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FINITE RELATIVE DETERMINATION FOR MORSE GERMS

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Introduction

Let $\varepsilon(n)$ the algebra of C^∞- germs at the origen from $\mathbb{R}^n$ to $\mathbb{R}$, and S a subspace of $\mathbb{R}^n$. We denote by $R_S^*(n) = R_S^*$ the group of diffeomorphisms which send S to S.

In his paper [1] P. Porto gives a geometric characterization for when a Morse germ $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ is 2- determined relative to $R_S^*(2)$, $S = y$-axis which is the transversality of S with the tangents of $f^{-1}(0)$. In this paper we give a version of this result for a Morse germ $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ and S an arbitrary subspace.

FINITE RELATIVE DETERMINATION FOR MORSE GERMS

Notation 1 We denote by $\varepsilon(n)$ the algebra of C^∞- germs at the origen from $\mathbb{R}^n$ to $\mathbb{R}$ and $m(n)$ its maximal ideal. Finally we denote by R_S^* the group of diffeomorphisms which send S to S.

Definition 2 Let f and g germs in $m(n)$. We say f and g are R_S^* equivalent if there exists $\phi \in R_S^*$ such that $g = f \circ \phi$.

Definition 3 (i) If k is a positive integer, we say that $f \in m(n)$ is k- determined relative to R_S^* if given $g \in m(n)$ such that $j^k f(0) = j^k g(0)$, then f and g are R_S^* equivalent.

(ii) f is finetely determined relative to R_S^* if there exists a positive integer k such that f is k- determined relative to R_S^*.

Theorem 4 (P. Porto) f and f/S are Morse germs if and only if f is 2- determined relative to R_S^*.

Let $< , >_k$ the interior product in $\mathbb{R}^n$ defined by

$$<(x_1, \ldots, x_n), (y_1, \ldots, y_n)>_k = \sum_{i=1}^{k} x_i y_i - \sum_{j=k+1} x_j y_j$$

We consider S an ℓ- dimensional subspace of $\mathbb{R}^n$ with $\{v_1, \ldots, v_\ell\}$ an orthogonal basis with respect to $< , >_k$ and A the $\ell \times \ell$ matrix which (i, j) term is given by $<v_i , v_j>_k$.

Let $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ with 2- jet at the origen given by $j^2f(0)(x) = \sum_{i=1}^{k} x_i^2 - \sum_{j=k+1}^{n} x_j^2 = <x, x>_k$, where $x = (x_1, \ldots, x_n)$.

Notation 5 We let $Z_2(f) = \{x \in \mathbb{R}^n \mid j^2f(0)(x) = 0\}$. If $p \in Z_2(f) - \{\bar{0}\}$ then the tangent space of $Z_2(f)-\{0\}$ at p, $T_p(Z_2(f)-\{\bar{0}\})$ is given by

$$T_p(Z_2(f)-\{\bar{0}\}) = \{y \in \mathbb{R}^n \mid <y,p>_k = 0\} .$$

We let $\mathbf{T} = \{T \mid T$ is a tangent space of $Z_2(f)-\{\bar{0}\}\}$.

Lemma 6 If S is transversal to every tangent space T in $\mathbf{T}$, then A is invertible.

Proof Suppose that A is not invertible, then there exists $c = (c_1, \ldots, c_\ell)$ not zero such that $A \cdot c = 0$, from which $<v_j, \sum_{i=1}^{\ell} c_i v_i>_k = 0$ for $j = 1, \ldots, \ell$ $(*)$ and since $\{v_j\}$ is ortnogonal we get $c_j <v_j, v_j>_k = 0$ for $j = 1, \ldots, \ell$.

It's clear that $p = \sum_{i=1}^{\ell} c_i v_i$ belongs to $Z_2(f)-\{0\}$ since $j^2f(0)(p) = <p,p>_k = \sum_{j=1}^{\ell} c_i^2 <v_i, v_i>_k = 0$. Moreover from $(*)$ it's clear that $v_j \in T = T_p(Z_2(f)-\{0\})$ $\forall j$, hence S is contained in T and we get a contradiction.

Theorem 7 Let S an ℓ dimensional subspace of $\mathbb{R}^n$ and $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ a Morse germ with $j^2f(0)(x) = \sum_{i=1}^{k} x_i^2 - \sum_{j=k+1}^{n} x_j^2$. If S is transversal to every tangent space T in $\mathbf{T}$, then f is 2- determined relative to S.

Proof Let $S = \langle v_1, \ldots, v_\ell \rangle$ then $j^2 f(0)(\sum_{i=1}^{\ell} c_i v_i) = \sum_{i=1}^{\ell} c_i^2 \langle v_i, v_i \rangle$. By lemma 6 A is invertible hence $j^2 f(0)|S$ is non degenerate and using theorem 4 we finish.

Lemma 8 Let S be an $(n-1)$-dimensional subspace of $\mathbb{R}^n$ and A the matrix constructed above. If A is invertible then S is transversal to every tangent space T in $\mathbf{T}$.

Proof Let $T = T_p(Z_2(f)-\{0\})$ not transversal to S, hence $T = S$ and since $p \in T_p(Z_2(f)-\{0\})$ we get $\langle p, v \rangle_k = 0$ for $v \in S$. Therefore A is not invertible.

Theorem 9 Let S be an $(n-1)$ dimensional subspace of $\mathbb{R}^n$ and $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ be a Morse germ with $j^2 f(0) = \sum_{i=1}^{k} x_i^2 - \sum_{j=k+1}^{n} x_j^2$. Then S is transversal to every space T in $\mathbf{T}$ if and only if f is 2- determined relative to R_S^*.

Proof The necessity was shown in theorem 7. For the sufficency, suppose there exists T in $\mathbf{T}$ such that $T = S$, hence by the previous lemma A is not invertible and then f/S is not a Morse germ, which contradicts theorem 4.

BIBLIOGRAPHY

[1] Porto P.F.S. and Loibel G.F. Relative Finite Determinacy and Relative Stability of Function Germs. Bol. Soc. Bras. Mat. Vol. 9, No. 2 (1978).