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On the extensions of some classical distributions

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On the Extensions of some Classical Distributions*

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ABSTRACT Some properties of polynomials associated with strong distribution functions are given, including conditions for the polynomials to satisfy a three term recurrence relation. Strong distributions that are extensions to the four classical distributions are given as examples.

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1. Introduction

We consider distribution functions whose moments exist for positive and negative values. That is functions $\psi(t)$ which are bounded and non-decreasing in $(-\infty,\infty)$ and for which the moments

$$\mu_n = \int_{-\infty}^{\infty} t^n \, d\psi(t)$$

are finite for $n = 0, \pm 1, \pm 2, \ldots$. Such functions have been described as strong distribution functions because they arise as solutions of strong moment problems (see [1],[2]). The distribution is called symmetric if all the odd order moments are zero and is called a positive half distribution if all the points of increase are on the positive real axis.

The Hankel determinants are defined by

$$H_r^{(m)} = \begin{vmatrix} \mu_m & \mu_{m+1} & \cdots & \mu_{m+r-1} \\ \mu_{m+1} & \mu_{m+2} & \cdots & \mu_{m+r} \\ \vdots & \vdots & & \vdots \\ \mu_{m+r-1} & \mu_{m+r} & & \mu_{m+2r-2} \end{vmatrix}$$

for all positive and negative m and $r \geq 1$, with

$$H_{-1}^{(m)} = 0 \quad \text{and} \quad H_0^{(m)} = 1.$$

For any strong distribution

$$H_r^{(2m)} > 0, \quad r \geq 0, \quad m = 0, \pm 1, \pm 2, \ldots.$$

In the case of a positive half distribution we also have

$$H_r^{(2m+1)} > 0, \quad r \geq 0, \quad m = 0, \pm 1, \pm 2, \ldots,$$

while for symmetric distributions

$$H_{2r+1}^{(2m+1)} = 0 \quad \text{and} \quad (-1)^r H_{2r}^{(2m+1)} > 0, \quad r \geq 0, \quad m = 0, \pm 1, \pm 2, \ldots.$$

The first of these latter results is because the columns of $H_j^{(k)}$ are linearly dependent if both j and k are odd. The second follows from the well known Jacobi identity

$$\left\{H_r^{(m)}\right\}^2 - H_r^{(m-1)} H_r^{(m+1)} + H_{r+1}^{(m-1)} H_{r-1}^{m+1} = 0.$$

2. **Polynomials related to strong distributions**

Given a strong distribution function $\psi(t)$ we define the polynomials $\{Q_n(z)\}_0^\infty$ by

$$\int_{-\infty}^{\infty} t^{-2[n/2]+s} Q_n(t)\, d\psi(t) = 0 \qquad 0 \le s \le n-1 \qquad (2.1)$$
$$= \gamma_n, \qquad s = n$$

for $n \ge 1$, with $Q_0(z) = 1$, and [x] denotes integer part of x.

In monic form the polynomials can be expressed as

$$Q_{2n}(z) = \frac{1}{H_{2n}^{(-2n)}} \begin{vmatrix} \mu_{-2n} & \cdots\cdots & \mu_0 \\ \vdots & & \vdots \\ \mu_{-1} & \cdots\cdots & \mu_{2n-1} \\ 1 \quad z & \cdots\cdots & z^{2n} \end{vmatrix}$$

$$Q_{2n+1} = \frac{1}{H_{2n+1}^{(-2n)}} \begin{vmatrix} \mu_{-2n} & \cdots\cdots & \mu_1 \\ \vdots & & \vdots \\ \mu_0 & \cdots\cdots & \mu_{2n+1} \\ 1 \quad z & \cdots\cdots & z^{2n+1} \end{vmatrix}$$

and, further,

$$\gamma_{2n} = H_{2n+1}^{(-2n)}/H_{2n}^{(-2n)}, \qquad \gamma_{2n+1} = H_{2n+2}^{(-2n)}/H_{2n+1}^{(-2n)}.$$

The existence of the polynomials is guaranteed by the positivity of $H_r^{(2m)}$, $r \geq 0$, $m = 0, \pm 1, \ldots$, and clearly all γ_k are positive. It is not difficult to show that the zeros of $Q_n(z)$ are real and distinct, for all values of $n \geq 1$.

A second sequence of polynomials is then defined in the usual way by

$$P_n(z) = \int_{-\infty}^{\infty} \frac{Q_n(z) - Q_n(t)}{z - t} \, d\psi(t), \qquad n \geq 0 \tag{2.2}$$

and clearly $P_n(z)$ is a polynomial of degree n-1 with leading coefficient μ_0.

Strong positive half distributions and strong symmetric distributions belong to those distribution functions for which the following result holds

Theorem: Let $\psi(t)$ be a strong distribution function such that

$$H_{2n}^{(-2n+1)} \neq 0, \qquad n \geq 0.$$

The polynomials $Q_n(z)$ and $P_n(z)$ each satisfy the three term recurrence relations

$$R_{2n}(z) = \left(z - \beta_{2n}\right) R_{2n-1}(z) - \alpha_{2n} R_{2n-2}(z)$$

$$R_{2n+1}(z) = \left[\left(1 + \alpha_{2n+1}\right) z - \beta_{2n+1}\right] R_{2n}(z) - \alpha_{2n+1} z^2 R_{2n-1}(z) \tag{2.3}$$

for $n \geq 1$ with $Q_0(z) = 1$, $Q_1(z) = z - \mu_1/\mu_0$, $P_0(z) = 0$ and $P_1(z) = \mu_0$. The coefficients are given by

$$\alpha_{2n} = \left\{ \frac{H_{2n}^{(-2n+1)}}{H_{2n-1}^{(-2n+2)}} \right\}^2 \frac{H_{2n-2}^{(-2n+2)}}{H_{2n}^{(-2n)}}, \quad \beta_{2n} = \frac{H_{2n}^{(-2n+1)} \, H_{2n-1}^{(-2n+1)}}{H_{2n-1}^{(-2n+2)} \, H_{2n}^{(-2n)}}$$

$$\alpha_{2n+1} = \frac{H_{2n+1} \, H_{2n-1}}{\left\{H_{2n}^{(-2n+1)}\right\}^2}, \qquad \beta_{2n+1} = \frac{H_{2n+1}^{(-2n+1)} \, H_{2n}^{(-2n)}}{H_{2n+1}^{(-2n)} \, H_{2n}^{(-2n+1)}}$$

for $n \geq 1$.

Proof: First for the odd index, write

$$A(z) = \left\{Q_{2n+1}(z) - zQ_{2n}(z)\right\} - \alpha_{2n+1} z \left\{Q_{2n}(z) - zQ_{2n-1}(z)\right\},$$

a polynomial of degree 2n at most, as

$$A(z) = -\beta_{2n+1} \, Q_{2n}(z) + B(z),$$

where $B(z)$ is some polynomial of degree $2n-1$ at most. Hence from (2.1) it follows that

$$\int_{-\infty}^{\infty} t^{s} B(t) \, d\psi(t) = \begin{cases} 0, & s = -2n, -2n+1, \ldots, -2 \\ -\gamma_{2n} - \alpha_{2n+1}(\gamma_{2n} - \gamma_{2n-1}), & s = -1. \end{cases}$$

Since $H_{2n}^{(-2n)}$ is non zero, then choosing α_{2n+1} such that

$$\gamma_{2n} + \alpha_{2n+1}(\gamma_{2n} - \gamma_{2n-1}) = 0$$

means that $B(z)$ is identically zero. This gives the required three term relation. Further, as γ_{2n} is positive, choosing α_{2n+1} in this way is possible only if $\gamma_{2n} - \gamma_{2n-1} \neq 0$. Expressing γ_{2n} and γ_{2n-1} in terms of the Hankel determinants and using the Jacobi identity we find that $\gamma_{2n} - \gamma_{2n-1} \neq 0$ if $H_{2n}^{(-2n+1)} \neq 0$. In this case α_{2n+1} can be given as in the theorem. With this choice of α_{2n+1} the value of β_{2n+1} can be found by considering the integral equation

$$\int_{-\infty}^{\infty} t^{-2n-1} A(z) \, d\psi(t) = -\beta_{2n+1} \int_{-\infty}^{\infty} t^{-2n-1} Q_{2n}(z) \, d\psi(t).$$

The expression for the even index are verified in a similar fashion by considering

$$Q_{2n}(z) - zQ_{2n-1}(z) = -\beta_{2n}Q_{2n-1}(z) - \alpha_{2n}Q_{2n-2}(z) + B(z),$$

where B(z) is some polynomial of degree 2n-3 at most.

Having established the recurrence relations for the $Q_n(z)$, we then use the definition (2.2) of $P_n(z)$ to show that they also satisfy the relations. □

The above recurrence relations indicate that the ratios $P_n(z)/Q_n(z)$ are, for n = 1,2,3,..., the successive convergents of the continued fraction.

$$\frac{\mu_0}{z-\beta_1} - \frac{\alpha_2}{z-\beta_2} - \frac{\alpha_3 z^2}{(1+\alpha_3)z-\beta_3} - \frac{\alpha_4}{z-\beta_4} - \frac{\alpha_5 z^2}{(1+\alpha_5)z-\beta_5} - \frac{\alpha_6}{z-\beta_6} - \cdots$$

From the definition of $P_n(z)$ we see that

$$\frac{P_n(z)}{Q_n(z)} = \int_{-\infty}^{\infty} \frac{1}{z-t}\, d\psi(t) - \frac{1}{Q_n(z)} \int_{-\infty}^{\infty} \frac{Q_n(t)}{z-t}\, d\psi(t).$$

Expanding the integrand in the second integral in inverse powers of z and using the orthogonality properties of $Q_n(z)$ yields

$$\frac{P_{2n}(z)}{Q_{2n}(z)} = \int_{-\infty}^{\infty} \frac{1}{z-t}\, d\psi(t) + O\left(\frac{1}{z^{2n+1}}\right) \qquad n \geq 1$$

and (2.4)

$$\frac{P_{2n+1}(z)}{Q_{2n+1}(z)} = \int_{-\infty}^{\infty} \frac{1}{z-t}\, d\psi(t) + O\left(\frac{1}{z^{2n+3}}\right) \qquad n \geq 0.$$

The symbol $O(1/z^r)$ denotes a power series in inverse powers of z starting with $1/z^r$.

Since

$$Q_{2n}(0) = \frac{H_{2n}^{(-2n+1)}}{H_{2n}^{(-2n)}}$$

then under the condition of the above theorem, $Q_{2n}(0) \neq 0$. On the other hand $Q_{2n+1}(0)$ may be zero, but, if it is, we can show from the linear system of equations yielded by (2.1) that $Q'_{2n+1}(0) \neq 0$. With these results we can expand the ratio $P_n(z)/Q_n(z)$ in powers of z and obtain

$$\frac{P_{2n}(z)}{Q_{2n}(z)} = \int_{-\infty}^{\infty} \frac{1}{z-t}\, d\psi(t) + O(z^{2n}) \qquad n \geq 1$$

and (2.5)

$$\frac{P_{2n+1}(z)}{Q_{2n+1}(z)} = \int_{-\infty}^{\infty} \frac{1}{z-t}\, d\psi(t) + O(z^{2n-1}) \qquad n \geq 0.$$

3. Examples

1. The strong Tchebycheff distribution. We first consider the distribution function $\psi_T(t)$ given by

$$d\psi_T(t) = \frac{|t|}{\sqrt{b^2-t^2}\,\sqrt{t^2-a^2}}, \qquad t \in B \equiv [-b,-a] \cup [a,b]$$

$$= 0, \qquad t \notin B$$

with $0 < a < b < \infty$.

In the limit as $a \to 0$ and $b \to 1$ the distribution becomes the Tchebycheff distribution and so we may view it as an extension to this distribution. Further, since $a > 0$, the function has finite moments of negative order and thus we refer to $\psi_T(t)$ as a strong Tchebycheff distribution. We have the following result.

Theorem II: For the strong Tchebycheff distribution function $\psi_T(t)$ defined above the polynomials $Q_n(z)$ and $P_n(z)$ satisfy the three term recurrence relation (2.3) with α_n and β_n given by

$$\beta_n = 0, \qquad \alpha_{2n} = \gamma, \qquad n \geq 1,$$

$$\alpha_3 = \frac{1}{2}\frac{\lambda^2}{\gamma}, \qquad \alpha_{2n+1} = \frac{1}{4}\frac{\lambda^2}{\gamma}, \qquad n \geq 2,$$

where $\gamma = ab$ and $\lambda = (b-a)$.

Proof: Consider the continued fraction

$$\frac{\mu_0^T}{z} - \frac{a_2}{z} - \frac{2a_1z^2}{(1+2a_1)z} - \frac{a_2}{z} - \frac{a_1z^2}{(1+a_1)z} - \frac{a_2}{z} - \cdots \qquad (3.1)$$

in which $a_2 = \gamma$ and $a_1 = \lambda^2/(4\gamma)$.

As the coefficients of (3.1) are bounded then the continued fraction converges uniformly to an analytic function over every bounded closed region in the upper half plane $Im(z) > 0$. See [4] theorem 9. Denoting this function by $F(z)$ then

$$F(z) = \frac{\mu_0^T}{z} - \frac{a_2}{z} - \frac{2a_1z^2}{(1+2a_1)z} - f(z),$$

where $f(z)$ is a 2-periodic continued fraction which can be written as

$$f(z) = \frac{a_2}{z} - \frac{a_1z^2}{(1+a_1)z} - f(z).$$

Solving for $f(z)$ yields

$$f(z) = \frac{(z^2+a_2) \pm \sqrt{(z^2+a_2)^2 - 4a_2(1+a_1)z^2}}{2z}.$$

If we now choose

$$a = \sqrt{a_2}\left\{\sqrt{1+a_1} - \sqrt{a_1}\right\},$$

and

$$b = \sqrt{a_2}\left\{\sqrt{1+a_1} + \sqrt{a_1}\right\},$$

then clearly $a_2 = \gamma$ and $a_1 = \lambda^2/(4\gamma)$ and we have

$$f(z) = \frac{1}{2z}\left\{(z^2+ab) \pm \sqrt{z^2-b^2}\,\sqrt{z^2-a^2}\right\}.$$

The function f(z) has two values but only one of them is appropriate since F(z) must take one value only. We note that Im F(z) < 0 whenever Im(z) > 0, see [4]. Consequently

$$f(z) = \frac{1}{2z}\left\{(z^2+ab) - \sqrt{z^2-b^2}\,\sqrt{z^2-a^2}\right\}$$

and

$$F(z) = \mu_0^T z/\left\{\sqrt{z^2-b^2}\,\sqrt{z^2-a^2}\right\}. \qquad (3.2)$$

The function F(z) can be written alternatively as

$$F(z) = \frac{\mu_0^T}{\pi}\int_B \frac{1}{z-t}\,\frac{|t|}{\sqrt{(b^2-t^2)}\,\sqrt{(t^2-a^2)}}\,dt,$$

a result given in Van Assche [5].

We can show that $\mu_0^T = \pi$ and hence

$$F(z) = \int_{-\infty}^{\infty} \frac{1}{z-t}\,d\psi_T(t). \qquad (3.3)$$

Hence the continued fraction converges to the Stieltjes function of the strong Tchebycheff distribution. Using results given in [3] we can then show that the convergents $P_n(z)/Q_n(z)$ of (3.1) satisfy (2.4) and (2.5) for this distribution. It is then easy to show that $Q_n(z)$ and $P_n(z)$ satisfy (2.1) and (2.2) respectively, see [2]. This completes the proof.

We can express a and b in terms of γ and λ,

$$b = \frac{\gamma}{a} = \lambda + \frac{\gamma}{\lambda +}\, \frac{\gamma}{\lambda +}\, \frac{\gamma}{\lambda + \dots}.$$

Also, by expanding the right hand side of (3.2) we see that the moments of $\psi_T(t)$ satisfy

$$\mu^T_{2n} = \frac{\pi}{4^n} \sum_{j=0}^{n} \sigma_j \sigma_{n-j} (a^2)^j (b^2)^{n-j},$$

$$\mu^T_{-2n-1} = \mu^T_{2n+1} = 0, \quad \mu^T_{-2n-2} = \mu^T_{2n}/(ab)^{2n+1},$$

for $n \geq 0$, where $\sigma_j = (2j)!/(j!)^2$.

2. The strong Legendre distribution. Next we consider

$$d\psi_{Le}(t) = dt, \qquad t \in B = [-b,-a] \cup [a,b]$$
$$= 0, \qquad t \notin B$$

again with $0 < a < b < \infty$.

The moments μ_n^{Le}, $n = 0, \pm 1, \pm 2, \dots$ of this distribution are easily found. As $\psi_{Le}(t)$ is a symmetric distribution function the polynomials $Q_n(z)$ and $P_n(z)$ each satisfy (2.3). Numerical evidence suggests that

$$\beta_n = 0, \quad \alpha_{2n} = \gamma, \quad \alpha_{2n+1} = \frac{\lambda^2}{\gamma} \cdot \frac{n^2}{4n^2-1}, \qquad n \geq 1, \qquad (3.4)$$

where $\gamma = ab$ and $\lambda = (b-a)$.

The coefficients of the continued fraction

$$\frac{\mu_0^{Le}}{z} - \frac{\alpha_2}{z} - \frac{\alpha_3 z^2}{(1+\alpha_3)z} - \frac{\alpha_4}{z} - \frac{\alpha_5 z^2}{(1+\alpha_5)z} - \dots$$

are bounded and hence the continued fraction converges uniformly over every bounded closed domain in the upper half plane $\mathrm{Im}(z) > 0$. (See [4]).

Hence, if the values in (3.4) are correct then the continued fraction converges to

$$\int_B \frac{1}{z-t}\, d\psi_{Le}(t).$$

In the case when z=i we would then have

$$\tan^{-1}\left(\frac{\lambda}{1+\gamma}\right) = \frac{\lambda}{1} + \frac{a_2}{1} + \frac{a_3}{1} + \frac{a_4}{1} + \ldots$$

where

$$a_{2n+2} = \frac{\gamma^2(4n^2-1)}{(\lambda^2+4\gamma)n^2-\gamma}, \qquad n \geq 0$$

$$a_{2n+1} = -\frac{\lambda^2 n^2}{(\lambda^2+4\gamma)n^2-\gamma}, \qquad n \geq 1.$$

Taking the even contraction leads, after some manipulation, to the well known expansion

$$\tan^{-1}x = \frac{x}{1} + \frac{1^2x^2}{3} + \frac{2^2x^2}{5} + \frac{3^2x^2}{7} + \ldots$$

A second result in support of (3.4) is the asymptotic behaviour of α_n. From an analysis similar to that given in Van Assche [5], of the three term recurrence relation (2.3), we find that

$$\sqrt{\alpha_{2n}}\left\{\sqrt{1+\alpha_{2n+1}} - \sqrt{\alpha_{2n+1}}\right\} \to a$$

$$\sqrt{\alpha_{2n}}\left\{\sqrt{1+\alpha_{2n+1}} + \sqrt{\alpha_{2n+1}}\right\} \to b$$

and clearly the expressions in (3.4) are compatible with these limits.

3. The strong Hermite Distribution. Thirdly we set

$$d\psi_H(t) = e^{-(t^2+a^2/t^2)/2}\,dt \qquad -\infty < t < \infty$$

with $0 < a < \infty$. In this case the moments μ_n^H satisfy

$$\mu_0^H = \sqrt{2\pi}/e^a, \quad \mu_{-2n-1}^H = \mu_{2n+1}^H = 0,$$

$$\mu_{-2n-2}^H = \mu_{2n}^H / a^{2n+1},$$

$$\mu_{2n+2}^H = (2n+1)\mu_{2n}^H + a^2\mu_{2n-2}^H,$$

for $n \geq 1$. We can also give $\mu_{2n}{}^H$ explicitly as

$$\mu_{2n}^H = \frac{\sqrt{2\pi}}{2^{3n} e^a} \sum_{r=0}^{n} \binom{2n+1}{2r+1} \sum_{s=0}^{r} \binom{r}{s} (8a)^s \frac{(2n-2s)!}{(n-s)!}, \quad n \geq 0.$$

The distribution function $\psi_H(t)$ is symmetric and hence the associated polynomials $Q_n(z)$ and $P_n(z)$ satisfy (2.3). Here computational evidence seems to suggest that

$$\beta_n = 0, \quad \alpha_{2n} = a \text{ and } \alpha_{2n+1} = \frac{n}{a}, \quad n \geq 1.$$

Again we do not have any analytic proof of this result. However as before, we are able to conjecture that it is true.

If the result is correct then

$$\int_{-\infty}^{\infty} \frac{1}{z-t} d\psi_H(t) = \frac{\mu_0^H}{z} - \frac{a}{z} - \frac{(1/a)z^2}{(1+1/a)z} - \frac{a}{z} - \frac{(2/a)z^2}{(1+2/a)z} - \frac{a}{z} - \dots$$

This continued fraction is uniformly convergent over all bounded closed regions in the half plane $\mathrm{Im}(z) > 0$. (See [4], theorem 9.) Hence, by taking the even part of this continued fraction, we find

$$\int_{-\infty}^{\infty} \frac{1}{z-t} d\psi_H(t) = \frac{\mu_0^H}{z^2-a} - \frac{z^2}{z^2-a} - \frac{2z^2}{z^2-a} - \frac{3z^2}{z^2-a} - \dots \quad (3.5)$$

Then substituting $z/(z^2-a) = \pm i$, we get

$$1 - \sqrt{\pi/2} \left\{ \int_0^{\infty} \frac{1}{1+t^2} e^{-\frac{1}{2}t^2} dt \right\}^{-1} = \frac{1}{1} + \frac{2}{1} + \frac{3}{1} + \frac{4}{1} + \dots$$

This expansion is correct, and can also be obtained from the

J-fraction expansion of

$$\int_{-\infty}^{\infty} e^{-t^2/2} dt/(z-t),$$

by letting z=i.

4. A strong Laguerre distribution.

Finally we consider

$$d\psi_{La}(t) = t^{-\frac{1}{2}} e^{-(t+a^2/t)/2} dt, \qquad 0 < t < \infty$$

$$= 0, \qquad -\infty < t \leq 0$$

with $0 < a < \infty$.

This is a positive half distribution and hence the polynomials $Q_n(z)$ and $P_n(z)$ do satisfy (2.3). It appears that

$$\beta_{2n-1} = 2n-1+a, \qquad \beta_{2n} = a,$$
$$\alpha_{2n-1} = (2n-2)/a, \qquad \alpha_{2n} = (2n-1)a,$$

for $n \geq 1$.

These results essentially follow from the strong Hermite case. Substituting z^2=z and t^2=u in (3.5) yields an M-fraction expansion for

$$\int_{0}^{\infty} t^{-1/2} e^{-1/2(t+a^2/t)} dt/(z-t).$$

The coefficients of this M fraction can then be used to derive the α_j and β_j.

The moments μ_n^{La} of this distribution function satisfy

$$\mu_n^{La} = \mu_{2n}^{H}$$

for all positive and negative n.

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