = (b^2-1)/b \ ,$$

$$\frac{\alpha_{2n+1}}{1+\alpha_{2n+1}} = \frac{\lambda^2 n^2}{(\lambda^2+4)n^2-1} < \frac{b^4}{(b^2+1)^2} \ , \qquad n\geq 1,$$

and

$$\frac{\alpha_{2n+2}}{1+\alpha_{2n+1}} = \frac{4n^2-1}{(\lambda^2+4)n^2-1} < \frac{4b^2}{(b^2+1)^2} \ , \qquad n\geq 1.$$

Hence with the choice of

$$\varepsilon_1=\varepsilon_2=0, \qquad \varepsilon=1/(b^2+1) \quad \text{and} \quad M=2b,$$

all the conditions required by theorem 4.2 are fulfilled. Thus, we can say that the associated distribution is unique and has all its points of increase inside the interval $[-2b,2b]$.

The distribution associated with this recurrence relation is $d\psi(t)=dt$ in $[-b,-1/b]\cup[1/b,b]$. This distribution actually has all its points of increase inside the interval $[-b,b]$. Here, we can also give the relation between λ and b in the following interesting manner:

$$b = \lambda + \frac{1}{\lambda +}\,\frac{1}{\lambda +}\,\frac{1}{\lambda +}\,\cdots .$$

5. An asymptotic case. In addition to satisfying the conditions of (1.2), (2.1) and (4.3), we now let α_n and β_n also to have the following asymptotic behaviour.

$$(5.1)\qquad \begin{aligned} &\lim_{n\to\infty} \alpha_{2n-1} = \alpha^{(1)}\ , \qquad && \lim_{n\to\infty} \beta_{2n-1} = \beta^{(1)}\ , \\ &\lim_{n\to\infty} \alpha_{2n} = \alpha^{(2)}\ , && \lim_{n\to\infty} \beta_{2n} = \beta^{(2)}\ . \end{aligned}$$

THEOREM 5.1. Let $Z_N \equiv \{z : z = z_r^{(n)}\ ,\ \ r=1,2,\dots,n;\ \ n\ge N\}$, *where* $z_r^{(n)}$ *are the zeros of* $Q_n(z)$, *and* $\mathbb{C}$ *the extended complex plane, then as* $n\to\infty$

$$(5.2)\qquad \frac{Q_{n+2}(z)}{Q_n(z)} \to R(z) = \frac{1}{2}\left[z^2 - v_1 z - v_2 + \sqrt{(z^2 - v_1 z - v_2)^2 - 4v_3^2 z^2}\ \right] ,$$

uniformly on every finite closed region of $\mathbb{C}\setminus Z_N$. *Here*

$$v_1 = \beta^{(1)} + (1+\alpha^{(1)})\beta^{(2)}\ , \qquad v_2 = \alpha^{(2)} - \beta^{(1)}\beta^{(2)} \quad \textit{and}$$

$$v_3^2 = \alpha^{(1)}\alpha^{(2)}\ .$$

Proof: From (1.1) we note that the even indexed polynomials satisfy

$$(5.3) \qquad Q_{2n+2}(z) = \xi_n^{(1)}(z)Q_{2n}(z) - \xi_n^{(2)}(z)Q_{2n-2}(z)$$

where $\quad \xi_n^{(2)}(z) = \alpha_{2n}\alpha_{2n+1}z^2(z-\beta_{2n+2})/(z-\beta_{2n})$,

$$\xi_n^{(1)}(z) = [(z-\beta_{2n+2})\{(1+\alpha_{2n+1})z - \beta_{2n+1}\} - \alpha_{2n+2} - \xi_n^{(2)}/\alpha_{2n}] .$$

Since the coefficients α_n and β_n satisfy the conditions (1.2), (2.1) and (4.3), the zeros of $Q_{2n}(z)$ lie inside the interval $[-M,M]$ for all $n \geq 1$. Hence for any $z \in [M,\infty)$, we arrive at the chain sequence $\{D_n(z)=d_n(z)(1-d_{n-1}(z))\}$, where

$$D_n(z) = \xi_n^{(2)}(z)/\{\xi_n^{(1)}(z)\xi_{n-1}^{(1)}(z)\} ,$$

with parameter sequence $\{d_n(z)\}$ given by

$$d_n(z) = 1-Q_{2n+2}(z)/\{\xi_n^{(1)}(z)Q_{2n}(z)\}$$

This parameter sequence is in fact the minimal parameter sequence, i.e., $d_0(z)=0$.

As α_n and β_n are asymptotic and again satisfy the condition (4.3), we find that $\{D_n(z)\}$ and hence $\{d_n(z)\}$ are convergent for any $z \in [M,\infty)$. (see Chihara [3], theorem 6.4, page 102 for latter). The convergence of $\{d_n(z)\}$ implies the convergence of $\{Q_{2n+2}(z)/Q_{2n}(z)\}$. Now to determine the limit of $\{Q_{2n+2}(z)/Q_{2n}(z)\}$ we devide (5.3) by $Q_{2n}(z)$ and then let $n\to\infty$ to obtain

(5.4) $\quad R(z) = (z^2 - v_1 z - v_2) - v_3^2\, z^2/R(z).$

Similarly we can also obtain the exact result for the odd indexed polynomials. Hence, for $z \in [M,\infty)$ it follows that $\{Q_{n+2}(z)/Q_n(z)\}$ converges to the limit $R(z)$ given by (5.4). The equation (5.4) gives a quadratic equation for $R(z)$ and we take (5.2) as the solution of this which tends to $+\infty$ as $z \to +\infty$.

Furthermore, since $Q_n(z)$, $n \geq N$ has no zeros in $\mathbb{C}\setminus Z_N$, the ratio $Q_n(z)/Q_{n+2}(z)$ is analytic in this region for $n \geq N$. Also since the zeros of $Q_n(z)$ are distinct and different form those of $Q_{n-1}(z)$,

$$\frac{Q_{n-1}(z)}{Q_n(z)} = \sum_{r=1}^{n} \frac{m_r^{(r)}}{z - z_r^{(n)}} , \qquad n \geq 1,$$

where $\quad m_r^{(n)} = \dfrac{Q_{n-1}(z_r^{(n)})}{Q_n'(z_r^{(n)})} = \dfrac{(Q_{n-1}(z_r^{(n)}))^2}{T_n(z_r^{(n)})}$.

The functions $T_n(z)$ are defined in section 2. These equations indicate thet $m_r^{(n)}$ are positive and $\sum_{r=1}^{n} m_r^{(n)} = 1$. Thus we can write for $z \in \mathbb{C}\setminus Z_N$ and for $n \geq N$

$$\left|\frac{Q_n(z)}{Q_{n+2}(z)}\right| = \left|\frac{Q_n(z)}{Q_{n+1}(z)}\right|\left|\frac{Q_{n+1}(z)}{Q_{n+2}(z)}\right| < 1/\delta^2 .$$

Here, δ is the minimum distance of Z_N from z. That is to say $Q_n(z)/Q_{n+2}(z)$ are uniformly bounded on every bounded closed region of $\mathbb{C}\backslash Z_N$. Therefore, by applying the Stieltjes-Vitali theorem one establishes the uniform convergence of $Q_n(z)/Q_{n+2}(z)$ and hence the required results of the theorem. □

THEOREM 5.2. *In theorem 5.1, if* $v_1=0$*, then as* $n\to\infty$

$$\frac{Q_n'(z)}{nQ_n(z)} \to \frac{R'(z)}{2R(z)} = \frac{1}{2\pi}\int_B \frac{1}{z-t}\,\frac{(|t|+\mu_1\mu_2/|t|)}{\sqrt{\mu_2^2-t^2}\,\sqrt{t^2-\mu_1^2}}\,dt ,$$

uniformly on every bounded closed region of $\mathbb{C}\backslash Z_N$*, where*

$$(5.5)\qquad \mu_1 = \sqrt{v_3^2+v_2}-v_3 \qquad\text{and}\qquad \mu_2 = \sqrt{v_3^2+v_2}+v_3 ,$$

and

$$B = [-\mu_2,-\mu_1]\cup[\mu_1,\mu_2]$$

Proof: The proof of $(Q_n'(z)/(nQ_n(z))\to R'(z)/2R(z))$ for $z\in\mathbb{C}\backslash Z_N$, can be realised from Van Assche [12]. From (5.2),

$$R'(z) = \frac{1}{2}\left[2z-v_1+\frac{1}{2}\left((z^2-v_1z-v_2)^2-4v_3^2z^2\right)^{-1/2}\right.$$

$$\left.\cdot\left(2(z^2-v_1z-v_2)(2z-v_1)-8v_3^2z\right)\right]$$

$$= \left[(2z-v_1)R(z)-2v_3^2z\right]/\left((z^2-v_1z-v_2)^2-4v_3^2z^2\right)^{1/2} .$$

This gives

$$\frac{R'(z)}{R(z)} = \left[(2z-v_1)-2v_3^2z/R(z)\right]/\left((z^2-v_1z-v_2)^2-4v_3^2z^2\right)^{1/2} .$$

However, from (5.2) and (5.4) one has

$$\frac{v_3^2 z^2}{R(z)} = \frac{1}{2}\left[z^2 - v_1 z - v_2 - \sqrt{(z^2 - v_1 z - v_2)^2 - 4v_3^2 z^2}\right]$$

Consequently

$$\frac{R'(z)}{2R(z)} = \frac{z + v_2/z}{2\sqrt{(z^2 - v_1 z - v_2)^2 - 4v_3^2 z^2}} + \frac{1}{2z}$$

In the case when $v_1 = 0$, it follows that $v_2 > 0$ and thus

$$\frac{R'(z)}{2R(z)} = \frac{z + \mu_1\mu_2/z}{2\sqrt{(z^2 - \mu_1^2)}\,\sqrt{(z^2 - \mu_2^2)}} + \frac{1}{2z} \quad ,$$

where μ_1 and μ_2 are defined as in (5.5). Now to complete the proof we point at the following results

$$\frac{1}{\pi}\int_B \frac{1}{z-t}\,\frac{|t|}{\sqrt{\mu_2^2 - t^2}\,\sqrt{t^2 - \mu_1}}\,dt = \frac{z}{\sqrt{z^2 - \mu_1^2}\,\sqrt{z^2 - \mu_2^2}} \quad ,$$

$$\frac{1}{\pi}\int_B \frac{1}{z-t}\,\frac{\mu_1\mu_2/|t|}{\sqrt{\mu_2^2 - t^2}\,\sqrt{t^2 - \mu_1^2}}\,dt = \frac{\mu_1\mu_2/z}{\sqrt{z^2 - \mu_1^2}\,\sqrt{z^2 - \mu_2^2}} + \frac{1}{z} \quad .$$

The first of these results is given in Van Assche [12]. The other can be casily obtained using the first.

We now look at the partial decomposition of the ratio $Q_n'(z)/(nQ_n(z))$.

$$\frac{Q_n'(z)}{nQ_n(z)} = \sum_{r=1}^{n} \frac{1/n}{z-z_r^{(n)}} = \int_{-\infty}^{\infty} \frac{1}{z-t}\, dF_n(t) .$$

The function $nF_n(t)$ can be interpreted as the number of zeros of $Q_n(z)$ less than or equal to t. Using the above theorem

$$F_n(t) \to \frac{1}{2} \int_{-\infty}^{t} \frac{|x| + \mu_1\mu_2/|x|}{\sqrt{\mu_2^2 - x^2}\ \sqrt{x^2 - \mu_1^2}}\, I_B\, dx ,$$

where I_B is the indicator function of the set $B=[-\mu_2,-\mu_1]\cup[\mu_1,\mu_2]$.

This result indicates that if the number of zeros of $Q_n(z)$ outside B is $O(n)$ then $O(n)/n$ tends to zero as $n\to\infty$.

Example: We look at example III of section 4. The coefficients are given by

$$\beta_n=0, \quad \alpha_{2n}=1 \quad \text{and} \quad \alpha_{2n+1}=\frac{\lambda^2 n^2}{4n^2-1} \qquad n\geq 1.$$

Hence, we have

$$\beta^{(1)}=\beta^{(2)}=0, \quad \alpha^{(2)}=1 \quad \text{and} \quad \alpha^{(1)}=\frac{\lambda^2}{4}=\frac{(b^2-1)^2}{4b^2}$$

This gives,

$$v_1=0, \quad v_2=1, \quad v_3=\frac{(b^2-1)}{2b}, \quad \mu_1=1/b \quad \text{and} \quad \mu_2=b$$

Since the associated distribution function is dt in $B=[-b,-1/b]\cup[1/b,b]$ the results are exactly as we expected.

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