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Stability and genericity of composition maps and applications

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STABILITY AND GENERICITY OF COMPOSITION OF MAPS AND APPLICATIONS

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A B S T R A C T

With the purpose of obtaining a unified viewpoint on some of the applications of singularity theory, we consider compositions of smooth maps $X \xrightarrow{g} Y \xrightarrow{f} Z$, and study the relation between their stability and that of g (resp. f) when f (resp. g) is fixed (under convenient restrictions on the maps f and g). We also study the analogous problem when one of the maps is substituted by a family of smooth maps. Finally we give a geometrical application related to the problem of stability of caustics.

STABILITY AND GENERICITY OF COMPOSITION OF MAPS AND APPLICATIONS

§0. INTRODUCTION

In various contexts of the applications of the singularity theory, eg. stability of caustics [8], Generic Geometry [9], [10], weak transversality [3], stability of the cut-locus in a Riemannian manifold [4], one has to analyze a situation which could be generalized as follows:

Given a composition of C^∞-maps

$$X \xrightarrow{g} Y \xrightarrow{f} Z$$

find the relations between the G-stability (G-genericity) of the map $f \circ g$ and the G'-stability (G'-genericity) of the map g when f is fixed, or viceversa, of the map f when g is fixed. Here G and G' are appropriate diffeomorphism groups, which for most of the applications purposes we may take as subgroups of A or K.

Looking at the literature, we see that for each particular case, the group G is well defined, usually $A(X,Z)$ as in [9], [14] or $K(X,Z)$ as in [3], [10]. Whereas G' is not explicitely given, but just suggested either by a concept of stability, or by a constructive process along the proofs in which the candidates for the orbits of $j^k G'$ in $J^k(X,Y)$ or in $J^k(Y,Z)$ are skettched ([3],[10]). Indeed, the knowledge of G' is not needed in any explicit way for carrying out the proofs of most of the stability and genericity theorems of the kind we refer to. Nevertheless, we think that the introduction of such a group may give a more enlightening and unifying viewpoint on various of the results already known, appart from inducing some other new results that will be presented in this paper.

We shall define a group G' which will depend on G and f or g in each case, so it will be denoted by G_f or G_g respectively. This G_f (resp. G_g) will be expected to be such that, for sufficiently well behaved fixed f (resp. g), for instance a submersion (resp. an immersion), G_f - stability of g (resp. G_g - stability of f) will imply G - stability of $f \circ g$.

We shall consider the problem of stability in §1 , and in §2 the problem of versality of the families obtained by composing a single map with a family of maps. In §3 we shall work out some geometrical applications.

N.A. Baas has also approached [2] the problem of studying the stability and genericity of composition of maps, although he does this from a different viewpoint to ours. Indeed, we prove that under some restrictions on the maps g and f , the A-stability of the composed map $f \circ g$ is equivalent to the structural stability of the pair (g,f) as defined by Baas.

Conventions, notations and basic facts:

We impose here some general restrictions to the problem in consideration:

a) X is a compact manifold;

b) $\dim X$, $\dim Z < \dim Y$;

c) the group G is either $A(X,Z)$ or $K(X,Z)$, although one expects that the obtained results can be adapted to others subgroups of $\mathcal{K}(X,Z)$.

All the maps and manifolds we consider along this work are smooth. The topologies on the function spaces are the corresponding Whitney C^{∞}-topologies.

We denote by $Subm^{\infty}(Y,Z)$ the subspace of all smooth submersions from Y to Z and by $Emb^{\infty}(X,Y)$ that of all smooth embeddings from X to Y .

We denote by $\mathcal{R}(M)$ the group of diffeomorphisms on a manifold M . Given $f \in C^{\infty}(X,Y)$, we understand by $Iso(f)$ the isotropy subgroup of f in $\mathcal{A}(X,Y) = \mathcal{R}(X) \times \mathcal{R}(Y)$ that is, $Iso(f) = \{ (h,h') \in \mathcal{A}(X,Y) : : h' \circ f \circ h^{-1} = f \}$. By abuse of language we also call $Iso(f)$ the subgroups $\{ h \in \mathcal{R}(X) : \exists\, h' \in \mathcal{R}(Y)$ with $h' \circ f \circ h^{-1} = f \}$ or $\{ h' \in \mathcal{R}(Y) : \exists\, h \in \mathcal{R}(X)$ with $h' \circ f \circ h^{-1} = f \}$ when no confusion can arise.

Given any group G of diffeos acting on a space $C^{\infty}(X,Y)$, we have induced group actions of the groups of germs of diffeos of G at appropriate points on the $\mathbb{R}$-algebra $\mathcal{E}_{x,y}(X,Y)$ of germs of maps in $C^{\infty}(X,Y)$ at some point x with fixed target y . The associated equivalence relations both among maps in $C^{\infty}(X,Y)$ and among germs in $\mathcal{E}_{x,y}(X,Y)$ will be indistinctly denoted by $\sim_G$. Given $g , g' \in C^{\infty}(X,Y)$ we shall denote by g_x the germ of g at $x \in X$, and by $g'_{x'}$, the germ of g' at $x' \in X$. Then we shall write $g_x \sim_G g'_{x'}$, whenever there are manifold charts: (ϕ,U) for X at x , (ϕ',U') for X at x' , (ψ,V) for Y at $g(x)$ and (ψ',V') for Y at $g'(x')$ such that $\phi(x) = \phi'(x') = = 0 \in \mathbb{R}^m$, $\psi(g(x)) = \psi'(g'(x')) = 0 \in \mathbb{R}^n$ and $\psi \circ g \circ \phi^{-1} \sim_G \psi' \circ g' \circ \phi'^{-1}$

as germs at 0 of maps from $\mathbb{R}^m$ to $\mathbb{R}^n$.

We also notice that the action of G on $C^\infty(X,Y)$ induces an action of the group j^kG of k-jets of elements of G on the space $J^k(X,Y)$ that will be denoted by $\sim_G$ too.

§1. GROUP ACTIONS AND STABILITY

Consider a composition of maps

$$\phi : X \xrightarrow{g} Y \xrightarrow{f} Z$$

and a group G acting on $C^\infty(X,Z)$. To say that $\phi = f \circ g$ is G-stable means that the G-orbit of $f \circ g$ is open in $C^\infty(X,Z)$. Or in other words, that there is a neighbourhood N of $\phi = f \circ g$ in $C^\infty(X,Z)$ such that $\forall \phi' \in N$, $\exists \theta \in G$ with $\phi' = \theta * \phi$, where $*$ denotes the action of G on $C^\infty(X,Z)$. This action defines an equivalence relation, $\sim_G$, on $C^\infty(X,Z)$. And this induces in turn, for each fixed $f \in C^\infty(Y,Z)$ another equivalence relation, $\sim_f$, on $C^\infty(X,Y)$, namely: $g \sim_f g' \iff f \circ g \sim_G f \circ g'$. The relation $\sim_f$ defines a stability criterium on $C^\infty(X,Y)$ studied by G. Wassermann [14] . It would be interesting to identify this stability criterium with the one associated to some group action on $C^\infty(X,Y)$.

In an analogous way, given a fixed $g \in C^\infty(X,Y)$, we can define a relation $\sim_g$ from $\sim_G$ by requiring that $f \sim_g f' \iff f \circ g \sim_G f \circ g'$. There is also a stability criterium associated to $\sim_g$ on $C^\infty(Y,Z)$, that we would like to relate to the standard one defined by some group action on $C^\infty(Y,Z)$.

We are thus looking for groups, that we denote by G_f and G_g respectively, whose orbits coincide with the equivalence classes of the relations $\sim_f$ and $\sim_g$ respectively. We shall restrict our study to the cases $G = A, K$.

The path that we shall follow consists in defining groups G_f and G_g and prove that they provide the correct setting for the local situation, when g is an embedding and f a submersion. For the global situation we shall have to content ourselves with some more restrictive results when $G = A$, namely:

a) $f \circ g \sim_{G*} f \circ g' \Rightarrow g \sim_{G_f} g' \Rightarrow f \circ g \sim_G f \circ g'$;

b) $f \circ g \sim_{G*} f' \circ g \Rightarrow f \sim_{G_g} f' \Rightarrow f \circ g \sim_G f' \circ g$;

where $G^* = \{ (h,k) \in G / k$ is in the component of $1_Z \}$.

In order to get the global equivalence one should prove that given $\phi, \phi' \in C^\infty(X,Z)$: $\phi \sim_G \phi' \Leftrightarrow \phi \sim_{G*} \phi'$, with, perhaps, some conditions on ϕ and ϕ' , like ϕ and ϕ' being near enough one to each other. We leave this question unanswered.

For $G = K$, both the local and the global situation are solved when g is an embedding and f a submersion.

We finally get:

$f \circ g$ is G-stable $\Leftrightarrow$ f is G_g-stable $\Leftrightarrow$ g is G_f-stable.

1.1 Consider a fixed map $f \in C^\infty(Y,Z)$. We define

$$G_f = \{ (h,\ell) \in A(X,Y) \; ; \; h \in R(X) , \; \ell \in Iso_G(f) \} ,$$

where we distinguish two different cases:

a) For $G = A(X,Z)$, $Iso_G(f)$ is the usual isotropy subgroup of f in $A(Y,Z)$.

Remark: This definition is suggested by the following commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{g} & Y & \xrightarrow{f} & Z \\ h \downarrow & & \downarrow \ell & & \downarrow k \\ X & \xrightarrow[g']{} & Y & \xrightarrow[f]{} & Z \end{array}$$

b) For $G = K(X,Z)$, as it is usual in the problems arising from the contact viewpoint we put $Z = \mathbb{R}^p$ and define

$$Iso_G(f) = \{ \ell \in R(Y) \; / \; \ell(f^{-1}(0)) \subseteq f^{-1}(0) \} \quad .$$

Observe that the group G_f acts on $C^\infty(X,Y)$ in the obvious way and we have the following.

Lemma 1.

a) For $G = A(X,Z)$: $g \sim_{G_f} g' \Rightarrow f \circ g \sim_G f \circ g'$, $\forall\, g,g' \in C^\infty(X,Y)$.

b) For $G = K(X,Z)$ and $f \in C^\infty(Y,Z)$ being a submersion at every point of $f^{-1}(0) \subset Y$:

$$g \sim_{G_f} g' \Rightarrow f \circ g \sim_G f \circ g' \;, \quad \forall\, g, g' \in C^\infty(X,Y) \;.$$

Proof.

a) If $g \sim_{G_f} g'$ then we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ h \downarrow & & \downarrow \ell \\ X & \xrightarrow[g']{} & Y \end{array}$$

with $(h,\ell) \in G_f$. So $\ell \in Iso_G(f)$ and we must also have another diagram

$$\begin{array}{ccc} Y & \xrightarrow{f} & Z \\ \ell \downarrow & & \downarrow k \\ Y & \xrightarrow[f]{} & Z \end{array}$$

and by combining both diagrams we get

$$\begin{array}{ccccc} X & \xrightarrow{g} & Y & \xrightarrow{f} & Z \\ h \downarrow & & \downarrow \ell & & \downarrow k \\ X & \xrightarrow[g']{} & Y & \xrightarrow[f]{} & Z \end{array}$$

and hence $f \circ g \sim_G f \circ g'$.

b) As above we know that $\exists\, (h,\ell) \in \Lambda(X,Y)$ such that $g' = \ell^{-1} \circ g \circ h$. Therefore we can write locally:

$$h_* (I [f \circ \ell^{-1} \circ g]) = I [f \circ g'] \qquad \text{(i)}$$

where $I[\ ]$ denotes the local ideal generated by the map inside the brackets. Now from [10, lemma 1.5 , pg. 118] it follows that the contact class of the composed map doesn't depend on the submersion f or $f \circ \ell^{-1}$. Consequently

$$I (f \circ \ell^{-1} \circ g) = I (f \circ g) \qquad \text{(ii)}$$

Now (i) and (ii) imply that $(f \circ g')_x \sim_K (f \circ g)_x$. That is, there are diffeo-germs .

$$h : (X,x) \longrightarrow (X,x')$$

and

$$k : (X \times \mathbb{R}^p, (x,z)) \longrightarrow (X \times \mathbb{R}^p, (x',z')$$

where $z = (f \circ g)(x)$, $z' = (f \circ g')(x)$, making commutative the diagram

$$\begin{array}{ccc} (X \times \mathbb{R}^p, (x,z)) & \xrightarrow{k} & (X \times \mathbb{R}^p, (x',z')) \\ \pi_1 \downarrow & & \downarrow \pi_1 \\ (X, x) & \xrightarrow[h]{} & (X, x') \end{array}$$

Notice that the diffeomorphism k may be chosen to be linear when restricted to each fiber of π_1 in $X \times \mathbb{R}^p$. That is $\pi_2 \circ k \mid \pi_1^{-1}(\alpha) \in$ $\in G\ell(\mathbb{R}^p)$, $\forall \alpha \in X$.

Now, by using a standard argument of partitions of unity on X we can obtain a global diffeomorphism $K : X \times \mathbb{R}^p \longrightarrow X \times \mathbb{R}^p$ and conclude that $f \circ g \sim_G f \circ g'$. □

Lemma 2. Let $f \in \text{Subm}^\infty(Y,Z)$, then for all $g, g' \in \text{Emb}^\infty(X,Y)$ we have:

a) For $G = A(X,Z)$:

i) $(f \circ g)_x \sim_G (f \circ g')_{x'} \iff g_x \sim_{G_f} g'_{x'}$;

ii) $f \circ g \sim_{G*} f \circ g' \implies g \sim_{G_f} g'$.

b) For $G = K(X,Z)$:

i) $(f \circ g)_x \sim_K (f \circ g')_{x'} \iff g_x \sim_{K_f} g'_{x'}$;

ii) $f \circ g \sim_C f \circ g' \iff g \sim_{C_f} g'$,
where, as usual $C = C(X,Z)$ is defined as the subgroup $\{ (1,k) \in K(X,Z) \}$.

Proof.

a) Necessity in i) is just Lemma 1. Let us prove the sufficiency: $(f \circ g)_x \sim_G (f \circ g')_{x'}$ means that we can find local diffeos $h : (X,x) \to (X,x')$ and $k : (Z,z) \longrightarrow (Z,z')$, where $(f \circ g)(x) = z$ and $(f \circ g')(x') = z'$, such that the following diagram commutes locally,

$$\begin{array}{ccccc} (X,x) & \xrightarrow{g} & (Y, g(x)) & \xrightarrow{f} & (Z,z) \\ h \downarrow & & & & \downarrow k \\ (X,x) & \xrightarrow[g']{} & (Y,g'(x')) & \xrightarrow[f]{} & (Z,z') \end{array} \qquad (1)$$

We now need to find $(h',\ell) \in G_f$ such that

$$\begin{array}{ccc} (X,x) & \xrightarrow{g} & (Y, g(x)) \\ h' \downarrow & & \downarrow \ell' \\ (X,x') & \xrightarrow[g']{} & (Y,g'(x')) \end{array}$$

be a locally commutative diagram.

Observe that f and $k \circ f$ are both local submersions at some neighbourhoods of $y' = g'(x')$ and $y = g(x)$ respectively. And thus there is a local diffeomorphism $\ell : (Y,y) \longrightarrow (Y,y')$ such that

$$k \circ f = f \circ \ell \qquad (2)$$

We now define $h' \in R(X)$ by conveniently modifying h in such a way that $(h',\ell) \in G_f$ (in a local sense). Firstly observe that from the commutativity of the diagram (1) it follows $f \circ g = k^{-1} \circ f \circ g' \circ h$, and from (2) we have $k \circ f \circ \ell^{-1} = f$. Therefore $f \circ g = f \circ \ell^{-1} \circ g' \circ h$. That is, in a small enough neighbourhood N_z of z in Z we can write

$$g^{-1}(f^{-1}(c)) = h^{-1} \circ g'^{-1} \circ \ell(f^{-1}(c)) \ , \quad \forall\, c \in N_z \ ,$$

i.e. $$h(g^{-1}(f^{-1}(c))) = g'^{-1}(\ell(f^{-1}(c))) \ , \quad \forall\, c \in N_z \ .$$

So we can define a diffeo-germ

$$T : (X,x) \longrightarrow (X,x')$$

as follows: given $a \in X$ sufficiently close to x, then $g(a) \in f^{-1}(c)$, for some $c \in N_z \subset Z$. Put $T(a) = (h^{-1} \circ g'^{-1} \circ \ell)(g(a))$.

The fact that g' is an embedding ensures that, T is a diffeo-germ at x.

If we take $h' = h \circ T$ it is not difficult to see that this is the local diffeo satisfying the required conditions.

ii) We can use an analogous argument to part i) but assuming that k is in the component of the identity, 1_Z, on Z. Hence $kf \sim_{R(Y)} f$ and we can get a globally defined diffeo $\ell \in R(Y)$, for g and g' are embeddings.

b) The proof of i) can be found in [10, Thm. 1.4]. The idea is the following: suppose first that $\dim X = \dim(f^{-1}(0))$. We can view Y locally as $\mathbb{R}^k \times \mathbb{R}^p$ and $f : \mathbb{R}^k \times \mathbb{R}^p \longrightarrow \mathbb{R}^p$ as the usual projection.

We can also consider locally, $g(X)$ and $g'(X)$ as the graphs of the maps $\phi : \mathbb{R}^k \longrightarrow \mathbb{R}^p$ and $\phi' : \mathbb{R}^k \longrightarrow \mathbb{R}^p$ respectively. (Observe that for this particular case we have $X \cong f^{-1}(0) = \mathbb{R}^k$). Now, we know from the hipothesis that $\phi \sim_K \phi'$. Hence any diffeomorphism $\ell \in R(\mathbb{R}^k \times \mathbb{R}^p)$ taking ϕ to ϕ' will carry $g(X)$ onto $g'(X)$ and leave $f^{-1}(0)$ invariant. Therefore, for the equidimensional case we have proven that $(f \circ g)_x \sim_K (f \circ g')_{x'}$ implies $g_x \sim_{K_f} g'_{x'}$. It also follows from this, that by taking $h = (g')^{-1} \circ \ell \circ g$ and $k = f \circ \ell$, we can obtain a diagram for the K_f-equivalence.

The case $\dim X \neq \dim f^{-1}(0)$ reduces to the equidimensional one by considering a suspension of the smaller dimensional manifold. Montaldi proves [10, Lemma 1.6] that the contact class is invariant by suspension.

ii) It is enough to prove that $(f \circ g) \sim_C (f \circ g') \Longrightarrow g \sim_{C_f} g'$ for near enough embeddings g and g'.

We know from i) that given a point $x \in X$, there is a diffeo ℓ taking a neighbourhood V_x of $g(x)$ in Y to a neighbourhood V'_x of $g'(x)$ in Y, such that $\ell(g(X) \cap V_x) = g'(X) \cap V'_x$, where ℓ leaves $f^{-1}(0)$ fixed.

By using the compacity of X and taking a convenient partition of unity, we can define a diffeo $\tilde{\ell}$ from a closed neighbourhood V of $g(X)$ onto a closed neighbourhood V' of $g(X')$ in Y, such that $\tilde{\ell}(g(X)) = g'(X)$ and $\tilde{\ell}(f^{-1}(0)) = f^{-1}(0)$.

Now, since g and g' are near enough to each other we can assume that $\tilde{\ell} : V \longrightarrow V'$ is near enough to the identity on Y restricted to V. Then it is easy to see that there is a diffeo $\ell \in R(Y)$ extending $\tilde{\ell}$ that satisfies the required conditions.

Remarks

1. We can obtain the result of b ii) above for K_f by imposing the condition that g and $g' \circ h$ must be sufficiently near, where h is

the source diffeo in the $\mathcal{K}$-equivalence between $(f \circ g)$ and $(f \circ g')$.

2. If Y is compact, (Y,Z,f) is a locally trivial fibre-bundle and the implication

$$f \circ g \sim_G f \circ g' \implies g \sim_{G_f} g'$$

will hold whenever there exists a bundle map $\ell : Y \to Y$ completing the diagram

$$\begin{array}{ccc} Y & \xrightarrow{f} & Z \\ \ell \downarrow & & \downarrow k \\ Y & \xrightarrow[f]{} & Z \end{array}$$

for all $k \in \mathcal{R}(Z)$. (We thank Prof. J. Daccach and Prof. C. Biasi for helpful conversations on this subject).

As in most applications of Lemma 2, one has $Y = \mathbb{R}^n$, we consider here the general case in which Y does not need to be compact.

3. Consider the map

$$f_* : C^\infty(X,Y) \longrightarrow C^\infty(X,Z)$$
$$g \longmapsto f \circ g \quad .$$

Lemma 1 above implies that

$$f_*(G_f\text{-orbit of } g) \subset G\text{-orbit of } f \circ g .$$

On the other hand from Lemma 2 it follows:

a) $f_*^{-1}(\mathcal{A}^*\text{-orbit of } f \circ g) \subset \mathcal{A}_f\text{-orbit of } g$.

b) $f_*^{-1}(\mathcal{C}\text{-orbit of } f \circ g) \subset \mathcal{C}_f\text{-orbit of } g$.

Definitions. We say that g is G_f-stable if the G_f-orbit of g is open in $C^\infty(X,Y)$.

We say that $\phi \in C^\infty(X,Z)$ is G-stable if the G-orbit of ϕ is open in $C^\infty(X,Z)$.

Theorem 1. Let $f \in \mathrm{Subm}^\infty(Y,Z)$, then given any $g \in \mathrm{Emb}^\infty(X,Y)$, $g \circ f$ is G-stable $\iff$ g is G_f-stable.

Proof. Consider the continuous map

$$ {}_r\Gamma_f^k : {}_rJ^k(X,Y) \longrightarrow {}_rJ^k(X,Z) \quad , $$

$$ {}_rj^k\, h(x) \longmapsto {}_rj^k(f \circ h)(x) \quad , $$

where $x = (x_1, \text{—}, x_r) \in X^{(r)}$ (see [6] for notation). The assumption that f is a submersion ensures that ${}_r\Gamma_f^k$ is transversal to all the ${}_rG^k$ - invariant submanifolds of ${}_rJ^k(X,Z)$, $\forall\, k$, $\forall\, r$. Then by using the composition

$$ {}_rj(f \circ g) : X \xrightarrow{\ {}_rj^k g\ } {}_rJ^k(X,Y) \xrightarrow{\ {}_r\Gamma_f^k\ } {}_rJ^k(X\,,\,Z) $$

$$ x \longmapsto {}_rj^k\, g(x) \longmapsto {}_rj^k(f \circ g)(x) $$

we have that given any ${}_rG^k$ - invariant submanifold W of ${}_rJ^k(X,Z)$:

$$ {}_rj^k(f \circ g) \pitchfork W \iff {}_rj^k g \pitchfork (\,{}_r\Gamma^k)^{-1}(W) \quad . $$

Suppose that g is G_f-stable and let $\phi = f \circ g$. Let $W_x \subset {}_rJ^k(X,Z)$ be the ${}_rG^k$ - orbit of ${}_rj^k\,\phi(x)$. Then we know from Lemma 1 that $(\,{}_r\Gamma_f^k)^{-1}(W_x)$ must contain the ${}_rG_f^k$ - orbit, ${}_r\Omega_x^k$ of ${}_rj^k\, g(x)$. Now,

$${}_r j^k g \pitchfork {}_r\Omega^k_x \Longrightarrow {}_r j^k g \pitchfork ({}_r\Gamma^k_f)^{-1}(W_x) \Longrightarrow {}_r\Gamma^k_f \circ j^k g \pitchfork W_x \ , \forall\, r \ , \ \forall\, k \ .$$

Hence ϕ is G-stable (see [6]) .

Reciprocally, let ϕ be G-stable. Then there is a neighbourhood N_ϕ of ϕ in $C^\infty(X,Z)$ contained in the G-orbit of ϕ . Let g' be an embedding sufficiently close to g , such that $f \circ g' \in N_\phi$.

Now, N_ϕ may be taken small enough such that $f \circ g' \sim_G \phi$ where the diffeomorphism in the G-equivalence is sufficiently close to the identity. Consequently, from lemma 2 if follows that $g' \sim_{G_f} g$ and hence g is G_f-stable.

Observe that for $G = K$ we consider $Z = \mathbb{R}^p$. Also notice that in this case the proof may be simplified by just considering transversality of jets instead of multijets. Moreover $\forall\ x\ ,\ x' \notin (f \circ g)^{-1}(0)$, $j^k(f \circ g)(x)$ and $j^k(f \circ g')(x')$ are in the same K-orbit. $\square$

Remark. We have used the fact that f is a submersion to ensure that a) $k \circ f \sim_{R(Y)} f$, for each diffeomorphism k near to 1_Z ; b) ${}_r\Gamma^k_f$ is transversal to all the ${}_rG^k$ - orbits in ${}_rJ^k(X,Z)$. It can be shown that condition b still holds in some other cases, for instance when $Z = \mathbb{R}$ and f is a Morse function.

Clearly, condition a does not hold in general when f is not a submersion. However, if f is any $A(Y,Z)$-stable map, we know that there exists a diffeomorphism k' near to 1_Z such that $k' \circ k \circ f \sim_{R(Y)} f$. We conjecture that the results in Theorem 1 above should hold under the hypothesis of the A-stability of f .

1.2 We now fix $g \in C^\infty(X,Y)$. We also consider here two cases:

a) $G = A$, then we define

$$G_g = \{ (\ell,k) \in G(Y,Z) : \ell(g(X)) \subset g(X) \}$$

b) $G = K$, then we put

$$G_g = \{ (\ell,H) \in G(Y,\mathbb{R}^p) : \ell(g(X)) \subset g(X) \} \quad ,$$

where each (ℓ,H) corresponds to a diagram

$$\begin{array}{ccccccc} Y & \xrightarrow{\iota} & Y \times \mathbb{R}^p & \xrightarrow{\pi_1} & Y & \xleftarrow{g} & X \\ \ell \downarrow & & \downarrow H & & \downarrow \ell & & \downarrow h \\ Y & \xrightarrow{\iota} & Y \times \mathbb{R}^p & \xrightarrow{\pi_1} & Y & \xleftarrow{g} & X \end{array} \quad .$$

Lemma 3. Given a 1 : 1 immersion g from X to Y and

i) $G = A(X,Z)$, then

$$f \sim_{G_g} f' \implies f \circ g \sim_G f' \circ g \quad , \quad \forall\ f, f' \in C^\infty(Y,Z) \quad .$$

ii) $G = K(X,Z)$, $Z = \mathbb{R}^p$, then

$$f \sim_{G_g} f' \implies f \circ g \sim_G f' \circ g \quad , \quad \forall\ f, f' \in C^\infty(Y,\mathbb{R}^p) \quad .$$

Proof.

i) If f and f' are in the same G_g-orbit, then we may find diffeomorphisms $\ell \in R(Y)$ and $k \in R(Z)$, such that $\ell(g(X)) \subset g(X)$. Hence, given $x \in X$, $g(x) \in Y$ we have $\ell(g(x)) = y \in g(X)$. Now, g is a 1 : 1 immersion and thus there exists a unique $x' \in X$, such that $g(x') = y$. We define $h(x) = x'$ and then we have that $(h,k) \in G$

and the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{f \circ g} & Z \\ h \downarrow & & \downarrow k \\ X & \xrightarrow[f' \circ g]{} & Z \end{array}$$

Therefore $f \circ g \sim_G f' \circ g$.

ii) If $G = K(X, \mathbb{R}^p)$ and $f \sim_{G_g} f'$. This implies the existence of $(\ell, H) \in K(Y, \mathbb{R}^p)$, with $\ell(g(X)) = g(X)$ making commutative the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{g} & Y & \xrightarrow{(id,f)} & Y \times \mathbb{R}^p & \xrightarrow{\pi_1} & Y \\ \downarrow & & \downarrow \ell & & \downarrow H & & \downarrow \ell \\ X & \xrightarrow{g} & Y & \xrightarrow{(id,f')} & Y \ \ \mathbb{R}^p & \xrightarrow{\pi_1} & Y \end{array}$$

and we define $h : X \to X$ by $h(x) = (g^{-1} \circ \ell \circ g)(x)$. Clearly h is a diffeomorphism and the pair $(h, \tilde{H})$, where $\tilde{H} = H|_{g(X) \times \mathbb{R}^p}$ defines the required K-equivalence between $f \circ g$ and $f' \circ g$. □

Lemma 4. Let g be a fixed $1:1$ immersion from X to Y . Then the following holds

i) if $G = A(X,Z)$: $f \circ g \sim_{G*} f' \circ g \implies f \sim_{G_g} f'$
$\forall f, f' \in Subm^\infty(Y,Z)$, close enough.

ii) if $G = K(X,Z)$, $Z = \mathbb{R}^p$,

a) $(f \circ g)_x \sim_K (f' \circ g)_{x'} \iff f_x \sim_{G_g} f'_{x'}$,
$\forall f, f' \in Subm^\infty(Y, \mathbb{R}^p)$.

b) $(f \circ g) \sim_{\mathcal{C}} (f' \circ g) \iff f \sim_{\mathcal{C}_g} f'$, $\forall\, f, f'$ sufficiently close submersions from Y to $\mathbb{R}^p$ Here $\mathcal{C}_g$ is the obvious subgroup of K_g .

Proof.

i) Let $G = A(X,Z)$ and G^* as previously defined. If $f \circ g \sim_{G^*} f' \circ g$ this means that there is a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{g} & Y & \xrightarrow{f} & Z \\ h \big\downarrow & & & & \big\downarrow k \\ X & \xrightarrow[g]{} & Y & \xrightarrow[f']{} & Z \end{array}$$

with $h \in R(X)$ and $k \in R(Z)$ lying in the component of 1_Y . Then provided that f and f' are close enough submersions we can assert that $k f$ and f' are right equivalent. That is, $\exists\, \ell \in R(Y)$ such that $k f = \ell f'$, where this ℓ can be taken in the component of the identity. Now, (ℓ, k) is not necessary an element of G_g , for it is not clear that $\ell(g(X)) \subset g(X)$. We shall hence find another diffeo $\ell' : Y \to Y$ satisfying:

1) $\ell'(\ell(g(X)) \subset g(X)$;

2) ℓ' carries the fibres of f' into the fibres of f' ; i.e. $\exists\, k' : Z \to Z$ diffeomorphism, such that $(\ell', k') \in \mathrm{Iso}(f')$.

First observe that $f' \circ \ell \circ g \sim_{G^*} f' \circ g$, for $1_Z \circ (f' \circ \ell \circ g) = (f' \circ g) \circ h$, with $(h, 1_Z) \in G^*$. So from, Lemma 2 we conclude that $\ell \circ g \sim_{G_{f'}} g$. That is, $\exists\, (h', \ell') \in G_{f'}$ such that $\ell' \circ (\ell \circ g) = g \circ h'$, with $\ell' \in Iso(f')$; so there is a diffeo $k' \in R(Z)$ satisfying $k' \circ f' = f' \circ \ell'$.

Hence the pair $(\ell' \circ \ell\ ,\ k' \circ k)$ is in G_g and $(k' \circ k)f = f' \circ (\ell' \circ \ell)$. Consequently $f \sim_{G_g} f'$.

ii) a) follows from [10, Thm 1.4]. Observe that this proof becomes analogous to that of Lemma 2 b i) by interchanging the rules of the 1 : 1 immersion g and the submersion f .

b) This proof is also similar to that of Lemma 2 b ii) in which we interchange $f^{-1}(0)$ with $g(X)$ and $g'(X)$ with $(f')^{-1}(0)$. □

Remark. The corresponding result for K_g in above lemma can be obtained with the additional hypothesis that the diffeomorphism h in the source of the K-equivalence between $f \circ g$ and $f' \circ g$ be close to the identity.

In a similar way to section 1.1 we may consider here a map

$$g^* : C^\infty(Y,Z) \longrightarrow C^\infty(X,Z)$$

$$f \longmapsto f \circ g$$

which is continuous, for X is compact (see [6 , pg. 49]) .

Then from Lemma 3 we deduce that g^* carries G_g-orbits into G-orbits. So the counter-image of a G-orbit through $(g^*)^{-1}$ must be a union of G_g-orbits.

Again the map f is said to be G_g-stable if its G_g-orbit is open in $C^\infty(Y,Z)$.

Theorem 2. Let $g \in \mathrm{Emb}^{\infty}(X,Y)$, then

$$f \circ g \text{ is } G\text{-stable} \iff f \text{ is } G_g\text{-stable} , \quad \forall f \in \mathrm{Subm}^{\infty}(Y,Z) .$$

Proof. Denote by $J^k_{g(X)}(Y,Z)$ the subset of k-jets of maps from Y to Z with source in $g(X) \subset Y$. Analogously ${}_rJ^k_{g(X)}$ represents the multijets of such maps with sources in $(g(X))^{(r)} \subset Y^{(r)}$. We can define a map

$$\Gamma^k_g : J^k_{g(X)}(Y,Z) \longrightarrow J^k(X,Z)$$

$$j^k h(y) \longmapsto j^k(h \circ g)(g^{-1}(y)) .$$

This is well defined because g is an immersion 1 : 1 . Moreover, it is a tedious but straightforward exercise to see that:

1) Γ^k_g is continuous, $\forall k$; and

2) Γ^k_g is a submersion $\forall k$.

And the same assertions can be made about the multijet version

$${}_r\Gamma^k_g : {}_rJ^k_{g(X)}(Y,Z) \longrightarrow {}_rJ^k(X,Z)$$

$$(j^k h_1(y_1), \text{—}, j^k h_r(y_r)) \longmapsto (j^k(h_1 \circ g)(g^{-1}(y)), \text{—}, j^k(h_r \circ g)(g^{-1}(y)))$$

$\forall k$, $\forall r \geq 1$.

Let us suppose that f is G_g-stable, then given any $y \in Y^{(r)}$ if we denote $\omega = {}_rj^k f(y)$ and W_ω is the ${}_rG^k_g$-orbit of ω in ${}_rJ^k(Y,Z)$, $k \geq 1$, $r \geq 1$ we can write

$${}_rj^k f \pitchfork_\omega W_\omega \qquad (*)$$

where ${}_rj^k f : Y^{(r)} \longrightarrow {}_rJ^k(Y,Z)$ is the multijet extension map of f.

Let $\phi = f \circ g$ and Ω_σ be the ${}_rG^k$-orbit of $\sigma = {}_rj^k\phi(x)$ with $x = g^{-1}(y)$, for some $y \in (g(X))^{(r)} \subset Y^{(r)}$. In order to prove that ϕ is G-stable it is enough to see that ${}_rj^k\phi \pitchfork_\sigma \Omega_\sigma$, for any $x \in X^{(r)}$, $\forall k$, $\forall r \geq 1$, (see [6]).

Observe that we can write ${}_rj^k\phi$ as the composition

$${}_rj^k\phi : X^{(r)} \xrightarrow{g^r} (g(X))^{(r)} \xrightarrow{{}_rjf|} {}_rJ^k_{g(X)}(Y,Z) \xrightarrow{{}_r\Gamma^k_g|} {}_rJ^k(X,Z),$$

where the vertical bar $|$ on the right of a map denotes the appropriate restriction of the considered map.

Notice that ${}_rJ^k_{g(X)}(Y,Z)$ is a union of ${}_rG^k_g$-orbits. In fact the orbits of G^k_g coincide with the G^k-orbits outside from $J^k_{g(X)}(Y,Z)$ and, on the other hand a k-jet with source in $g(X)$ cannot be G^k_g-equivalent to another k-jet with source off $g(X)$.

Now, from $(*)$ we have that

$$T_y\, {}_rj^k f\,(T_y Y^{(r)}) + T_\omega W_\omega = T_\omega\, {}_rJ^k(Y,Z) .$$

But this implies that

$$T_y\, {}_rj^k f\,(T_x g^r(T_x X^{(r)})) + T_\omega W_\omega = T_\omega\, {}_rJ^k_{g(X)}(Y,Z)$$

where $x = (g^r)^{-1}(y)$.

And by applying $T_\omega\, {}_r\Gamma^k_g$ on both sides we get

$$T_\omega\, {}_r\Gamma^k_g [\, T_y\, {}_rj^k f\,(T_x g^r(T_x X^{(r)}))\,] + T_\omega\, {}_r\Gamma^k_g (T_\omega W_\omega) =$$

$$= T_\omega\, {}_r\Gamma^k_g (T_\omega\, {}_rJ^k_{g(X)}(Y,Z)) .$$

That is,

$$T_x\, {}_rj^k\phi(T_xX) + T_\omega\, {}_r\Gamma^k_g(T_\omega W_\omega) = T_\sigma\, {}_rJ^k(X,Z) \quad ,$$

for ${}_r\Gamma^k_g$ being a submersion implies that

$$T_\omega\, {}_r\Gamma^k_g(T_\omega\, {}_rJ^k_{g(X)}(Y,Z)) = T_\sigma\, {}_rJ^k(X,Z) \quad .$$

Now from Lemma 3 it follows that ${}_r\Gamma^k_g(W_\omega) \subset \Omega_\sigma$ and hence we have

$$T_x\, {}_rj^k\phi(T_xX) + T_\sigma\, \Omega_\sigma = T_\sigma\, {}_rJ^k(X,Z) \quad ,$$

i.e. ${}_rj^k\phi \pitchfork_\sigma \Omega_\sigma$, as we wanted to show.

Reciprocally, suppose that $\phi = f \circ g$ is G-stable. This means that the G-orbit of ϕ contains some open neighbourhood N_ϕ of ϕ . Let f' be in a sufficiently small neighbourhood V_f of f such that f' is a submersion too and $f' \circ g \in N_\phi$. Then $f' \circ g \sim_G \phi$. In fact we can take V_f small enough such that $f' \circ g \sim_{G*} \phi$. Then from Lemma 4 we can conclude that $f' \sim_{G_g} f$. Hence V_f is contained in the G_g-orbit of f and therefore this is an open G_g-orbit . Consequently f is G_g-stable.

As in Thm. 1 we must remark that, for the case $G = K(X,\mathbb{R}^p)$, this proof can be simplified by just considering jet extensions instead of multijets. □

Note. It would be interesting to extend the results in Thms. 1 and 2 for any group G (with perhaps some convenient restrictions) acting on $C^\infty(X,Z)$, so that A and K could be particular cases.

Corollary 1. Given $f \in \mathrm{Subm}^\infty(Y,Z)$ and $g \in \mathrm{Emb}^\infty(X,Y)$,

$$f \text{ is } G_g\text{-stable} \iff g \text{ is } G_f\text{-stable}.$$

Proof. It follows immediately from Thms. 1 and 2 . □

Following N. A. Baas [2] we define a couple $(g,f) \in L = C^\infty(X,Y) \times \times C^\infty(Y,Z)$ to be structurally stable iff there exists a neighbourhood N of (g,f) in L such that for any $(g',f') \in N$, $\exists (h,\ell,k) \in \mathrm{Diff}(X) \times \times \mathrm{Diff}(Y) \times \mathrm{Diff}(Z)$ making commutative the diagram

$$\begin{array}{ccccc} X & \xrightarrow{g} & Y & \xrightarrow{f} & Z \\ h\big\downarrow & & \big\downarrow\ell & & \big\downarrow k \\ X & \xrightarrow[g']{} & Y & \xrightarrow[f']{} & Z \end{array}$$

Corollary : Given $f \in \mathrm{Subm}^\infty(Y,Z)$ and $g \in \mathrm{Emb}^\infty(X,Y)$,

$$(g, \ ) \text{ is structurally stable} \iff f \circ g \text{ is } A\text{-stable} .$$

Proof. For the necessity, we shall show that

$$(g,f) \text{ is structurally stable} \iff f \text{ is } A_g\text{-stable}$$

and then from Thm. 2 if follows that $f \circ g$ must be A-stable.

If (g,f) is structurally stable, then we can find a neighbourhood N of (g,f) in L , satisfying the requirement of the above definition. Observe that we can take N to be of the form $N_g \times N_f$, with N_g a neighbourhood of g in $C^\infty(X,Y)$ and N_f a neighbourhood of f in $C^\infty(Y,Z)$. Now the commutativity of the above diagram means that $g' = \ell \circ g \circ h^{-1}$ and $f' = k \circ f \circ \ell^{-1}$. But this implies that

$f' \circ g' = k \circ (f \circ g) \circ h^{-1}$, that is $f' \circ g' \sim_A f \circ g$.

With this we have proven that $\forall f' \in N_f$, $f' \sim_{A_g} f$ and hence that f is A_g-stable.

Let's see the sufficiency . Suppose that $f \circ g$ is A-stable. From Thm. 1 we know that g is A_f-stable and hence there is a neighbourhood V_g of g in $C^\infty(X,Y)$ such that $\forall g' \in V_g : g' \sim_{A_f} g$. In other words, we must have a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{g} & Y & \xrightarrow{f} & Z \\ h \downarrow & & \downarrow \ell & & \downarrow k \\ X & \xrightarrow[g']{} & Y & \xrightarrow[f]{} & Z \end{array}$$

with $h \in \mathrm{Diff}(X)$, $\ell \in \mathrm{Diff}(Y)$ and $k \in \mathrm{Diff}(Z)$.

Now, given $g' \in V_g$, we must have that g' is an embedding and it is also A_f-stable. Hence from Corollary 1 it follows that f is $A_{g'}$-stable. Therefore we can find a neighbourhood V_f of f in $C^\infty(Y,Z)$ such that $\forall f' \in V_f$, $f' \sim_{A_{g'}} f$, i.e. , there is a commutative diagram:

$$\begin{array}{ccccc} X & \xrightarrow{g'} & Y & \xrightarrow{f} & Z \\ h' \downarrow & & \downarrow \ell' & & \downarrow k' \\ X & \xrightarrow[g']{} & Y & \xrightarrow[f']{} & Z \end{array}$$

and from both we get

$$\begin{array}{ccccc} X & \xrightarrow{g} & Y & \xrightarrow{f} & Z \\ (h' \circ h) \downarrow & & \downarrow \ell' \circ \ell & & \downarrow k' \circ k \\ X & \xrightarrow[g']{} & Y & \xrightarrow[f']{} & Z \end{array}$$

which implies that (g,f) is equivalent in the sense of Baas [2] to (g',f') and this is $\forall\ (g',f') \in V_g \times V_f$. Hence (g,f) is structurally stable. □

§2. VERSALITY OF FAMILIES OF COMPOSED MAPS

We shall study along this section families of compositions of the two following kinds:

a) $X \times U \xrightarrow{H} Y \xrightarrow{f} Z$, $H \in C^\infty(X \times U, Y)$, $f \in Subm^\infty(Y,Z)$;

and

b) $X \times U \xrightarrow{h \times 1_U} Y \times U \xrightarrow{F} Z$, $h \in Imm^\infty(X,Y)$, $F \in C^\infty(Y \times U, Z)$,

being U a parameter manifold in both cases and 1_U the identity map on U.

For case a we have the following results:

a1) If f is a fixed submersion and H varies among the C^∞-families of immersions from X to Y, then

$$H \text{ is } G_f\text{-versal} \iff f \circ H \text{ is } G\text{-versal}.$$

a2) If H is a fixed C^∞-family of immersions and $\dim U$ is small enough (e.g. $\dim U \leq 6$ for $Z = \mathbb{R}$ and $G = A$) then $\exists$ residual subset F of $Subm^\infty(Y,Z)$ such that $\forall\ f \in F$, the family $f \circ H$ is G-versal.

And the following holds for case b:

b1) If h is a fixed immersion and F varies among the C^∞-families of submersions from Y to Z, then

$$F \text{ is } G_h\text{-versal} \iff F \circ (h \times 1_U) \text{ is } G\text{-versal}.$$

b2) If F is a fixed C^∞-family of submersions from Y to Z and dim U is small enough, then $\exists$ residual subset $H \subset \mathrm{Imm}^\infty(X,Y)$ such that $\forall\, h \in H$, the family $F\circ(h \times 1_U)$ is G-versal.

Remark. Actually just results a1 and b1 will be treated from our group actions viewpoint. Nevertheless, we have included a2 and b2 in order to exhibit a more complete picture of the possibilities that arise with this kind of compositions. We give followingly the proofs of a1, a2 and b1. A proof of b2 for $G = K$ can be found in (J.A. Montaldi [10]) and for $G = A$ in (G. Wassermann [14]). It is also mentioned in [10] (without proof) that result a2 is true.

a1) Let f be a fixed submersion from Y to Z and consider the compositions

$$\Phi : X \times U \xrightarrow{H} Y \xrightarrow{f} Z .$$

We denote,

$$j_1^k \Phi : X \times U \longrightarrow J^k(X,Z)$$

$$(x,u) \longmapsto j^k \Phi_u(x)$$

and

$$j_1^k H : X \times U \longrightarrow J^k(X,Y)$$

$$(x,u) \longmapsto j^k H_u(x) .$$

Note. When $G = K$ it would be enough to ask f to be a submersion at $f^{-1}(0)$.

Proposition 1. Given any (G^k-invariant) submanifold S of $J^k(X,Z)$ the subset $\{ H \in C^\infty(X \times U, Y) : j_1^k \Phi \pitchfork S \}$ is residual in $C^\infty(X \times U, Y)$ with the Whitney C^∞-topology.

Proof. Consider the map

$$\Gamma_f^k : J^k(X,Y) \longrightarrow J^k(X,Z)$$
$$j^k h(x) \longmapsto j^k(f \circ h)(x) .$$

It is not hard to show that the fact that f is a submersion implies that Γ_f^k is a submersion too, $\forall k$. Therefore Γ_f^k is transversal to any submanifold of $J^k(X,Z)$, in particular to S . And hence $\Omega = (\Gamma_f^k)^{-1}(S)$ is a submanifold of $J^k(X,Y)$ which is G_f - invariant (this follows from lemma 1 in §1) . Now,

$$j_1^k H \pitchfork \Omega \iff \Gamma_f^k \circ j_1^k H \pitchfork S$$
$$\iff j_1^k \Phi \pitchfork S \qquad (\text{for} \quad \Gamma_f^k \circ j_1^k H = j_1^k \Phi) .$$

And the required result follows immediately from the following version of the Thom transversality Theorem whose proof can be found in (Bruce [3]) : Let Ω be a submanifold of the jet space $J^k(X,Y)$. There is a residual subset of smooth maps $H \in C^\infty(X \times U, Y)$ such that the jet extension $j_1^k H : X \times U \to J^k(X,Y)$ is transverse to Ω . □

Corollary 1. Let $f : Y \to Z$ be a fixed submersion. Then, for any C^∞-family of immersions H from X to Y we have

$$f \circ H \text{ is a } G\text{-versal family} \iff H \text{ is a } G_f\text{-versal family.}$$

Proof. It follows easily from Proposition 1 above together with Lemma 2 in §1 and the characterization of versality in terms of transversality to the orbits of the corresponding group actions. □

Remarks. Observe that in general given a G^k-orbit S in $J^k(X,Z)$, the submanifold $(\Gamma_f^k)^{-1}(S)$ may contain more than one G_f-orbit. So when we consider $H : X \times U \to Y$ as a C^∞-family of maps (not necessarily immersions) from X to Y, we can only say that, when the dimension of the parameter manifold U is small enough (see [10] for an analysis of the relevance of this dimension), there is a residual subset of families $H \in C^\infty(X \times U, Y)$ such that $f \circ H$ is G-versal.

Notice that when $G = K$ the relevant orbits are those in $J^k_{f^{-1}(0)}(Y,Z)$, for the complement of this subset in $J^k(Y,Z)$ is a unique K-orbit.

a2) Let $H \in C^\infty(X \times U, Y)$ be a fixed family of immersions from X to Y.

Proposition 2. Given any (G^k-invariant) submanifold W of $J^k(X,Z)$, the subset $\tilde{R}_W = \{ f \in C^\infty(Y,Z) : j_1^k \Phi \pitchfork W \}$; with Φ as above, is residual in $C^\infty(Y,Z)$ with the Whitney C^∞-topology.

Proof. Let's define for each $u \in U$ a map

$$\Gamma_u^k : J^k_{H_u(X)}(Y,Z) \longrightarrow J^k(X,Z)$$

$$j^k h(H_u(a)) \longmapsto j^k(h \circ H_u)(a) \quad .$$

Notice that these must be submersions $\forall u$, for all the maps H_u are immersions (see proof of Theorem 2 in §1).

And hence we can define a map from the disjoint union $\bigsqcup_{u \in U} J^k_{H_u(X)}(Y,Z)$ to $J^k(X,Z)$, as

$$\Gamma^k : \bigsqcup_{u \in U} J^k_{H_u(X)}(Y,Z) \longrightarrow J^k(X,Z)$$

$$j^k h(H_u(a)) \longmapsto j^k(h \circ H_u)(a) \quad ,$$

which is also a submersion.

Observe that $\bigsqcup_{u \in U} J^k_{H_u(X)}(Y,Z)$ is a submanifold of $J^k(Y,Z) \times U$ which is stratified by the pull-backs of the G-orbits in $J^k(X,Z)$ by the submersions Γ_u , $u \in U$.

Now, the jet extension map $j^k_1 \Phi$ is also given by the composition

$$j^k_1\Phi : X \times U \xrightarrow{H} \bigsqcup_{u \in U} H_u(X) \xrightarrow{j^k f \,|\, \bigsqcup H_u(X)} \bigsqcup_{u \in U} J^k_{H_u(X)}(Y,Z) \xrightarrow{\Gamma} J^k(X,Z)$$

$$(x,u) \longmapsto H_u(x) \longmapsto j^k f(H_u(x)) \longmapsto j^k(f \circ H_u)(x).$$

Then given a G^k-invariant submanifold W in $J^k(X,Z)$, the pull--back $(\Gamma^k)^{-1}(W)$ is a submanifold of $\bigsqcup_{u \in U} J^k_{H_u(X)}(Y,Z)$. In fact it is a union of pull-backs of W by the maps Γ_u . It is also a submanifold of $J^k(Y,Z) \times U$.

We can use at this point the following variation of the Thom transversality theorem as given in [3] : If C is a submanifold of $J^k(Y,Z) \times U$, then the subset

$$T_C = \{ f \in C^\infty(Y,Z) : (j^k f \times 1_U) \pitchfork C \}$$

is residual in $C^\infty(Y,Z)$ with the C^∞-Whitney topology.

Now a similar argument to the one used along the proof of Theorem 2 in §1 shows that $\forall f \in T_C$, $j_1^k(f \circ H) \pitchfork W$. And the required result follows from the fact that $T_C \subset \tilde{R}_W$. □

b1) Let h be a fixed immersion 1 : 1 from X to Y. If $F \in C^\infty(Y \times U, Z)$ is a family of submersions from Y to Z with parameters in U, we denote their composition by

$$\Phi : X \times U \xrightarrow{h \times 1_U} Y \times U \xrightarrow{F} Z .$$

The jet extensions of the families F and Φ will be respectively written as $j_1^k F$ and $j_1^k \Phi$.

Proposition 3. With h, F and Φ as above and for any G^k-invariant submanifold S of $J^k(X, \ )$ we have that the subset $\{ F \in C^\infty(Y \times U, Z) : : j_1^k \Phi \pitchfork S \}$ is residual n $C^\infty(Y \times U, Z)$ with the Whitney C^∞-topology.

Proof. Let's denote by $J^k_{h(X)}(Y,Z)$ the subset of k-jets in $J^k(Y,Z)$ with sources in h(X), following the above notations. It is a submani fold of $J^k(Y,Z)$. As in Theorem 2 in §1, we have that the maps

$$\Gamma_h^k : J^k_{h(X)}(Y,Z) \longrightarrow J^k(X,Z)$$

$$j^k\phi(h(x)) \longmapsto j^k(\phi \circ h)(x)$$

are submersions and hence transversal to any submanifold of $J^k(X,Z)$. Now we can express $j_1^k \Phi$ as the following composition

$$j_1^k\Phi : X\times U \xrightarrow{h\times 1_U} h(X)\times U \xrightarrow{j_1^kF \mid h(X)\times U} J^k_{h(X)}(Y,Z) \xrightarrow{\Gamma_h^k} J^k(X,Z)$$

$$(x,u) \longmapsto (h(x),u) \longmapsto j_1^k F_u(h(x)) \longmapsto j_1^k(F_u \circ h)(x).$$

Given a G^k-invariant submanifold $S \subset J^k(X,Z)$, the pull-back $(\Gamma_h^k)^{-1}(S)$ is a submanifold of $J^k_{h(x)}(Y,Z)$ and hence of $J^k(Y,Z)$ that we shall denote by Ω. Now, the version of the Thom transversality theorem specified in Proposition 1 above tells us that the subset $T_\Omega = \{ F \in C^\infty(Y\times U, Z) : j_1^k F \pitchfork \Omega \}$ is residual in $C^\infty(Y\times U, Z)$ with the Whitney C^∞-topology. But we have that for any $F \in T_\Omega$,

$$T_{(y,u)} j_1^k F(T_{(y,u)}(Y\times U)) + T_{j_1^kF_u(y)}\Omega = T_{j_1^k F_u(y)} J^k(Y,Z),$$

$\forall (y,u) \in Y\times U$ such that $j_1^k F_u(y) \in \Omega$. Which (similarly to the proof of Thm. 2 in §1) implies that

$$T_{(h(x),u)} j_1^k F(T_{(h(x),u)}(h(X)\times U)) + T_{j_1^k F_u(y)}\Omega = T_{j_1^k F_u(y)} J^k_{h(X)}(Y,Z).$$

By applying now $T_{j_1^k F_u(h(x))}\Gamma_h^k$ on both sides we get that $j_1^k\Phi \pitchfork_{(x,u)} S$, $\forall (x,u) \in X\times U$ such that $j_1^k\Phi_u(x) \in S$, that is $j_1^k\Phi \pitchfork S$ as required. □

Corollary 2. Let h be a fixed injective immersion from X to Y. Then for any smooth family of submersions F from Y to Z we have,

$$F\circ(h\times 1_U) \text{ is } G\text{-versal} \iff F \text{ is } G_h\text{-versal}.$$

Proof. It follows as a consequence of Proposition 3 above, Lemma 4 in

§1 , and the characterization of versality of families in terms of transversality to the orbits of the considered group actions.

Comments.

1. J. W. Bruce obtains in [3] a result which is similar, in some sense to our Proposition 1 in §2 . The difference resides in the fact that we study here not only the finite-singularity-type but the versality of the composition. On the other hand we restrict our attention to smooth families H of immersions, whereas Bruce's result holds for arbitrary smooth families of maps.

As a consequence of a2 we have that $\forall\, u \in U$ the local submanifolds $H_u(X)$ are weakly transversal in Bruce's sense to $f^{-1}(z)$, $\forall\, z \in Z$, for all submersion $f \in \mathcal{F}$. Then all the Bruce's considerations in [3, pg. 116] apply. As a particular case we may consider the following

$$S^1 \times \mathbb{R}^p \xrightarrow{H} \mathbb{R}^3 \xrightarrow{f} \mathbb{R}$$

to conclude that, given any p-parameter family H of curves in $\mathbb{R}^3 (p \leq 6)$, there exists a residual subset $\mathcal{F}$ in $\mathrm{Subm}^\infty(\mathbb{R}^3 , \mathbb{R})$ such that all the surfaces $f^{-1}(t)$, $t \in \mathbb{R}$, are tangent to each curve of the family at most at a finite set of points.

2. An extensive geometrical study of the implications of result b2 for $G = K$ in low dimensions can be found in [10].

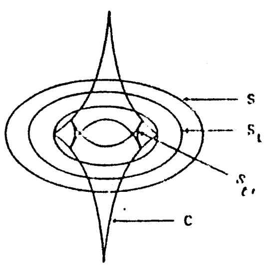
§3. AN APPLICATION TO STABILITY OF CAUSTICS

We shall restrict our attention to applications of case b in §2. In particular we shall consider compositions of type

$$\Phi : X \times Y \xrightarrow{h \times 1_Y} Y \times Y \xrightarrow{d} \mathbb{R} ,$$

being h embeddings of a manifold X into a manifold Y , and d families of submersions from Y to $\mathbb{R}$ with parameters in Y .

Suppose that there is a Riemannian metric on the manifold Y such that $d : Y \times Y \to \mathbb{R}$ is the induced distance map ($d(y_1,y_2)$ = length of the shortest geodesic joining y_1 to y_2 , locally, or globally provided Y is complete) . Let h be an embedding of a manifold X into Y . Then $S = h(X)$ is a submanifold of Y that we shall call <u>initial wave front</u>. If we start at this initial wave front and walk along the normal geodesics to S starting at each point of S , during a fixed small enough time t , we obtain a hypersurface S_t called <u>wave front at the time t</u> . We shall reach a moment at which these S_t beguin to have singularities, i.e. they are not smooth hypersurfaces anymore. If we join all these singularities we get the evolute, that we shall also call <u>caustic</u> C of S (see figure below) .



Wave fronts evolution and caustic

In terms of singularities we can also characterize the caustic as follows: Let $\Sigma\tilde{\Phi}$ be the singular set of the map

$$\tilde{\Phi} : X \times Y \longrightarrow \mathbb{R} \times Y$$

$$(x,y) \longmapsto (\phi(x,y) , y) \quad .$$

When d comes from the usual euclidean metric for $Y = \mathbb{R}^n$, then it is easy to see that $\Sigma\tilde{\Phi}$ is precisely the "normal bundle" NS of $S = h(X)$ in $\mathbb{R}^n$ (see [11]) . Now consider the restriction to $\Sigma\tilde{\Phi}$ of the projection $\pi_2 : X \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$. The image by π_2 of the singular set $\Sigma(\pi_2 \mid \Sigma\tilde{\Phi})$ is the evolute or caustic of S in $\mathbb{R}^n$. In general for submanifolds of $\mathbb{R}^n$ the definition of evolute may also be given as that of the locus of the centres of the spheres with higher order contact with S . In this sense see the work of Montaldi [10] based on the analysis of the contact types of the functions of the family

$$\Psi : X \times \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}$$

$$(x , y , t) \longmapsto (h(x) , y , t) \longmapsto d(h(x),y) - t \quad .$$

From a slightly different and more general viewpoint we could consider d as being locally defined by a " propagation law " induced from some P.D.E. or Hamiltonian function on T^*Y . For instance if we consider the wave equation $(\Delta - \frac{\partial^2}{\partial t^2}) u = 0$ in $\mathbb{R}^3$, and we assume a sinusoidal solution $u(x,y,z,t) = a(x,y,z)\, e^{-\nu t + \nu\phi(x,y,z)}$ then the phase function ϕ must satisfy the eikonal equation of the geometical optics,

$$(\frac{\partial\phi}{\partial x})^2 + (\frac{\partial\phi}{\partial y})^2 + (\frac{\partial\phi}{\partial z})^2 = 1 \quad .$$

This corresponds exactly to the propagation along the normal rays in the euclidean space $\mathbb{R}^3$, so we obtain again the above model. The submanifold S gives the initial states in the propagation process, thus justifying the name of initial wave front. The caustics are defined in a similar way to the above one. Again, for the particular case of the wave equation, these have been classically studied by the Geometrical Optics (see [7]).

A question that has raised the interest of various mathematicians is that of stability and genericity of caustics. R. Thom proposed in [13] a model, based on the theory of unfoldings of smooth functions, for the versality of a caustic. This was rigorously treated by K. Jänich [8]. Afterwards G. Wassermann proved [14] that versality of the caustic is equivalent to stability under perturbations of the initial wave front alone. V. I. Arnold [1] and V. M. Zakalyukin [15] have also aborded the problem of stability of caustics and wave fronts by analysing the stable singularities of lagrangian and legendrian maps. An alternative viewpoint would rely on fixing the initial wave front and thinking about the evolution of the caustics (and succesive wave fronts) under perturbations of the "distance family" d . The goal of this section is thus the study of the stability and genericity concepts for caustics defined in this way. In fact, it seems interesting to investigate whether the stability with respect to perturbations of d alone implies versality of the caustic (and hence stability with respect to variations of the initial wave front), at least in small enough dimensions, namely $\dim Y \leq 6$. The first step consists, of course, in specifying which kind of perturbations of d will be considered. For instance, one can ask d to be defined from a Riemannian metric, as done by M.A. Buchner [4] in order to study the stability of the cut locus, or perhaps, we could also include some more general perturbations, and allow d to be induced from a homogeneous Hamiltonian on T^*Y. However, these cases, being more complicated, will be let to a separate study. We shall treat here instead a simpler situation that fits better with the techniques presented in sections 1 and 2. In this context we shall think of the family d as a smooth function

$$d : Y \times Y \longrightarrow \mathbb{R}$$

satisfying:

1) $d(y_1,y_2) \geq 0$, $\forall\, y_1,y_2 \in Y$ (positive) ;

2) $d(y_1,y_2) = 0 \iff y_1 = y_2$, $\forall\, y_1,y_2 \in Y$ (non-degenerate ;

3) $d(y_1,y_2) = d(y_2,y_1)$, $\forall\, y_1,y_2 \in Y$ (symmetric) ;

4) $d(y_1,y_3) \leq d(y_1,y_2) + d(y_2,y_3)$, $\forall\, y_1,y_2,y_3 \in Y$ (satisfying the triangle inequality) ;

5) the function
$$d_b : Y \longrightarrow \mathbb{R}$$
$$y \longmapsto d(y,b) ,$$
is a smooth submersion $\forall\, b \in Y$.

A function d with such properties will be called a smooth distance function on the manifold Y and the pair (Y,d) will be called a metric manifold. Let us denote by $\mathcal{M}(Y)$ the set of all smooth distances on Y. This set is topologized by the restriction of the Whitney C^∞-topology on $C^\infty(Y \times Y, \mathbb{R})$.

Notice that condition 5 ensures that all the d-spheres ,

$$S_d(b,\varepsilon) = \{ y \in Y : d(y,b) = \varepsilon \} = d_b^{-1}(\varepsilon)$$

are smooth 1-codimensional submanifolds of Y .

Given an embedding $h : X \to Y$ let us consider the composition

$$\Phi : X \times Y \xrightarrow{h \times 1_Y} Y \times Y \xrightarrow{d} \mathbb{R}$$

and the "distance-map"

$$\tilde{\Phi} : X \times Y \longrightarrow \mathbb{R} \times Y$$

$$(x,y) \longmapsto (\Phi(x,y) , y)$$

By analogy to the euclidean case studied by Porteus and Montaldi we define the <u>d-normal bundle of S in Y</u> as the subset

$$\Sigma\tilde{\Phi} = \{ (x,y) \in X \times Y : \text{rank } D\tilde{\Phi}(x,y) = \dim Y \} .$$

So a pair (x,y) is in the d-normal bundle of S iff the d-sphere $S_d(y , |h(x)-y|)$ has contact of order higher than one with the submanifold $S = h(X)$ at the point $h(x)$.

We can say that "generically" the subset $\Sigma\tilde{\Phi}$ is a submanifold of $X \times Y$. Here, the genericity refers to that of the map Φ as a family, in the sense that we know that if Φ is an $\mathcal{A}$-versal family , then $\Sigma\tilde{\Phi}$ must be a submanifold.

The <u>caustic</u> C associated to Φ is defined as in the euclidean case above, that is C is the image by the projection π_y of the singular set $\Sigma(\pi_y | \Sigma\tilde{\Phi})$ of the restriction map $\pi_y | \Sigma\tilde{\Phi}$. Following Jänich, a caustic C is said to be versal provided the family Φ is A-versal.

As a consequence of result b1 in section 2, we have that for a fixed initial wave front $S = h(X)$ in the manifold Y and for $\dim Y \leq 6$, there is a residual subset F in $C^{\infty}(Y \times Y, \mathbb{R})$ such that $\forall F \in F$, the composition $F \circ (h \circ 1_Y)$ is A-versal. In our particular case we want $\Phi = d \circ (h \circ 1_Y)$ to be a distance function on $S \times Y$ and d to to vary among the elements of $M(Y)$. In this sense we have the following:

Theorem. Given a fixed embedding $h : X \to Y$ and an A-invariant submanifold $\Omega \subset J^k(X, \mathbb{R})$, there is a dense subset $\mathcal{D}_{\Omega}$ in $M(Y)$ such that $\forall d \in \mathcal{D}_{\Omega}$ $j_1^k(d \circ (h \circ 1_Y)) \pitchfork \Omega$.

Proof. Let $\mathcal{D}_{\Omega} = \{ d \in M(Y) : j_1^k(d \circ (h \circ 1_Y)) \pitchfork \Omega \}$. In order to prove density, we show that $\forall d \in M(Y)$ there is a sequence $\{d_t\} \subset \mathcal{D}_{\Omega}$ that converges to d in the topology of $M(Y)$.

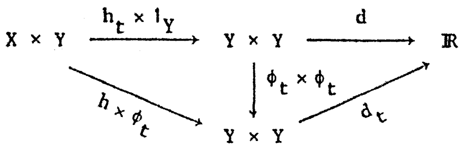
Consider the composition

$$\Phi : X \times Y \xrightarrow{h \times 1_Y} Y \times Y \xrightarrow{d} \mathbb{R} ,$$

either $j_1^k(d \circ (h \circ 1_Y)) \pitchfork \Omega$, in which case $d \in \mathcal{D}_{\Omega}$ and there is nothing to prove, or $j_1^k(d \circ (h \circ 1_Y)) \not\pitchfork \Omega$. Now, in the second case from the result specified in b.2 it follows that there is a sequence $\{h_t\}$ converging to h in the Whitney C^{∞}-topology on $\mathrm{Emb}^{\infty}(X,Y)$ such that $j_1^k(d \circ (h_t \times 1_Y)) \pitchfork \Omega$ $\forall t$.

We can work in a small enough neighbourhood of the embedding h such that $\forall t$ there is a diffeomorphism $\phi_t : Y \to Y$ with $h = \phi_t \circ h_t$ and such that $\{\phi_t\}$ converges to 1_Y.

Then we have a commutative diagram



where we define $d_t = d(\phi_t^{-1} \times \phi_t^{-1})$.

Let's prove now that $d_t \in \mathcal{D}_\Omega \quad \forall\, t$.

First we notice that $d_t \in M(Y)$, $\forall\, t$. In fact, it is not difficult to check that d_t satisfies all the conditions 1 - 5 in the definition of smooth distance function , $\forall\, t$. So let's show that $j_1^k(d_t \circ (h \times 1_Y)) \pitchfork \Omega \quad \forall\, t$. To see this observe that from commutativity of the above diagram

$$d_t \circ (h \times \phi_t) = d \circ (h_t \times 1_Y) \quad .$$

Now, $j_1^k(d \circ (h_t \times 1_Y)) \pitchfork \Omega$ and hence $j_1^k(d_t \circ (h \times \phi_t)) \pitchfork \Omega$.

And since ϕ_t is a diffeomorphism so is $1_X \times \phi_t : X \times Y \longrightarrow X \times Y$ and it is immediate to see that

$$j_1^k(d_t \circ (h \times \phi_t)) \pitchfork \Omega \iff j_1^k(d_t \circ (h \times 1_Y)) \pitchfork \Omega$$

as required.

Finally the fact that $\{\phi_t\}$ converges to 1_Y implies that $\{d_t\}$ converges to d in the Whitney C^∞-topology restricted to $M(Y)$. □

The following is an immediate consequence:

Corollary. Let $\dim Y \leq 6$. With the above hypothesis, there is a dense subset $\mathcal{D}$ in $M(Y)$., such that $\forall\, d \in \mathcal{D}$, the family $d \circ (h \times 1_Y)$ is $\mathcal{A}$-versal, and hence the corresponding caustic is versal.

Finally, if we define the caustic C of $\Phi = d \circ (h \times 1_Y)$ to be stable with respect to variations of the distance function whenever d is $\mathcal{A}_h$-versal, we shall have as a consequence of result b.1 in section 2, that C is versal iff C is stable w.r.t. variations of the distance function.

Remarks.

1. A distance function d on Y satisfying the above conditions 1-5 does not need to be always induced from a Riemannian metric on Y in contrast to what happens in the Euclidean case. In fact, metric manifolds generalize the complete Riemannian manifolds, for a Riemannian metric on a complete manifold Y always defines a globally smooth distance function on Y (see [12]). Or even, they can somehow be considered as quite a wild generalization of Finsler spaces (see [5]). As mentioned before our plans for a future work consist in imposing appropriate conditions on the perturbations of d as to obtain stability theorems for these more restrictive cases.

2. Observe that throughout all this section we have worked exclusivelly with the group $\mathcal{A}$. In a similar way, by working with the group $\mathcal{K}$, we could substitute the family d by another family,

$$D : Y \times Y \times \mathbb{R} \longrightarrow \mathbb{R}$$

$$(y_1, y_2, r) \longmapsto d(y_1, y_2) - r$$

of submersions at zero (and such that d satisfies conditions 1-4 above). Then we would obtain analogous results that could be interpreted as density of the families of d-spheres whose contacts with a fixed submanifold S of Y are non-degenerate.

BIBLIOGRAPHY

[1] V.I. ARNOLD 1973 Normal forms of functions close to degenerate critical points, the Weyl groups A_k, D_k, E_k, and Lagrangian singularities, *Funct. Anal. Appl.* 6 , 254-272.

[2] N.A. BAAS 1974 *Structural Stability of Composed Mappings*, Preprint, Institute for Advanced Study, Princeton, N.J.

[3] J.W. BRUCE 1986 On transversality. *Proc. Edinburgh Math. Soc.* 29, 115-123.

[4] M.A. BUCHNER 1977 Stability of the Cut Locus in Dimensions Less than or Equal to 6 . *Invent. Math.* 43 , 199-231.

[5] H. BUSEMANN 1955 *The Geometry of geodesics* , Acad. Press. Inc. Publ., New York.

[6] M. GOLUBITSKY and V. GUILLEMIN 1973 *Stable Mappings and Their Singularities.* GTM 14 , Springer-Verlag, New York.

[7] V. GUILLEMIN and S. STERNBERG 1977 *Geometric Asymptotics.* Math. Surveys 14 , AMS , Providence R.I.

[8] K. JÄNICH 1974 Caustics and Catastrophes. *Math. Ann.* 209 , 161-180.

[9] E.J.N. LOOIJENGA 1974 *Structural stability of families of C^∞-functions and the canonical stratification of $C^\infty(N)$.* Doctoral Thesis, Univ. of Amsterdam.

[10] J.A. MONTALDI 1983 *Contact , with applications to submanifolds of $\mathbb{R}^n$* . Ph. D. Thesis, Univ. of Liverpool.

[11] I.R. PORTEOUS 1971 The normal singularities of a submanifold. *J. Diff. Geom.* 5 , 543-564.

[12] M.M. POSTNIKOV 1967 *The Variational theory of geodesics.* W.B. Saunders Co. Filadelphia & London.

[13] R. THOM 1972 *Stabilité structurelle et morphogénèse.* Benjamin , Reading (Mass.).

[14] G. WASSERMANN 1975 Stability of Caustics. *Math. Ann.* 216, 43-50.

[15] V.M. ZAKALYUKIN 1976 Lagrangian and Legendrian singularities. *Funct. Anal. Appl.* 10 : 1(1976), 23-31.